Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 1003–1010

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Characterization of Maximal Subsemigroups of the Injective Transformation Semigroups with equal Gap and Defect

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Abstract Suppose that X is an infinite set and $I(X)$ is the symmetric inverse semigroup defined on X. It is known that the semigroup $A(X)$ consisting of all $\alpha \in I(X)$ such that $|X \setminus \operatorname{dom} \alpha| = |X \setminus \operatorname{ran} \alpha|$ is a factorisable inverse subsemigroup of $I(X)$. In 2009, all maximal subsemigroups of $A(X)$ has been described when X is uncountable. So, it is an obvious question to ask what happens when X is countably infinite. In this paper, we answer this question by classifying all prime ideals of $A(X)$ and apply these results to characterize all maximal subsemigroups of $A(X)$ for an arbitrary infinite set X. Our results generalize and simplify the results obtained in 2009.

MSC: 20M20; 20M18

Keywords: transformation semigroup; maximal subsemigroup; inverse semigroup; factorisable semigroup; prime ideal

Submission date: 31.10.2019 / Acceptance date: 18.01.2022

1. Introduction

Suppose X is a non-empty set, and let $I(X)$ denote the *symmetric inverse semigroup* on X under composition (see [1, p.29]): that is, the set of all injective mappings α whose *domain*, $\operatorname{dom} \alpha$, and *range*, $X\alpha$ (or $\operatorname{ran} \alpha$) are subsets of X. We denote the composition of maps by jutaposition, and we compose maps from left to right. As usual, $|X|$ denotes the cardinality of X and we write $X \setminus Y = \{x \in X : x \notin Y\}$, where Y is a set. We also write

$$g(\alpha) = |X \setminus \operatorname{dom} \alpha|, \quad d(\alpha) = |X \setminus X\alpha| \text{ and } r(\alpha) = |X\alpha|,$$

and refer to these cardinals as the *gap*, the *defect* and the *rank* of α, respectively. And, as usual, $G(X)$ denotes the group of permutations of X: that is, the set of all bijective mappings from X to itself. We write

$$A(X) = \{\alpha \in I(X) : g(\alpha) = d(\alpha)\}$$

Published by The Mathematical Association of Thailand.

and call this the *injective transformation semigroup with equal gap and defect* defined on X. We observe that, if $\alpha \in G(X)$, then $\alpha \in I(X)$ with $g(\alpha) = d(\alpha) = 0$. Hence, $G(X) \subseteq A(X) \subseteq I(X)$ and we also note that $A(X) = I(X)$ when X is finite. In addition, the set of all idempotents in $I(X)$ and $A(X)$ coincide, this set consists of all identity transformations on a subset A of X, written as id_A. In particular, id_X is the identity of $I(X)$ and $A(X)$, and $\emptyset$ denotes the empty (one-to-one) mapping, which acts as a zero for $I(X)$ and $A(X)$.

The study of the semigroup $A(X)$ goes back to a 1974 paper of Chen and Hsieh [2]. They showed that $A(X)$ is a *factorisable* inverse semigroup (that is, $A(X) = GE$, where G is a subgroup and E is the set of idempotents of $A(X)$) and any inverse semigroup can be embedded in some $A(X)$. In [3, p.238], Jampachon, Saichalee and Sullivan remarked that, if $\alpha \in A(X)$ and $\alpha\beta = \mathrm{id}_X$ for some $\beta \in A(X)$ then $g(\alpha) = 0$, therefore $\alpha \in G(X)$. That is, $G(X)$ is the group of units in $A(X)$. They also showed that $A(X)$ is not a *locally factorisable semigroup* (a semigroup S is locally factorisable if eSe is factorisable for every idempotent e of S).

The semigroup $A(X)$ can be regarded as a model for all factorisable inverse semigroups since any factorisable inverse semigroup S can be embedded in $A(S)$, this was proved by Sanwong and Sullivan in [4]. In the same paper, the authors also characterized Green's relations and ideals on $A(X)$, and they used this to describe all maximal subsemigroups of $A(X)$ when X is uncountable. In one case, they showed that some maximal subsemigroups of $A(X)$ are induced by maximal subsemigroups of $G(X)$.

The purpose of this paper is to provide a description of all maximal subsemigroups of A(X) when X is an arbitrary infinite set. In section 2, we describe all prime ideals of $A(X)$. In section 3, the results of section 2 are applied to determine all maximal subsemigroups of $A(X)$ for any infinite set X, this result is a generalization of [4, Theorem 13].

