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Fixed Point Theorems of Wardowski Type Mappings in S_b-Metric Spaces

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Abstract In this paper, we introduced the concept of (α, F)-contractive pair in the structure of S_b-metric spaces. And we generalized the Wardowski type mappings in this structure and proved the fixed point and common fixed point theorems for (α, F)-contractive pair with some suitable conditions. Furthermore, we give some examples for the guarantee of the main theorems.

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1. Introduction

Recently Ali et al. [7] presented the idea of (α, F)-contractive mappings on the platform of uniform spaces and in [28], Souayah and Malakii produced some fixed point results on the platform of S_b-metric spaces. Motivated by the above mentioned work, some fixed point results on (α, F)-contractive mappings on the platform of S_b-metric spaces have been reported in the present article. Some new Wardowski type notions of (α, F)-contractive mappings in the new setting of S_b-metric space have been introduced along with some novel results of fixed point theorems on self-mappings in this space. This important extension may be helpful in the generalization of many existing results. The existence and uniqueness of fixed point by using such novel notions has also been discussed. The proposed results have been support by including some relevant examples.

The notion of b-metric space was presented by Branciari [11] as a generalization of the metric space by using the weaker triangular condition. This notion was initiated by Bourbaki [14], and Bakhtin [9] based on Banach contraction. Banach contraction principle has obtained an important place in this century of science and technology. Its importance and significance can be noticed from its numerous extensions and applications in many

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fields. Banach [10] states the contraction principle in the way that in a complete metric space (X, d), every self-mapping t satisfying:

$$d(ta, tb) \leq \lambda d(a, b),\ \forall\, a, b \in X \text{ and } \lambda \in (0, 1),$$

has a unique fixed point.

Definition 1.1. [11] Let X be a non-empty set and $m \geq 1$ a real number. A mapping $d_b\colon X \times X \to [0, +\infty)$ is called a b-metric if for all $a, b, c \in X$, it satisfies the following conditions:

BM1. $d_b(a, b) \geqslant 0$,
BM2. $d_b(a, b) = 0 \Leftrightarrow a = b$,
BM3. $d_b(a, b) = d_b(b, a)$,
BM4. $d_b(a, c) \leq m[d_b(a, b) + d_b(b, c)]$.

In this case, the pair (X, d) is called a b-metric space.

Effectiveness of metric fixed point theory is based on two things, one is to modify the contraction condition and second is to amend the structure of metric space [1–8, 13, 17–24, 27–33]. In literature we can see massive contribution in both directions by different mathematician. For example a lot of work has been done by using the platform of b-metric spaces [3, 6–8, 12, 15, 28, 30]. Where as, in many results contraction is replaced by cyclic contraction and by α-ψ-admissible type mappings.
Samet *et al*. [26], Salimi et al. [25] and Ali *et al*. [7] have also given some nice extensions by using the notions of (α,F)-admissible and (α,ψ)-admissible type mappings. In this paper, we consider (α,F)-contraction on S_b-metric spaces to produce some fixed point results. Furthermore, we have also given some examples to illustrate the main results. Our results generalize several existing results given by many authors in literature.
Now we give some basic definitions and results to provide a base for our main results.

Definition 1.2. [11] Consider that (X, d_b) is a b-metric space, then the following holds:

a. a sequence $\{a_n\}$ in the space is called Cauchy if $d(a_n, a_m) \to 0$ whenever $m, n \to \infty$,
b. a sequence $\{a_n\}$ in the space is called convergent if there exists $a \in X$ such that $d(a_n, a) \to 0$ whenever $n \to \infty$ and it is expressed as

$$\lim_{n\to\infty} a_n = a.$$

c. the space X is called complete if every Cauchy sequence is convergent in X.

Definition 1.3. [11] A mapping $F\colon \mathbb{R}^+ \to \mathbb{R}$ is said to be an F-mapping, if the following conditions are observed:

a. F is strictly increasing function, that is, as for all $a, b \in \mathbb{R}^+$, if $a < b$ holds that $F(a) < F(b)$,
b. for each sequence $\{b_n\}$ that belongs to positive real numbers $\mathbb{R}^+$,

$$\lim_{n\to\infty} b_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(b_n) = -\infty,$$

c. there is a real number $m \in (0, 1)$ such that

$$\lim_{a\to\, 0^+} a^m F(a) = 0.$$

Definition 1.4. [11] Let (X, d_b) be a complete b-metric space. The mapping $t\colon X \to X$ is called (α,F)-contractive for the two mappings $\alpha\colon X \times X \to [0, \infty)$ and $F\colon \mathbb{R}^+ \to \mathbb{R}$ if it satisfies the following for $\tau > 0$:

$$\tau + F(\alpha(a, b)d(ta, tb)) \leq F(d(a, b)) \ \ \forall\, a, b \in X.$$

Definition 1.5. [11] A mapping $t\colon X \to X$ is called α-admissible if for all $a, b \in X$, such that $\alpha(a, b) \geq 1$ implies that $\alpha(ta, tb) \geq 1$.

