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Convergence of Three-Step Iteration Scheme for Common Fixed Point of Three Berinde Nonexpansive Mappings

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Abstract The purpose of this paper is to establish weak and strong convergence theorems of three-step iterations for three Berinde nonexpansive mappings in Banach space. The results obtained in this paper extend and improve the recent ones announced by Phuengrattana and Suantai [6] and S. Kosol [7].

MSC: 47H09; 47H10.

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1. Introduction and Preliminaries

Throughout this paper, K be a nonempty convex subset of a Banach space X and $T : K \to K$ be a mapping. We denote by $F(T)$ the set of fixed points of T. We denote by $F = \bigcap_{i=1}^{3} F(T_i)$ the set of common fixed points of $T_i : K \to K, i = 1, 2, 3$.

A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in X$.

T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in X$ and $p \in F(T)$.

A mapping T is said to be *Berinde nonexpansive* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\|$$

for all $x, y \in K$.

In 2003, Berinde [1] introduced a new type of contraction as above, called weak contraction and proved a fixed point theorem for this type of mapping in a complete metric space by showing that Picard iteration $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converge strongly to its fixed point.

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In 2000, Noor [2] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme gives better numerical results than the Mann-type(one-step) [4] and the Ishikawa-type (two-step)[5] approximate iterations. In 2013, Phuengrattana and Suantai [6] introduced the following iterative method for weak contraction.

$$\begin{cases} z_n = a_n T x_n + (1-a_n)x_n \\ y_n = b_n T z_n + (1-b_n)z_n \\ x_{n+1} = \alpha_n T y_n + \beta_n T z_n + (1-\alpha_n-\beta_n)x_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $x_1 \in K$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_n+\beta_n\}$ in $[0,1]$ satisfy certain conditions. Let $T_i : K \to K, i = 1,2,3$ be mappings.

By studying the following iteration process:

$$\begin{cases} z_n = a_n T_1 x_n + (1-a_n)x_n \\ y_n = b_n T_2 z_n + (1-b_n)x_n \\ x_{n+1} = (1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $x_1 \in K$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ in $[0,1]$. S. Kosol [7] proved the weak and strong convergence theorems of the above iterative method for approximating a common fixed point of Berinde nonexpansive mappings in a Banach space.

Inspired and motiveted by these facts, we introduce and study a new class of iterative for three Berinde nonexpansive mappings in this paper. The scheme is defined as follows.

Let K be a nonempty convex subset of a Banach space X and $T_i : K \to K, i = 1,2,3$ be mappings. Then for arbitrary $x_1 \in K$,the following iteration scheme is studied:

$$\begin{cases} z_n = a_n T_1 x_n + (1-a_n)x_n \\ y_n = b_n T_2 z_n + c_n T_2 x_n + (1-b_n-c_n)x_n \\ x_{n+1} = \alpha_n T_3 y_n + \beta_n T_3 z_n + \gamma_n T_3 x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n, \end{cases} \tag{1.3}$$

$\forall n \in \mathbb{N}$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n+c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n+\beta_n+\gamma_n\}$ in $[0,1]$ satisfy certain conditions.

The aim of this paper is to introduce and study convergence problem of the three-step iterative sequence (1.3) for three Berinde type nonexpansive mappings in a real Banach space. The results presented in this paper generalize and extend some recent Phuengrattana and Suantai [6] and S.Kosol [7].

The following lemma will be needed in proving our main results.

A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta(\varepsilon) = inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\} > 0.$$

for all $0 < \varepsilon \leq 2$ $\Big($i.e., $\delta(\varepsilon)$ is a function $(0,2] \to (0,1)\Big)$.

Recall that a Banach space X is said to satisfy *Opial's condition* [8] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

A mapping $T : K \to X$ is said to be demiclosed with respect to $y \in X$ if, for for each sequence $\{x_n\}$ in K and each $x \in X$, $x_n \to x$ weakly and $Tx_n \to y$ strongly imply that $x \in K$ and $Tx = y$.

A mapping $T : K \to X$ is said to be semi-compact if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly $x^* \in K$.

In what follows, the following lemmas will be needed in proving our main results.