In this paper, all notations and terminologies will be from [1] and [5] unless specified otherwise.

2. Prime Ideals

In this section, we give a characterization of prime ideals of $A(X)$. We first recall that, an ideal P of a semigroup S is termed *prime* if $ab \in P$ implies $a \in P$ or $b \in P$, equivalently, $S\backslash P$ is a semigroup. Let S be a semigroup and $\emptyset \neq T \subseteq S$. Then $\langle T \rangle$ denotes the subsemigroup of S generated by T, and, recall that a proper non-empty subsemigroup M of S is *maximal* in S if, whenever $M \subseteq N \subseteq S$ for some subsemigroup N of S, we have $M = N$ or $N = S$. Throughout this paper, to demonstrate the maximality of M, we show $\langle M, \alpha \rangle = S$ for all $a \in S\backslash M$, or equivalently, select $a, b \in S\backslash M$ and show that a can be expressed as a finite product of b and elements in M.

The following result has been proved in [6, Lemma 3.1] which may be rewritten as follows.

Lemma 2.1 ([6], Lemma 3.1)**.** *Let S be a semigroup and suppose that P is a prime ideal of S. Then the following statements hold:*

(*a*) *for any maximal subsemigroup M of $S\backslash P$, $M \cup P$ is a maximal subsemigroup of S;*

(*b*) *for any maximal subsemigroup N of S such that $(S\backslash P)\backslash N \neq \emptyset$ and $(S\backslash P) \cap N \neq \emptyset$, $(S\backslash P) \cap N$ is a maximal subsemigroup of $S\backslash P$.*

From Lemma 2.1, in order to determine some maximal subsemigroups of $A(X)$, we first classify all of its prime ideals. To do this, we recall from [4, Theorem 5] that, for $|X| = n \geq \aleph_0$ and for each cardinal p, let p' denote the successor of p (note that, when p is finite, $p' = p + 1$). Then the ideals of $A(X)$ are precisely the sets

$$A(r, d) = \{\alpha \in A(X) : r(\alpha) < r \text{ and } d(\alpha) \geq d\}$$

where $1 \leq r \leq n'$ and $0 \leq d \leq n$. The authors also showed in [4, p.331] that, these ideals form a chain and each of these containment is proper as follows:

$$\{\emptyset\} = A(1, n) \subset A(2, n) \subset ... \subset A(n, n) \subset$$

$$A(n', n) \subset ... \subset A(n', \aleph_0) \subset ... \subset A(n', 1) \subset A(n', 0) = A(X).$$

In what follows, for any subset Y of X, we write $Y = A \dot{\cup} B$ for Y is a disjoint union of A and B. We also need the fact: if $\alpha \in I(X)$ and $Y, Z \subseteq X$, then $\alpha^{-1} \in I(X)$ and $(Y \backslash Z)\alpha = Y\alpha \backslash Z\alpha$ (as usual, we interpret $Y\alpha$ as $(Y \cap \operatorname{dom} \alpha)\alpha$), this fact will be used without further mention. We may now describe all the prime ideals of $A(X)$.

Theorem 2.2. *Suppose that $|X| = n \geq \aleph_0$. The proper prime ideals of $A(X)$ are precisely the sets $A(n', k)$ where $k = 1$ or $\aleph_0 \leq k \leq n$.*

Proof. If $k = 1$, then $A(n', 1) = \{\alpha \in A(X) : d(\alpha) \geq 1\}$. Thus,

$$A(X) \backslash A(n', 1) = \{\alpha \in A(X) : d(\alpha) = 0\} = G(X),$$

which is a subsemigroup (also a subgroup) of $A(X)$. Hence $A(n', 1)$ is a prime ideal of $A(X)$. Next, suppose that $\aleph_0 \leq k \leq n$ and let $\alpha, \beta \in A(X) \backslash A(n', k)$. We note that

$$A(X) \backslash A(n', k) = \{\alpha \in A(X) : d(\alpha) < k\}.$$

Then $g(\alpha) < k$ and $g(\beta) < k$. Thus,

$$|X\alpha \setminus \operatorname{dom} \beta| \leq |X \setminus \operatorname{dom} \beta| = g(\beta) < k.$$

