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Best Proximity Coincidence Point Theorem for G-proximal Generalized Geraghty Auxiliary Function in a Metric Space with Graph G

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Abstract In a complete metric space endowed with a directed graph G, we investigate the best proximity coincidence points of a pair of mappings that is G-proximal generalized auxiliary function. We show that the best proximity coincidence point is unique if any pair of two best proximity coincidence points is an edge of the graph G. In addition, we provide an example as well as corollaries that are pertinent to our main theorem.

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1. Introduction

In 1973, M. A. Geraghty [1] presented the notion of contractive mapping based on the class of mappings $F : [0,\infty) \to [0,1)$ such that

$$\lim_{n\to\infty} F(u_n) = 1 \quad \implies \quad \lim_{n\to\infty} u_n = 0.$$

Inspired by the work of M. A. Geraghty, M. I. Ayari [2] considered the class $\mathcal{B}$ of all mappings $F : [0,\infty) \to [0,1]$ such that

$$\lim_{n\to\infty} F(u_n) = 1 \quad \implies \quad \lim_{n\to\infty} u_n = 0$$

to achieve existence result and prove uniqueness outcome for best proximity points in the case of α-proximal Geraghty non-self mappings on closed subsets in complete metric spaces. Moreover, E. Karapinar, T. Abdeljawa, and F. Jarad [3] were using a class of mappings that are auxiliary functions to apply new fixed point theorems on fractional

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and ordinary differential equations. Besides, there are some research concerning with P-property, for example, see [4–7].

Additionally, J. Jachymski [8] viewed the Banach contraction principle for mappings on a metric space provided with a graph as an example of how graph theory might be used to fixed point theory. See, for example, [9–12] for work on fixed point theorems for mappings of some spaces furnished with a graph.

C. Klanarong and S. Suantai [13], on the other hand, examined the best proximity point theorems using a G-proximal generalized contraction in a complete metric space furnished with a graph G. Then, to consider the presence of a best proximity coincidence point, A. Khemphet [14] defined a G-proximal generalized Geraghty mapping.

We are interested in identifying the best proximity coincidence points for G-proximal generalized auxiliary functions defined on closed subsets of complete metric spaces equipped with a directed graph G, as inspired by the previous work.

2. Preliminaries and definitions

Standard terminologies and notations in topology and analysis used throughout this work are assumed to be defined as usual.

For a metric space (Y, d) and nonempty subsets I, J of Y, let us define the following notions.

$$D(I, J) = \inf\{d(u, v) : u \in I, v \in J\};$$
$$I_0 = \{u \in I : D(I, J) = d(u, v) \text{ for some } v \in J\};$$
$$J_0 = \{v \in J : D(I, J) = d(u, v) \text{ for some } u \in I\}.$$

Notice that if $u \in I_0$, then there is $v \in J$ such that $D(I, J) = d(u, v)$, which implies $v \in J_0$. Similar observation is also true for the case J_0. Next, we recall the concepts of best proximity points and best proximity coincidence points as follows.

Definition 2.1. [15, 16] Assume (Y, d) is a metric space, and $\varnothing \neq I, J \subseteq Y$. Suppose that $P : I \to J$ and $q : I \to I$ are functions and $u^* \in I$. Then

(1) u^* will be called a **best proximity point** of P whenever $d(u^*, Pu^*) = D(I, J)$;
(2) u^* will be called a **best proximity coincidence point** of the pair (P, q) whenever $d(qu^*, Pu^*) = D(I, J)$.

Definition 2.2. [7] Assume (Y, d) is a metric space, and $\varnothing \neq I, J \subseteq Y$ such that I_0 is nonempty. Then we say that the pair (I, J) has the **weak P-property** whenever for all $u_1, u_2 \in I_0$ and $v_1, v_2 \in J_0$,

$$d(u_1, v_1) = d(u_2, v_2) = D(I, J) \quad \Longrightarrow \quad d(u_1, u_2) \leq d(v_1, v_2).$$

In the succeeding definition, we introduce the concept of metric spaces endowed with directed graphs.

Definition 2.3. [8] Assume (Y, d) is a metric space, and diagonal Δ is a set defined by $\Delta := \{(u, u) : u \in Y\}$. The metric space (Y, d) is said to be **endowed with a directed graph** $G = (V(G), E(G))$ if G is a directed graph such that the vertex set $V(G)$ consists of all elements in Y and the edge set $E(G)$ contains the diagonal Δ.