Lemma 1.1 ([9])**.** *Let $k > 1$ be a fixed numbers and X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^k \leq \lambda\|x\|^k + (1 - \lambda)\|y\|^k - \omega_k(\lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

Lemma 1.2 ([10])**.** *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|)$$

for all $x, y, z \in B_r$ and $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 1.3 ([11])**.** *Let X be a uniformly convex Banach space and $B_R := \{x \in X : \|x\| \leq R\}, R > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \nu w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2$$

$$-\frac{1}{3}\nu(\lambda g(\|x - w\|) + \mu g(\|y - w\|) + \xi g(\|z - w\|))$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, \nu \in [0, 1]$ with $\lambda + \mu + \xi + \nu = 1$.

Lemma 1.4 ([12], Lemma 2.7)**.** *Let X be a Banach space which satisfies Opial's condition and let x_n be a sequence in X. Let $q_1, q_2 \in X$ be such that $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. If $\{x_{n_k}\}$, $\{x_{n_j}\}$ are the subsequences of $\{x_n\}$ which converge weakly to $q_1, q_2 \in X$, respectively. Then $q_1 = q_2$.*

2. Main results

In this section, we prove the three-step iterative scheme given in (1.3) to converge to a common fixed point for Berinde nonexpansive mappings in uniformly convex Banach space.

Lemma 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X, $T_i : K \to K, i = 1,2,3$ be quasi-nonexpansive mappings. Assume that $F = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by (1.3). Then for each $p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ exits.*

Proof. For any $p \in F$, using (1.3) and $T_i : K \to K, i = 1,2,3$ are quasi-nonexpansive mappings,

$$\begin{aligned} \|z_n - p\| &= \|a_nT_1x_n + (1-a_n)x_n - p\| \\ &\leq \|a_n(T_1x_n - p) + (1-a_n)(x_n - p)\| \\ &\leq a_n\|T_1x_n - p\| + (1-a_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.1}$$

$$\begin{aligned} \|y_n - p\| &= \|b_nT_2z_n + c_nT_2x_n + (1-b_n-c_n)x_n - p\| \\ &\leq \|b_n(T_2z_n - p) + c_n(T_2x_n - p) + (1-b_n-c_n)(x_n - p)\| \\ &\leq b_n\|T_2z_n - p\| + c_n\|T_2x_n - p\| + (1-b_n-c_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_nT_3y_n + \beta_nT_3z_n + \gamma_nT_3x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n - p\| \\ &\leq \|\alpha_n(T_3y_n - p) + \beta_n(T_3z_n - p) + \gamma_n(T_3x_n - p) \\ &\quad + (1-\alpha_n-\beta_n-\gamma_n)(x_n - p)\| \\ &\leq \alpha_n(\|T_3y_n - p\|) + \beta_n(\|T_3z_n - p\|) + \gamma_n(\|T_3x_n - p\|) \\ &\quad + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| \\ &\leq \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\quad + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\|. \end{aligned} \tag{2.3}$$

Substituting (2.1) and (2.2) into (2.3) and simplifying, we have $\|x_{n+1} - p\| \leq \|x_n - p\|$. This implies that $\{\|x_n - p\|\}$ is bounded and nonincreasing for all $p \in F$. Hence we have that $\lim_{n\to\infty} \|x_n - p\|$ exits. ∎

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X, $T_i : K \to K, i = 1,2,3$ be Berinde nonexpansive and quasi-nonexpansive mappings, p is a common fixed point of $T_i, i = 1,2,3$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence in K defined by (1.3), and parameters satisfy one of the following conditions:*

(1) *If* $0 < \liminf_n \alpha_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$,
(2) *If* $0 < \liminf_n \alpha_n$ *and* $0 \leq \limsup_n b_n \leq \limsup_n(b_n + c_n) < 1$,
(3) *If* $0 < \liminf_n \beta_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$,
(4) *If* $0 < \liminf_n \gamma_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$,
(5) *If* $0 < \liminf_n(\alpha_n b_n + \beta_n)$, *and* $0 < \liminf_n a_n \leq \limsup_n a_n < 1$

then $\lim_{n\to\infty}\|T_1x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2z_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_3z_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_3y_n - x_n\| = 0$, $\lim_{n\to\infty}\|z_n - x_n\| = 0$, $\lim_{n\to\infty}\|y_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$.