Consequently,

$$\begin{aligned} |\operatorname{dom} \alpha \setminus \operatorname{dom}(\alpha\beta)| &= |[X\alpha \setminus (X\alpha \cap \operatorname{dom} \beta)]\alpha^{-1}| \\ &= |(X\alpha \setminus \operatorname{dom} \beta)\alpha^{-1}| \\ &= |X\alpha \setminus \operatorname{dom} \beta| < k. \end{aligned}$$

Hence,

$$d(\alpha\beta) = g(\alpha\beta) = |X \setminus \operatorname{dom}(\alpha\beta)| = |X \setminus \operatorname{dom} \alpha| + |\operatorname{dom} \alpha \setminus \operatorname{dom}(\alpha\beta)| < k,$$

that is, $A(X) \backslash A(n', k)$ is a subsemigroup of $A(X)$. Therefore, $A(n', k)$ is a prime ideal. Conversely, suppose that $A(r, d)$ is a proper prime ideal of $A(X)$. If $r < n'$, then a subsemigroup $A(X) \backslash A(r, d)$ contains all transformatons of $A(X)$ with rank n. We write $X = C \dot{\cup} D$ where $|C| = |D| = n$. Then $\operatorname{id}_C, \operatorname{id}_D \in A(X) \backslash A(r, d)$, and thus $\emptyset = \operatorname{id}_C \cdot \operatorname{id}_D \in A(X) \backslash A(r, d)$, which is a contradiction since every ideal of $A(X)$ always contains $\emptyset$. Hence $r = n'$. Now, if $d = 0$, then $A(r, d) = A(n', 0) = A(X)$, a contradiction, so, we deduce that $d \neq 0$. Suppose that d is finite and $d \geq 2$. We write $X = E \dot{\cup} F$ where $|E| = n, |F| = d$ and choose distinct $x, y \in F$. Then $d(\operatorname{id}_{E \cup \{x\}}) = d(\operatorname{id}_{E \cup \{y\}}) = d - 1$, that is, $\operatorname{id}_{E \cup \{x\}}, \operatorname{id}_{E \cup \{y\}} \in A(X) \backslash A(n', d)$. Therefore, $\operatorname{id}_E = \operatorname{id}_{E \cup \{x\}} \cdot \operatorname{id}_{E \cup \{y\}} \in A(X) \backslash A(n', d)$, a contradiction since $d(\operatorname{id}_E) = d$. Hence, $d = 1$ or d is infinite. ∎

In view of Theorem 2.2, when $n = \aleph_0$, there are only two proper prime ideals of $A(X)$, which are $A(n', 1)$ and $A(n', \aleph_0)$. On the other hand, when $n > \aleph_0$, if there exist cardinals r, s such that $\aleph_0 < r < s < n$, then we have a chain of all proper prime ideals of $A(X)$ as follows:

$$A(n', n) \subset ... \subset A(n', s) \subset A(n', r) \subset ... \subset A(n', \aleph_0) \subset A(n', 1),$$

these ideals play an important role in characterizing all maximal subsemigroups of $A(X)$, which will be shown in the next section.

To close this section, we prove following simple result, which establishes a relation between an ideal and a maximal subsemigroup of an arbitrary semigroup. This will be useful when we determine the maximal subsemigroups of $A(X)$.

Lemma 2.3. *Let S be a semigroup and suppose that I is a proper ideal of S. If M is a maximal subsemigroup of S, then either $I \subseteq M$ or $S \backslash I \subseteq M$.*

Proof. Suppose that $I \nsubseteq M$. Then there exists $\alpha \in I \backslash M$. It follows that $\langle M, \alpha \rangle = S$ by the maximality of M. Thus, for each $\beta \in S \backslash I$, we can write $\beta = \gamma_1 \gamma_2 ... \gamma_k$ where $\gamma_i \in M \cup \{\alpha\}$ for all $i = 1, ..., k$. Since $\beta \notin I$, we have $\gamma_i \neq \alpha$ for all i. Therefore $\beta = \gamma_1 \gamma_2 ... \gamma_k \in M$, that is $S \backslash I \subseteq M$. ∎

3. Maximal Subsemigroups

In what follows, if $\alpha \in I(X)$ is non-zero, then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran} \alpha = \{x_i\}$, $x_i \alpha^{-1} = \{a_i\}$ and $\operatorname{dom} \alpha = \{a_i : i \in I\}$.

Refer to a remark in [7, p.157] that, there are infinitely many maximal subsemigroups of $G(X)$ which is not a group. In [4], the authors showed that, when X is uncountable, these maximal subsemigroups of $G(X)$ give rise to a class of maximal subsemigroups of $A(X)$. It is not surprising that, this result also holds for an arbitrary infinite set X, which is shown as follows.