It is worth mentioning that, in this work, we will assume $E(G)$ contains no parallel edges. Thenceforth, we denote a metric space (Y, d) endowed with directed graph $G = (V(G), E(G))$ by $G - (Y, d)$.

Definition 2.4. [8] Let $G - (Y, d)$.

(1) A function $P : Y \to Y$ will be called **G-continuous at** $u \in Y$ whenever for any sequence $\{u_n\}$ in Y such that $(u_n, u_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$,

$$u_n \to u \quad \Longrightarrow \quad Pu_n \to Pu.$$

In addition, P will be called **G-continuous** whenever it is G-continuous at every $u \in Y$.

(2) We will say that the edge set $E(G)$ has the **transitive property** whenever

$$(u, v), (v, w) \in E(G) \quad \Longrightarrow \quad (u, w) \in E(G)$$

for all $u, v, w \in Y$.

Definition 2.5. [14] Assume $G - (Y, d)$, and $P : I \to J$ and $q : I \to I$ are functions. We will say that P is **G-proximal edge preserving with respect to** q whenever for all $x_1, x_2, u_1, u_2 \in I$ with $(x_1, x_2) \in E(G)$, it is true that

$$d(qu_1, Px_1) = d(qu_2, Px_2) = D(I, J) \quad \Longrightarrow \quad (u_1, u_2) \in E(G).$$

3. Main Results

In this section, we extend the definition of G-proximal generalized Geraghty mappings to the case of G-proximal generalized auxiliary functions.

To begin with, for a metric space (Y, d), we let $\mathcal{A}(Y)$ be the class of auxiliary functions $F : Y \times Y \to [0, 1]$ such that

$$\lim_{n\to\infty} F(u_n, v_n) = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} d(u_n, v_n) = 0$$

for all sequences $\{u_n\}$ and $\{v_n\}$ in Y.

Definition 3.1. Assume $G - (Y, d)$, and $P : I \to J$ and $q : I \to I$ are functions. Then we say that the pair (P, q) is a **G-proximal generalized auxiliary function** if the following conditions hold.

(1) P is G-proximal edge preserving with respect to q; and

(2) There exists $F \in \mathcal{A}(Y)$ such that for any $u, v \in I$ with $(u, v) \in E(G)$,

$$d(Pu, Pv) \leq F(qu, qv)d(qu, qv).$$

Theorem 3.2. *Assume $G - (Y, d)$ such that Y is complete, and $E(G)$ has the transitive property. Suppose that (I, J) is a pair of closed subsets $\varnothing \neq I, J \subseteq Y$ having the weak P-property. Let $P : I \to J$ and $q : I \to I$ be functions such that q is an isometry and the pair (P, q) is a G-proximal generalized auxiliary function. Furthermore, suppose that the following conditions hold.*

(i) $P(I_0) \subseteq J_0$ and $I_0 \subseteq q(I_0)$;

(ii) There exist $u, v \in I_0$ with $d(qu, Pv) = D(I, J)$ and $(v, u) \in E(G)$; and

(iii) Either (a) or (b) is true:

(a) P is G-continuous on I;

(b) For every sequence $\{u_n\}$ in I with $(u_n, u_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $u_n \to u^*$ for some $u^* \in I$, then there exists a subsequence $\{u_{n(t)}\}$ of $\{u_n\}$ with $(u_{n(t)}, u^*) \in E(G)$ for each $t \in \mathbb{N}$.

Then the pair (P, q) has a best proximity coincidence point. In addition, if $(u^*, v^*) \in E(G)$ for some best proximity coincidence points $u^*, v^* \in I$, then (P, q) has a unique best proximity coincidence point.