Proof. By Lemma 2.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exits for any $p \in F$. Then the sequence $\{\|x_n - p\|\}$ is bounded. By quasi-nonexpansiveness of $T_i : K \to K, i = 1, 2, 3$, there exists $R > 0$ such that $\{x_n - p\}$, $\{T_1x_n - p\}$, $\{T_2z_n - p\}$, $\{T_3y_n - p\}$, $\{T_3z_n - p\}$, $\{T_3x_n - p\} \subset B_R$. By (1.3) and Lemma 1.2 , we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|a_nT_1x_n + (1 - a_n)x_n - p)\|^2 \\
&\le \|a_nT_1x_n + (1 - a_n)x_n - p)\|^2 \\
&\le \|a_n(T_1x_n - p) + (1 - a_n)(x_n - p)\|^2 \\
&\le a_n\|T_1x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 - a_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \\
&\le a_n\|x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 - a_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \\
&= \|x_n - p\|^2 - a_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \\
\|z_n - p\|^2 &\le \|x_n - p\|^2 - a_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big).
\end{aligned}$$

Thus we have

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - a_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big).$$

Now by (1.3) and Lemma 1.3 , we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_nT_2z_n + c_nT_2x_n + (1 - b_n - c_n)x_n - p\|^2 \\
&\le \|b_nT_2z_n + c_nT_2x_n + (1 - b_n - c_n)x_n - p\|^2 \\
&\le b_n\|z_n - p\|^2 + c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\
&\quad - b_n(1 - b_n - c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
&\quad - b_na_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \\
&\le b_n\|x_n - p\|^2 + c_n\|x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 \\
&\quad - b_n(1 - b_n - c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
&\quad - b_na_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big).
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|^2 \le\; & \|x_n - p\|^2 - b_n(1 - b_n - c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
& -b_na_n(1 - a_n)\Big(g(\|T_1x_n - x_n\|)\Big).
\end{aligned}$$

Moreover, by (1.3) and Lemma 1.4, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n T_3 y_n + \beta_n T_3 z_n + \gamma_n T_3 x_n + (1-\alpha_n-\beta_n-\gamma_n)x_n - p\|^2 \\
&\leq \alpha_n\|T_3y_n - p\|^2 + \beta_n\|T_3 z_n - p\|^2 + \gamma_n\|T_3x_n - p\|^2 + \Big(1-\alpha_n-\beta_n-\gamma_n\Big)\|x_n-p\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_3z_n - x_n)\| \\
&\quad + \gamma_n g(\|T_3x_n - x_n\|)\Big) \\
&\leq \alpha_n\|y_n - p\|^2 + \beta_n\|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \Big(1-\alpha_n-\beta_n-\gamma_n\Big)\|x_n-p\|^2 \\
&\quad - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_3z_n - x_n)\| \\
&\quad + \gamma_n g(\|T_3x_n - x_n\|)\Big) \\
&\leq \|x_n - p\|^2 - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_3z_n - x_n)\| \\
&\quad + \gamma_n g(\|T_3x_n - x_n\|)\Big) - \alpha_n b_n(1-b_n-c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
&\quad - \alpha_n b_n a_n(1-a_n)\Big(g(\|T_1x_n - x_n\|)\Big) - \beta_n a_n(1-a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \\
&= \|x_n - p\|^2 - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g(\|T_3y_n - x_n\|) + \beta_n g(\|T_3z_n - x_n)\| \\
&\quad + \gamma_n g(\|T_3x_n - x_n\|)\Big) - \alpha_n b_n(1-b_n-c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
&\quad - (\alpha_n b_n + \beta_n)a_n(1-a_n)\Big(g(\|T_1x_n - x_n\|)\Big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \quad \leq \quad & \|x_n - p\|^2 - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\Big(\alpha_n g(\|T_3y_n - x_n\|) \\
& + \beta_n g(\|T_3z_n - x_n)\| + \gamma_n g(\|T_3x_n - x_n\|)\Big) \\
& - \alpha_n b_n(1-b_n-c_n)\Big(g(\|T_2z_n - x_n\|)\Big) \\
& - (\alpha_n b_n + \beta_n)a_n(1-a_n)\Big(g(\|T_1x_n - x_n\|)\Big).
\end{aligned}$$