Lemma 3.1. *Suppose that $|X| = n \geq \aleph_0$. Then each one of the following types M_1, M_2 and M_3 is a maximal subsemigroup of $A(X)$.*

(a) *$M_1 = A(n', 1) \cup T$, where T is a maximal subsemigroup of $G(X)$;*
(b) *$M_2 = (A(X) \backslash A(n', d)) \cup A(n', d')$, where $d = 1$ or $\aleph_0 \leq d < n$;*
(c) *$M_3 = (A(X) \backslash A(n', n)) \cup A(n, n)$.*

Proof. (a) Since $A(n', 1)$ is a prime ideal of $A(X)$ and its complement equals $G(X)$, it is a direct consequent from Lemma 2.1(a) that $A(n', 1) \cup T$ is a maximal subsemigroup of $A(X)$ for each maximal subsemigroup T of $G(X)$.

(b) Suppose that $d = 1$ or $\aleph_0 \leq d < n$. Then, by Theorem 2.2, $A(n', d)$ is a prime ideal. So, $A(X) \backslash A(n', d)$ is a subsemigroup of $A(X)$. Since $A(n', d')$ is an ideal of $A(X)$, it is easily seen that $M_2 = (A(X)\backslash A(n', d)) \cup A(n', d')$ is a subsemigroup of $A(X)$. To show the maximality of M_2, we let $\delta, \epsilon \in A(X)\backslash M_2$. Here, note that

$$M_2 = \{\alpha \in A(X) : d(\alpha) \neq d\},$$

so $d(\delta) = g(\delta) = d(\epsilon) = g(\epsilon) = d$. Since $d < n$, we have $r(\delta) = r(\epsilon) = n$. We now write

$$\delta = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \epsilon = \begin{pmatrix} b_i \\ y_i \end{pmatrix} \text{ where } i \in I, |I| = n$$

and suppose $X\backslash \operatorname{dom} \delta = \{p_j\}, X\backslash X\delta = \{q_j\}, X\backslash \operatorname{dom} \epsilon = \{u_j\}$ and $X\backslash X\epsilon = \{v_j\}$ where $|J| = d$. Then define $\gamma, \mu \in G(X) \subseteq M_2$ by

$$\gamma = \begin{pmatrix} b_i & u_j \\ a_i & p_j \end{pmatrix}, \mu = \begin{pmatrix} x_i & q_j \\ y_i & v_j \end{pmatrix}.$$

Clearly, $\epsilon = \gamma\delta\mu$ and hence M_2 is maximal in $A(X)$.

(c) Similar to the proof of (b), since $A(n', n)$ is a prime ideal and $A(n, n)$ is an ideal of $A(X)$, we have $M_3 = (A(X)\backslash A(n', n)) \cup A(n, n)$ is a subsemigroup of $A(X)$ and we note that, in fact,

$$M_3 = \{\alpha \in A(X) : d(\alpha) < n \text{ or } r(\alpha) < n\}.$$

To prove the maximality of M_3, we let $\delta, \epsilon \in A(X)\backslash M_3$. Thus, $d(\delta) = r(\delta) = d(\epsilon) = r(\epsilon) = n$. The rest of the proof follows in a manner similar to the proof of (b), the mappings δ, ϵ, γ and μ can be written in the same way as we did in the proof of (b), except that, in this case, we let $|J| = |I| = n$. Then $\gamma, \mu \in G(X) \subseteq M_3$ and $\epsilon = \gamma\delta\mu$, which proves maximality of M_3. ∎

In order to prove our main theorem, we also need the following two lemmas.

Lemma 3.2. *Suppose that $|X| = n \geq \aleph_0$. Let M be a maximal subsemigroup of $A(X)$. Then $A(n', 2)\backslash A(n', \aleph_0) = \{\alpha \in A(X) : 2 \leq d(\alpha) < \aleph_0\} \subseteq M$.*