Proof. By (ii), there are $u_0, u_1 \in I_0$ which satisfy $D(I, J) = d(qu_1, Pu_0)$ and $(u_0, u_1) \in E(G)$. Then, by (i) and the definition of J_0, there are $u_2 \in I_0$ with $d(qu_2, Pu_1) = D(I, J)$ and $(u_1, u_2) \in E(G)$ since P is G-proximal edge preserving with respect to q. Repeating this procedure, we have a sequence $\{u_n\}$ in I_0, which for each $n \in \mathbb{N}$,

$$d(qu_n, Pu_{n-1}) = D(I, J) \text{ and } (u_{n-1}, u_n) \in E(G). \tag{3.1}$$

Because (I, J) has the weak P-property, we obtain that

$$d(qu_n, qu_{n+1}) \leq d(Pu_{n-1}, Pu_n) \text{ for each } n \in \mathbb{N}.$$

Next, we claim that $\lim_{n\to\infty} d(u_{n-1}, u_n) = 0$. Since (P, q) is a G-proximal generalized auxiliary function and q is an isometry, for each $n \in \mathbb{N}$, we get that

$$\begin{aligned} d(u_n, u_{n+1}) &= d(qu_n, qu_{n+1}) \\ &\leq d(Pu_{n-1}, Pu_n) \\ &\leq F(qu_{n-1}, qu_n) d(qu_{n-1}, qu_n) \\ &= F(qu_{n-1}, qu_n) d(u_{n-1}, u_n) \\ &\leq d(u_{n-1}, u_n). \end{aligned} \tag{3.2}$$

Thus, the sequence $\{d(u_{n-1}, u_n)\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(u_{n-1}, u_n) = r$. We will prove that $r = 0$. Suppose on the contrary that $r > 0$. From (3.2), taking $n \to \infty$ provides

$$1 \leq \lim_{n\to\infty} F(qu_{n-1}, qu_n) \leq 1.$$

So $\lim_{n\to\infty} F(qu_{n-1}, qu_n) = 1$. By the definition of F, we have

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = \lim_{n\to\infty} d(qu_{n-1}, qu_n)$$

must be zero. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(u_{n-1}, u_n) = 0. \tag{3.3}$$

Now, we prove that $\{u_n\}$ is Cauchy. Suppose on the contrary that $\{u_n\}$ is not Cauchy. Then there is $\epsilon_0 > 0$ such that there are subsequences $\{u_{n(t)}\}$ and $\{u_{m(t)}\}$ of $\{u_n\}$ with $m(t) > n(t) \geq t$ for each $t \in \mathbb{N}$ and

$$d(u_{n(t)}, u_{m(t)}) \geq \epsilon_0. \tag{3.4}$$

We may choose the smallest $m(t)$ satisfying (3.4) for each $t \in \mathbb{N}$ so that

$$d(u_{n(t)}, u_{m(t)-1}) < \epsilon_0.$$

Because of the triangle inequality, for each $t \in \mathbb{N}$, we get that

$$\begin{aligned} \epsilon_0 &\leq d(u_{n(t)}, u_{m(t)}) \\ &\leq d(u_{n(t)}, u_{m(t)-1}) + d(u_{m(t)-1}, u_{m(t)}) \\ &< \epsilon_0 + d(u_{m(t)-1}, u_{m(t)}). \end{aligned}$$

From (3.3), letting $t \to \infty$ provides

$$\lim_{t\to\infty} d(u_{n(t)}, u_{m(t)}) = \epsilon_0. \tag{3.5}$$

Since $\{u_{n(t)}\}$ and $\{u_{m(t)}\}$ are subsequences of $\{u_n\}$, by (3.1), for each $t \in \mathbb{N}$,

$$d(qu_{n(t)+1}, Pu_{n(t)}) = D(I,J) \quad \text{and} \quad d(qu_{m(t)+1}, Pu_{m(t)}) = D(I,J).$$

Since (I,J) has the weak P-property, we have that

$$d(qu_{n(t)+1}, qu_{m(t)+1}) \leq d(Pu_{n(t)}, Pu_{m(t)}).$$

Also, from (3.1), $(u_{n(t)}, u_{n(t)+1}) \in E(G)$ for each $t \in \mathbb{N}$. Next, by the transitive property of $E(G)$, we have $(u_{n(t)}, u_{m(t)}) \in E(G)$. Since (P,q) is a G-proximal generalized auxiliary function and q is an isometry, consider

$$\begin{aligned} d(u_{n(t)+1}, u_{m(t)+1}) &= d(qu_{n(t)+1}, qu_{m(t)+1}) \\ &\leq d(Pu_{n(t)}, Pu_{m(t)}) \\ &\leq F(qu_{n(t)}, qu_{m(t)}) d(qu_{n(t)}, qu_{m(t)}) \\ &= F(qu_{n(t)}, qu_{m(t)}) d(u_{n(t)}, u_{m(t)}) \\ &\leq d(u_{n(t)}, u_{m(t)}). \end{aligned}$$