From the last inequality, we have

$$\alpha_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T_3y_n - x_n\|) \leq 3\Big(\|x_n - p\|^2 - \|x_{n+1}-p\|^2\Big), \tag{2.4}$$

$$\beta_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T_3z_n - x_n\|) \leq 3\Big(\|x_n - p\|^2 - \|x_{n+1}-p\|^2\Big), \tag{2.5}$$

$$\gamma_n(1-\alpha_n-\beta_n-\gamma_n)g(\|T_3x_n - x_n\|) \leq 3\Big(\|x_n - q\|^2 - \|x_{n+1}-p\|^2\Big), \tag{2.6}$$

and

$$(1-b_n-c_n)(b_n\alpha_n)(g(\|T_2z_n - x_n\|) \leq \Big(\|x_n - p\|^2 - \|x_{n+1}-p\|^2\Big). \tag{2.7}$$

$$(\alpha_n b_n + \beta_n)a_n(1-a_n)\Big(g(\|T_1x_n - x_n\|)\Big) \leq \Big(\|x_n - p\|^2 - \|x_{n+1}-p\|^2\Big). \tag{2.8}$$

By condition

$$0 < \liminf_n \alpha_n \leq \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1,$$

there exists a positive integer n_0 and $\delta, \delta' \in (0,1)$ such that $0 < \delta < \alpha_n$ and $\alpha_n + \beta_n + \gamma_n < \delta' < 1$ for all $n \geq n_0$. Then it follows from (2.4) that

$$(\delta(1-\delta')) \lim_{n\to\infty} \alpha_n(1-\alpha_n-\beta_n-\gamma_n) g(\|T_3 y_n - x_n\|) \leq 3\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big),$$

for all $n \geq n_0$. Thus, for $m \geq n_0$, we write

$$\sum_{n=n_0}^{m} g(\|T_3 y_n - x_n\|) \leq \frac{3}{(\delta(1-\delta'))} \sum_{n=n_0}^{m} \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big) \leq \frac{3}{(\delta(1-\delta'))} \Big(\|x_{n_0} - p\|^2\Big).$$

Letting $m \to \infty$, we have $\sum_{n=n_0}^{m} g(\|T_3 y_n - x_n\|) < \infty$ so that

$$\lim_{n\to\infty} g(\|T_3 y_n - x_n\|) = 0.$$

From g is continuous strictly increasing with $g(0) = 0$ and (1) then we have

$$\lim_{n\to\infty} \|T_3 y_n - x_n\| = 0.$$

By using a similar method for inequalities (2.5), (2.6),(2.7) and (2.8) we have,

$$\begin{aligned}
\lim_{n\to\infty} \|T_3 z_n - x_n\| &= 0,\\
\lim_{n\to\infty} \|T_3 x_n - x_n\| &= 0,\\
\lim_{n\to\infty} \|T_2 z_n - x_n\| &= 0,\\
\lim_{n\to\infty} \|T_1 x_n - x_n\| &= 0.
\end{aligned}$$

Also note that

$$\|z_n - x_n\| \leq \|a_n T_1 x_n + (1-a_n)x_n - x_n\| \leq a_n \|T_1 x_n - x_n\| \underset{n\to\infty}{\to 0}.$$

Since T_2 is Berinde nonexpansive mapping, then we obtain,

$$\begin{aligned}
\|T_2 x_n - x_n\| &\leq \|T_2 x_n - T_2 z_n\| + \|T_2 z_n - x_n\|\\
&\leq \|x_n - T_2 z_n\| + \|z_n - x_n\| + L\|x_n - T_2 z_n\| \underset{n\to\infty}{\to 0}.
\end{aligned}$$

Also note that

$$\begin{aligned}
\|y_n - x_n\| &\leq \|b_n T_2 z_n + c_n T_2 x_n + (1-b_n-c_n)x_n - x_n\|\\
&\leq b_n \|T_2 z_n - x_n\| + c_n \|T_2 x_n - x_n\|.
\end{aligned}$$

Then we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|T_3x_n - x_n\| = 0, \quad \lim_{n\to\infty}\|T_2x_n - x_n\| = 0 \text{ and } \lim_{n\to\infty}\|T_1x_n - x_n\| = 0.$$

∎

In the next result, we prove our strong convergence theorem as follows.