Theorem 1.6. *[16] Let (X, d_b) be a complete b-metric space with constant $s \geq 1$ and b-metric is continuous functional. Let $t\colon X \to X$ be a mapping with contraction constant $m \in [0, 1)$ where as $sm < 1$. Then t has a unique fixed point.*

Let us recall S_b-metric space.

Definition 1.7. [11] Let X be a nonempty set X. A function $d : X^3 \to [0, \infty)$ is called an S-metric if it satisfies the following for all $a, b, c, t \in X$:

$S1.$ $S(a, b, c) = 0 \Leftrightarrow a = b = c,$
$S2.$ $S(a, a, b) = S(b, b, a),$
$S3.$ $S(a, b, c) \leq S(a, a, t) + S(b, b, t) + S(c, c, t).$

In this case, the pair (X, S) is called an S-metric space.

Definition 1.8. [11] Let X be a non-empty set X. A function $S_b : X^3 \to [0, \infty)$ is called an S_b-metric if it satisfies the following for all $a_1, a_2, a_3, t \in X$:

$S_b1.$ $S_b(a_1, a_2, a_3) = 0 \Leftrightarrow a_1 = a_2 = a_3,$
$S_b2.$ $S_b(a_1, a_1, a_2) = S_b(a_2, a_2, a_1),$
$S_b3.$ $S_b(a_1, a_2, a_3) \leq b[S_b(a_1, a_1, t) + S_b(a_2, a_2, t) + S_b(a_3, a_3, t)].$

In this case, the pair (X, S_b) is called an S_b-metric space.

Example 1.9. Consider a nonempty set X with $card(X) \geq 5$ also assume $X = X_1 \cup X_2$ is partition of X with $card(X_1) \geq 4$. Let $b \geq 1$ and for all $a_1, a_2, a_3 \in X$, we define

$$S_b(a_1, a_2, a_3) = \begin{cases} 0 \text{ for } a_1 = a_2 = a_3 = 0; \\ 3b \text{ for } (a_1, a_2, a_3) \in X_1^3; \\ 1 \text{ for } (a_1, a_2, a_3) \in X_1^3. \end{cases}$$

Then we can easily see that the pair (X, S_b) is an S_b-metric space with coefficient $m \geq 1$.

Example 1.10. Let $X = R$ be a set of real numbers and $S_b : X^3 \to [0, \infty)$ be a function with the metric defined as:

$$S_b(a, b, c) = |a - c| + |b - c|.$$

Then it is easy to verify that (X, S_b) is an S_b-metric space.

Definition 1.11. [11] In an S_b-metric space (X, S_b) the following statements are satisfied:

a. any sequence $\{a_n\}$ in the space is called convergent [28] if there exists $a \in X$ such that $S_b(a_n, a_n, a) \to 0$ whenever $n \to \infty$ and the limit of sequence is expressed as $\lim_{n\to\infty} a_n = a$,

b. any sequence $\{a_n\}$ in the space is called an S_b-Cauchy [28] if $S_b(a_n, a_n, a_m) \to 0$ whenever $m, n \to \infty$,

c. the space X is called complete [28] if every S_b-Cauchy sequence is convergent to a point $a \in X$ such that $\lim_{n,m\to\infty} S_b(a_n, a_n, a_m) = \lim_{n\to\infty} S_b(a_n, a_n, a) = a$.

2. Main Results

To give the main results of this paper, we first present some new definition in our setting.

Definition 2.1. Let (X, S_b) be an S_b-metric space with a continuity of functional S_b. Let $t\colon X \to X$ be a self mapping and $\alpha\colon X \times X \to [0, \infty)$. Then t is called (α, F, S_b)-contractive mapping for some constant $\tau > 0$ such that it satisfies the following condition for each $a, b \in X$:

$$\tau + F(\alpha(a, b)S_b(ta, ta, tb)) \leq F(S_b(a, a, b)) \tag{2.1}$$

with $\min\{\alpha(a, b)S_b(ta, ta, tb)), S_b(a, a, b)\} > 0$.

Theorem 2.2. *Let (X, S_b) be an S_b-metric space with S_b a continuous functional. Let $t\colon X \to X$ be an (α, F)-contractive mapping for $\tau > 0$ such that it satisfies the following conditions for each $a, b \in X$:*

i.: *t is α-admissible,*
ii.: *there is $a_0 \in X$ such as $\alpha(a_0, ta_0) \geq 1$ and $\alpha(ta_0, a_0) \geq 1$,*
iii.: *t is continuous.*

Then t has a fixed point w.