Similarly, by (3.5), we can conclude that

$$1 \leq \lim_{t\to\infty} F(qu_{n(t)}, qu_{m(t)}) \leq 1.$$

Then $\lim_{t\to\infty} F(qu_{n(t)}, qu_{m(t)}) = 1$. By the definition of the auxiliary function F, it is true that $\lim_{t\to\infty} d(u_{n(t)}, u_{m(t)}) = \lim_{t\to\infty} d(qu_{n(t)}, qu_{m(t)}) = 0$. This contradicts (3.5) because ϵ_0 is positive. Thus, $\{u_n\}$ is a Cauchy sequence in the closed subset I of (Y,d), which is complete. So, we obtain $u^* \in I$ such that $\lim_{n\to\infty} u_n = u^*$.

Next, suppose that the condition (a) holds. Since $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, by the G-continuity of P and the continuity of q on I,

$$\lim_{n\to\infty} Pu_n = Pu^* \quad \text{and} \quad \lim_{n\to\infty} qu_n = qu^*.$$

Therefore, $\lim_{n\to\infty} d(qu_{n+1}, Pu_n) = d(qu^*, Pu^*)$. From (3.1), we have that

$$\lim_{n\to\infty} d(qu_{n+1}, Pu_n) = D(I,J).$$

Thus, $d(qu^*, Pu^*) = D(I,J)$ because the limit is unique.

On the other hand, suppose that the condition (b) holds. There exists a subsequence $\{u_{n(t)}\}$ of $\{u_n\}$ with $(u_{n(t)}, u^*) \in E(G)$ for all $t \in \mathbb{N}$. Since (P,q) is a G-proximal generalized auxiliary function and q is an isometry,

$$\begin{aligned} d(Pu_{n(t)}, Pu^*) &\leq F(qu_{n(t)}, qu^*) d(qu_{n(t)}, qu^*) \\ &= F(u_{n(t)}, u^*) d(u_{n(t)}, u^*) \\ &\leq d(u_{n(t)}, u^*) \end{aligned} \tag{3.6}$$

By the triangle inequality,

$$\begin{aligned} d(qu^*, Pu^*) &\leq d(qu^*, qu_{n(t)+1}) + d(qu_{n(t)+1}, Pu_{n(t)}) + d(Pu_{n(t)}, Pu^*) \\ &= d(u^*, u_{n(t)+1}) + d(qu_{n(t)+1}, Pu_{n(t)}) + d(Pu_{n(t)}, Pu^*). \end{aligned}$$

Then

$$d(qu^*, Pu^*) - d(u^*, u_{n(t)+1}) - d(qu_{n(t)+1}, Pu_{n(t)}) \leq d(Pu_{n(t)}, Pu^*).$$

From (3.6), we have

$$d(qu^*, Pu^*) - d(u^*, u_{n(t)+1}) - d(qu_{n(t)+1}, Pu_{n(t)}) \leq d(u_{n(t)}, u^*).$$

By letting $t \to \infty$, by (3.1), we get $d(qu^*, Pu^*) - D(I, J) \leq 0$ so $d(qu^*, Pu^*) \leq D(I, J)$. Note that since $qu^* \in I$ and $Pu^* \in J$, we have that $D(I, J) \leq d(qu^*, Pu^*)$. Then we can conclude that $d(qu^*, Pu^*) = D(I, J)$.

Finally, let u^* and v^* be the best proximity coincidence points of (P, q) such that $(u^*, v^*) \in E(G)$. Then $d(qu^*, Pu^*) = d(qv^*, Pv^*) = D(I, J)$. Since (I, J) has the weak P-property, it is true that $d(qu^*, qv^*) \leq d(Pu^*, Pv^*)$. Because q is an isometry and (P, q) is a G-proximal generalized auxiliary function and q is an isometry, it follows that

$$\begin{aligned} d(u^*, v^*) &= d(qu^*, qv^*) \\ &\leq d(Pu^*, Pv^*) \\ &\leq F(qu^*, qv^*) d(qu^*, qv^*) \\ &= F(qu^*, qv^*) d(u^*, v^*) \\ &\leq d(u^*, v^*). \end{aligned}$$

If $d(u^*, v^*) > 0$, then $F(qu^*, qv^*) = 1$. By the property of the auxiliary function F, we have $d(u^*, v^*) = 0$. This is a contradiction. As a result, $d(u^*, v^*) = 0$ which implies $u^* = v^*$. Thence, (P, q) has a unique best proximity coincidence point. ∎

In the next part, we consider an example relating to our main theorem.