Theorem 2.3. *Let X be a real uniformly convex Banach space and K be a nonempty closed convex subset of X and $T_i : K \to K, i = 1,2,3$ be Berinde nonexpansive and quasi-nonexpansive mappings. Assume that $p \in F$ is a common fixed point of $T_i, i = 1,2,3$ and let $\{x_n\}$ be a sequence in K defined by (1.3), where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \in \mathbb{N}$, and satisfy the conditions of Lemma 2.2. If, in addition, T_1 or T_2 is semi-compact then $\{x_n\}$ converges strongly to a common fixed point of $T_i, i = 1,2,3$.*

Proof. By Lemma 2.1, $\{\|x_n - p\|\}$ is bounded. It follows by our assumption that T_1 is semi-compact, there exists a subsequence $\{T_1x_{n_k}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_k} \longrightarrow q*$ as $k \to \infty$.

$$\begin{aligned}\|T_1q* - T_1x_{n_k}\| &\le \|x_{n_k} - q*\| + L\|q* - T_1x_{n_k}\| \\ &\le \|x_{n_k} - q*\| + L(\|q* - x_{n_k}\| + \|x_{n_k} - T_1x_{n_k}\|) \longrightarrow 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Therefore, we have $\|T_1x_{n_k} - T_1q*\| \longrightarrow 0$ which implies that $x_{n_k} \longrightarrow q*$ as $k \to \infty$. Again by Lemma 2.2, we have

$$\|q* - T_1q*\| \le \lim_{k\to\infty}(\|x_{n_k} - q*\| + \|x_{n_k} - T_1x_{n_k}\| + T_1x_{n_k} - T_1q*\|) = 0.$$

Then we have $q* = T_1q*$. By using a similar method, $q* = T_2q*$ and then we have $q* = T_3q*$.

It follows that $q* \in F$. Moreover, since $\lim_{n\to\infty}\|x_n - q*\|$ exists, then $\lim_{n\to\infty}\|x_{n_k} - q*\| = 0$, that is, $\{x_n\}$ converges strongly to a fixed point $q* \in F$. Moreover, $\lim_{n\to\infty}\|y_n - x_n\| = 0$ and $\lim_{n\to\infty}\|z_n - x_n\| = 0$ as proved in Lemma 2.2 and it follows $\lim_{n\to\infty}\|y_n - q*\| = 0$ and $\lim_{n\to\infty}\|z_n - q^*\| = 0$. This completes the proof. ∎

Finally, we prove the weak convergence of the iterative scheme (1.3) for Berinde nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.4. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. and $T_i : K \to K, i = 1,2,3$ be Berinde nonexpansive and quasi-nonexpansive mappings. Assume that $p \in F$ is a common fixed point of $T_i, i = 1,2,3$ and let $\{x_n\}$ be a sequence in K defined by (1.3) where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n + \gamma_n\}$ in $[0,1]$ for all $n \in \mathbb{N}$, and satisfy the conditions of Lemma 2.2.*

Then $\{x_n\}$ converges weakly to a common fixed point of $T_i, i = 1,2,3$.

Proof. It follows from Lemma 2.2 that $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty}\|T_1x_n - x_n\| = 0$. Since X is a uniformly convex Banach space and $\{\|x_n - p\|\}$ is bounded, we may assume that $\{x_n\} \to q_1$ weakly as $n \to \infty$, without loss of generality. Then we have $q_1 \in F$. We assume that q_1 and q_2 are weak limits of the subsequences

$\{x_{n_k}\}$, $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By definition of demiclosed , $q_1, q_2 \in F$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. It follows from Lemma 1.5 that $q_1 = q_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $T_i : K \to K, i = 1, 2, 3$. Moreover, $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\lim_{n\to\infty} \|z_n - x_n\| = 0$ as proved in Lemma 2.2 and $\{x_n\} \to q_1$ weakly as $n \to \infty$, $\{y_n\} \to q_1$ weakly as $n \to \infty$ and $\{z_n\} \to q_1$ weakly as $n \to \infty$. This completes the proof. ∎

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