Proof. If $A(n', 2) \subseteq M$, then $A(n', 2)\backslash A(n', \aleph_0) \subseteq A(n', 2) \subseteq M$, and the lemma is proved. On the other hand, if $A(n', 2) \nsubseteq M$, then Lemma 2.3 implies that

$$A(X)\backslash A(n', 2) = \{\alpha \in A(X) : d(\alpha) < 2\} \subseteq M.$$

Now, let $\alpha \in A(n', 2)\backslash A(n', \aleph_0)$, say $d(\alpha) = t$ where $2 \leq t < \aleph_0$. Since t is finite, it follows that $r(\alpha) = n$. We write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, i \in I \text{ and } |I| = n,$$

and suppose

$$X\backslash \operatorname{dom} \alpha = \{b_1, b_2, ..., b_t\} \text{ and } X\backslash X\alpha = \{y_1, y_2, ..., y_t\}.$$

Now, we define

$$\sigma = \begin{pmatrix} a_i & b_1 \\ x_i & y_1 \end{pmatrix}, \tau = \begin{pmatrix} x_i & y_2 \\ x_i & y_2 \end{pmatrix}$$

and ensure that $\sigma, \tau \in A(X)$ with $d(\sigma) = d(\tau) = t - 1$. If $t - 1 = 1$, then $\sigma, \tau \in A(X)\backslash A(n', 2) \subseteq M$. Thus, $\alpha = \sigma\tau \in M$. Otherwise, if $t - 1 \geq 2$, then the above argument can be repeated a finite number of times to show that $\sigma = \sigma_1\sigma_2...\sigma_r$ and $\tau = \tau_1\tau_2...\tau_r$ for some positive integer r and $d(\sigma_j) = d(\tau_j) = 1$ for all $j = 1, ..., r$. Thus, $\alpha = \sigma\tau = \sigma_1...\sigma_r\tau_1...\tau_r \in M$. That is, $A(n', 2)\backslash A(n', \aleph_0) \subseteq M$ as required. ∎

By using some ideas of the proof of Lemma 3.2, we can prove the following result.

Lemma 3.3. *Suppose that* $|X| = n \geq \aleph_0$. *Let* M *be a maximal subsemigroup of* $A(X)$. *Then* $A(n,n) \subseteq M$.

Proof. Suppose that $A(X)\backslash A(n,n) \subseteq M$ and let $\beta \in A(n,n)$. Then $r(\beta) < n$ and $d(\beta) = n$. We write

$$\beta = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, i \in I \text{ and } |I| = r(\beta).$$

We also write

$$X\backslash \operatorname{dom}\beta = \{c_j\} \dot\cup \{d_j\} \text{ and } X\backslash X\beta = \{e_j\} \dot\cup \{f_j\} \text{ where } |J| = n,$$

and define

$$\lambda = \begin{pmatrix} a_i & c_j \\ x_i & e_j \end{pmatrix}, \mu = \begin{pmatrix} x_i & f_j \\ x_i & f_j \end{pmatrix}.$$

Now, $g(\lambda) = d(\lambda) = g(\mu) = d(\mu) = n$ and $r(\lambda) = r(\mu) = n$, so $\lambda, \mu \in A(X)\backslash A(n,n) \subseteq M$. It follows that $\beta = \lambda\mu \in M$, and thus $A(n,n) \subseteq M$. Hence,

$$A(X) = (A(X)\backslash A(n,n)) \cup A(n,n) \subseteq M,$$

which is a contradiction. Therefore, $A(X)\backslash A(n,n) \not\subseteq M$ and it follows by Lemma 2.3 that $A(n,n) \subseteq M$. ∎

Now, we are in a position to state and prove the main result of this paper.

Theorem 3.4. *Suppose that* $|X| = n \geq \aleph_0$. *If* M *is a maximal subsemigroup of* $A(X)$, *then* M *equals one of the following sets:*

(*a*) $M_1 = A(n', 1) \cup T$, *where* T *is a maximal subsemigroup of* $G(X)$;
(*b*) $M_2 = (A(X)\backslash A(n', d)) \cup A(n', d')$, *where* $d = 1$ *or* $\aleph_0 \leq d < n$;
(*c*) $M_3 = (A(X)\backslash A(n', n)) \cup A(n, n)$.