Example 3.3. Let $Y = \mathbb{R}^2$ be a complete metric space equipped with the metric d given by

$$d((u, v), (a, b)) = \sqrt{(u - a)^2 + (v - b)^2},$$

where $(u, v), (a, b) \in \mathbb{R}^2$. Let

$$I = \{(4, v) : 0 \leq v \leq 6\} \quad \text{and} \quad J = \{(-4, v) : 0 \leq v \leq 3\}.$$

Then I and J are closed subsets of $\mathbb{R}^2$. It can be shown that (I, J) has the weak P-property with $D(I, J) = 8$. In addition,

$$I_0 = \{(4, v) : 0 \leq v \leq 3\} \quad \text{and} \quad J_0 = \{(-4, v) : 0 \leq v \leq 3\}.$$

Next, define $P : I \to J$ such that, for each $v \in [0, 6]$,

$$P(4, v) = (-4, \ln(v + 1)),$$

and define q to be the identity function on I so that $q(I) = I$. Then

$$P(I_0) = \{(-4, v) : 0 \leq v \leq \ln 4\} \subseteq J_0.$$

Let $G = (V(G), E(G))$ be a directed graph with $V(G) = Y$ and

$$E(G) = \{((u, v), (a, b)) \in \mathbb{R}^2 \times \mathbb{R}^2 : u \geq a \text{ and } v \geq b\}.$$

We obtain that $E(G)$ has the transitive property. In fact, P is G-continuous on I. In addition, it can be proved that the condition (ii) in Theorem 3.2 is satisfied.

Next, we will show that (P, q) is a G-proximal generalized auxiliary function. First, we need to prove that P is G-proximal edge preserving with respect to q. To see this, let $(4, u), (4, v), (4, a), (4, b) \in I$ with $((4, u), (4, v)) \in E(G)$ and

$$d(q(4, a), P(4, u)) = d(q(4, b), P(4, v)) = d(I, J).$$

That is,

$$d((4,a),(-4,\ln(u+1))) = d((4,b),(-4,\ln(v+1))).$$

Then

$$a = \ln(u+1) \quad \text{and} \quad b = \ln(v+1).$$

Observe that $u \geq v$ implies $a \geq b$ so $((4,a),(4,b)) \in E(G)$. We can conclude that P is G-proximal edge preserving with respect to q.

Next, define $F : Y \times Y \to [0,1]$ by

$$F(u,v) = \begin{cases} 1 & \text{if } u = v; \\ \frac{\ln(1+d(u,v))}{d(u,v)} & \text{if } u \neq v. \end{cases}$$

Finally, let $(4,u),(4,v) \in I$ such that $((4,u),(4,v)) \in E(G)$, i.e., $u \geq v$. If $u = v$, then we are done.

If $u > v$, consider

$$\begin{aligned} d(P(4,u),P(4,v)) &= d((-4,\ln(u+1)),(-4,\ln(v+1))) \\ &= |\ln(u+1) - \ln(v+1)| \\ &= \left|\ln\left(\frac{u+1}{v+1}\right)\right| \\ &= \left|\ln\left(1 + \frac{u-v}{v+1}\right)\right| \\ &\leq \ln(1+|u-v|) \\ &= \frac{\ln(1+|u-v|)}{|u-v|}|u-v| \\ &= F((4,u),(4,v))d((4,u),(4,v)) \\ &= F(q(4,u),q(4,v))d(q(4,u),q(4,v)). \end{aligned}$$

As a result, P is a G-proximal generalized auxiliary function. By Theorem 3.2, (P,q) has a best proximity coincidence point in I. In fact, it can be checked that $(4,0)$ is a best proximity coincidence point of (P,q).

4. Consequence

In our last part, we consider some corollaries related to our main result. Since every G-proximal generalized Geraghty mapping, see [14] for the definition, is a G-proximal generalized auxiliary function, we can conclude that the following results, which are true for the case of G-proximal generalized Geraghty mappings, are also true for our case by applying Theorem 3.2.

Definition 4.1. [14] Assume $G - (Y,d)$, $\varnothing \neq I, J \subseteq Y$, and $P : I \to J$ and $q : I \to I$ are functions. We say that the pair (P,q) is a ***G*-proximal generalized mapping** if the following conditions hold.

(1) P is G-proximal edge preserving with respect to q; and
(2) There is $C \in [0,1)$ such that for any $u,v \in I$ with $(u,v) \in E(G)$, it is true that

$$d(Pu,Pv) \leq Cd(qu,qv).$$

By setting $F(u,v) = C$ for all $u, v \in Y$, we obtain that every G-proximal generalized mapping is a G-proximal generalized auxiliary function. Hence, we could apply Theorem 3.2 to the following corollary.

Corollary 4.2. *Assume $G-(Y,d)$ such that Y is complete, and $E(G)$ has the transitive property. Suppose that (I,J) is a pair of closed subsets $\varnothing \neq I, J \subseteq Y$ having the weak P-property. Let $P : I \to J$ and $q : I \to I$ be functions such that q is an isometry and the pair (P,q) is a G-proximal generalized mapping. Furthermore, suppose that the following conditions hold.*

- *(i) $P(I_0) \subseteq J_0$ and $I_0 \subseteq q(I_0)$;*
- *(ii) There exist $u, v \in I_0$ with $d(qu, Pv) = D(I,J)$ and $(v,u) \in E(G)$; and*
- *(iii) Either (a) or (b) is true:*
  - *(a) P is G-continuous on I;*
  - *(b) For every sequence $\{u_n\}$ in I with $(u_n, u_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $u_n \to u^*$ for some $u^* \in I$, then there exists a subsequence $\{u_{n(t)}\}$ of $\{u_n\}$ with $(u_{n(t)}, u^*) \in E(G)$ for each $t \in \mathbb{N}$.*

Then the pair (P,q) has a best proximity coincidence point. In addition, if $(u^*, v^*) \in E(G)$ for some best proximity coincidence points $u^*, v^* \in I$, then it is true that (P,q) has a unique best proximity coincidence point.

Definition 4.3. [14] Assume $G-(Y,d)$, and $P : I \to J$ and $q : I \to I$ are functions. We say that the pair (P,q) is a **G-proximal type R mapping** if the following conditions hold.

(1) P is G-proximal edge preserving with respect to q; and
(2) For all $u, v \in I$ with $(u,v) \in E(G)$, we have

$$d(Pu, Pv) \leq \frac{d(qu, qv)}{d(qu, qv) + 1}.$$

By setting $F(u,v) = \dfrac{1}{d(u,v)+1}$ for any $u, v \in Y$, we obtain that every G-proximal type R mapping is also a G-proximal generalized auxiliary function. Hence, we may apply Theorem 3.2 to the following result.

Corollary 4.4. *Assume $G-(Y,d)$ such that Y is complete, and $E(G)$ has the transitive property. Suppose that (I,J) is a pair of closed subsets $\varnothing \neq I, J \subseteq Y$ having the weak P-property. Let $P : I \to J$ and $q : I \to I$ be functions such that q is an isometry and the pair (P,q) is a G-proximal type R mapping. Furthermore, suppose that the following conditions hold.*

- *(i) $P(I_0) \subseteq J_0$ and $I_0 \subseteq q(I_0)$;*
- *(ii) There exist $u, v \in I_0$ with $d(qu, Pv) = D(I,J)$ and $(v,u) \in E(G)$; and*
- *(iii) Either (a) or (b) is true:*
  - *(a) P is G-continuous on I;*
  - *(b) For every sequence $\{u_n\}$ in I with $(u_n, u_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, if $u_n \to u^*$ for some $u^* \in I$, then there exists a subsequence $\{u_{n(t)}\}$ of $\{u_n\}$ with $(u_{n(t)}, u^*) \in E(G)$ for each $t \in \mathbb{N}$.*

Then the pair (P,q) has a best proximity coincidence point. In addition, if $(u^*, v^*) \in E(G)$ for some best proximity coincidence points $u^*, v^* \in I$, then it is true that (P,q) has a unique best proximity coincidence point.

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