Proof. From Lemma 3.1, we have proved that M_1, M_2 and M_3 are maximal subsemigroups of $A(X)$. To complete the proof of the theorem, let M be a maximal subsemigroup of $A(X)$. Since $A(n', 1)$ is a prime ideal, by Lemma 2.3, we have either $A(n', 1) \subseteq M$ or $G(X) = A(X)\backslash A(n', 1) \subseteq M$. If $A(n', 1) \subseteq M$, then $G(X) \cap M \neq \emptyset$ (otherwise, $M = A(n', 1)$, which is not a maximal subsemigroup of $A(X)$ by Lemma 3.1(a)) and $G(X)\backslash M \neq \emptyset$ (otherwise, $M = A(X)$, which is a contradiction). Then, Lemma 2.1(b) implies that $G(X) \cap M$ is a maximal subsemigroup of $G(X)$. Now we can write

$$M = A(n', 1) \cup (G(X) \cap M).$$

It follows that M is a maximal subsemigroup of type M_1. In the latter case, when $G(X) \subseteq M$, there exists $\alpha \in A(X)\backslash M$ and $g(\alpha) \neq 0$. Furthermore, Lemma 3.2 showed that M contains all transformations of $A(X)$ such that their gap are finite and are greater than 1, this implies $g(\alpha) = 1$ or $\aleph_0 \leq g(\alpha) \leq n$. Then we consider the following cases:

Case 1: $g(\alpha) = d$ (say) where $d = 1$ or $\aleph_0 \leq d < n$. Then $\alpha \in A(n', d)\backslash M$ and it follows by Lemma 2.3 that $A(X)\backslash A(n', d) \subseteq M$. In addition, $g(\alpha) = d$ also implies $\alpha \in (A(X)\backslash A(n', d'))\backslash M$. Using Lemma 2.3 again, we have that $A(n', d') \subseteq M$. We now see that $(A(X)\backslash A(n', d)) \cup A(n', d') \subseteq M$ and equality follows by Lemma 3.1(b). Thus, M is a maximal subsemigroup of type M_2.

Case 2: $g(\alpha) = n$. Similar to case 1, we see that $\alpha \in A(n', n)\backslash M$. Consequently, $A(X)\backslash A(n', n) \subseteq M$ by Lemma 2.3. Moreover, since M always contains $A(n, n)$ (by Lemma 3.3), it follows that $(A(X)\backslash A(n', n)) \cup A(n, n) \subseteq M$ and equality follows by Lemma 3.1(c). Hence, in this case, we deduce that M is a maximal subsemigroup of type M_3. This completes the proof. ∎

In [4, p.333], for the cardinals d and e such that $0 \leq d \leq e \leq n = |X|$, the authors defined the sets

$$S[d, e] = \{\alpha \in A(X) : d \leq d(\alpha) \leq e\} \text{ and}$$
$$S[d, e) = \{\alpha \in A(X) : d \leq d(\alpha) < e\}.$$

Furthermore, $S(d, e]$ and $S(d, e)$ were defined in a similar way. They showed in [4, Theorem 13] that, when $n > \aleph_0$, there are only four types of maximal subsemigroups of $A(X)$, which are:

(1) $T \cup S[1, n]$, where T is a maximal subsemigroup of $G(X)$;
(2) $G(X) \cup S(1, n]$;
(3) $G(X) \cup S[1, d) \cup S(d, n]$, where $\aleph_0 \leq d < n$;
(4) $G(X) \cup S[1, n) \cup A(n, n)$.

In view of Theorem 3.4, our result is a generalization of [4, Theorem 13] since (1) is the maximal subsemigroups of type M_1, (2) and (3) are the maximal subsemigroups of type M_2 when $d = 1$ and $\aleph_0 \leq d < n$, respectively, and (4) is equal to M_3. Moreover, as a specialization of Theorem 3.4 by letting $n = \aleph_0$, the infinite cardinal d in Theorem 3.4(b) does not exist (as it depends on infinite cardinals less than n). Then the following corollary is immediate.

Corollary 3.5. *Suppose that $|X| = \aleph_0$. Then the maximal subsemigroups of $A(X)$ are precisely the sets:*

(a) $M_1 = A({\aleph_0}', 1) \cup T$, *where T is a maximal subsemigroup of $G(X)$;*
(b) $M_2 = (A(X)\backslash A({\aleph_0}', 1)) \cup A({\aleph_0}', 2) = \{\alpha \in A(X) : d(\alpha) \neq 1\}$;
(c) $M_3 = (A(X)\backslash A({\aleph_0}', \aleph_0)) \cup A(\aleph_0, \aleph_0) = \{\alpha \in A(X) : d(\alpha) < \aleph_0 \text{ or } r(\alpha) < \aleph_0\}$.

Acknowledgements

The author would like to thank the referees for their comments and suggestions on the manuscript. The author also acknowledges the support and facilities of Chiang Mai Rajabhat University.

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