Proof. We take $a_0 \in X$ and $a_1 \in ta_0$ such that $\alpha(a_0, a_1) \geq 1$. If we find that $a_0 = a_1$, then the proof is obvious. Now suppose that $a_0 \neq a_1$. If we have $a_1 = ta_1$, then a_1 is a fixed point. Suppose that $a_1 \neq ta_1$ and $\alpha(a_n, a_{n+1}) \geq 1$and $\alpha(a_{n+1}, a_n) \geq 1$ for all $n \in N \cup \{0\}$.
From the Definition 2.1, for all $n \in N$, we have

$$\begin{aligned}\tau + F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) &= \tau + F(S_b(ta_n, ta_n, ta_{n+1})) \\ &\leq \tau + F(\alpha(a_{n+1}, a_n)S_b(ta_n, ta_n, ta_{n+1})) \\ &\leq F(S_b(a_n, a_n, a_{n+1})).\end{aligned}$$

We know that for 1-st iteration

$$\begin{aligned}\tau + F(S_b(a_1, a_1, a_2)) =& \tau + F(S_b(ta_0, ta_0, ta_1)) \\ &\leq \tau + F(\alpha(a_1, a_0)S_b(ta_0, ta_0, ta_1)) \\ &\leq F(S_b(a_0, a_0, a_1)).\end{aligned}$$

Hence we have:
$\tau + F(S_b(a_1, a_1, a_2)) \leq F(S_b(a_0, a_0, a_1)) \ \forall\, a_1, a_2, \in X.$
Likewise for 2-nd iteration, we have:

$$\begin{aligned}\tau + F(S_b(a_2, a_2, a_3)) =& \tau + F(S_b(ta_1, ta_1, ta_2)) \\ &\leq \tau + F(\alpha(a_2, a_1)S_b(ta_1, ta_1, ta_2)) \\ &\leq F(S_b(a_1, a_1, a_2)).\end{aligned}$$

That is:
$\tau + F(S_b(a_2, a_2, a_3)) \leq F(S_b(a_1, a_1, a_2)) \ \forall\, a_2, a_3, \in X.$
As the mapping t is α-admissible, so we have $\alpha(a_0, a_1) = \alpha(a_0, ta_0) \geq 1$

$\Rightarrow \alpha(a_1, a_2) = \alpha(ta_0, ta_1) \geq 1$.
Consequently, we have as under

$$\alpha(a_n, a_{n+1}) \geq 1, \quad \forall\, n \in N \cup \{0\}. \tag{2.2}$$

Therefore, we have inductively, for 2-th iteration

$$n\tau + F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) \leq F(S_b(a_0, a_0, a_1))$$

and

$$F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) \leq F(S_b(a_0, a_0, a_1)) - n\tau. \tag{2.3}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) \leq \lim_{n\to\infty}[F(S_b(a_0, a_0, a_1)) - n\tau].$$
$$\Rightarrow \lim_{n\to\infty} F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) = -\infty.$$

Then by b. of Definition 1.3, we have

$$\lim_{n\to\infty} S_b(a_{n+1}, a_{n+1}, a_{n+2}) = 0. \tag{2.4}$$

Let's define $S_{b,n} = S_b(a_{n+1}, a_{n+1}, a_{n+2})$.
By using c. of F-function there exists $k \in (0, 1)$ such that (2.4) becomes

$$\lim_{n\to\infty} S_{b,n}^k F(S_{b,n}) = 0. \tag{2.5}$$

Using the above notation, (2.3) may be expressed as:

$$F(S_{b,n}) - F(S_{b,0}) \leq -n\tau.$$

Using the c., (2.3) can be expressed as:

$$S_{b,n}^k F(S_{b,n}) - S_{b,n}^k F(S_{b,0}) \leq -S_{b,n}^k n\tau$$
$$\Rightarrow \lim_{n\to\infty} [S_n^k F(S_{b,n}) - S_{b,n}^k F(S_{b,0})] \leq \lim_{n\to\infty} -S_{b,n}^k n\tau$$
$$\Rightarrow \lim_{n\to\infty} nS_{b,n}^k \tau = 0$$
$$\Rightarrow \lim_{n\to\infty} nS_{b,n}^k = 0 \quad as \quad \tau > 0.$$

Since $\lim_{n\to\infty} nS_{b,n}^k = 0$, there is a $n_0 \in N$ such that $nS_{b,n}^k \leq 1$ for all $n \geq n_0$.

$$\Rightarrow S_{b,n}^k \leq 1/n \Rightarrow S_{b,n} \leq \frac{1}{n^{1/k}}.$$

To prove that $\{a_n\}$ is a Cauchy for some $m > n$, consider

$$S_b(a_n, a_n, a_m) \leq k[S_b(a_n, a_n, a_t) + S_b(a_n, a_n, a_t) + S_b(a_m, a_m, a_t)]$$
$$\leq k[2S_{b,n} + S_{b,m}]$$

$$\leq k[2\frac{1}{n^{1/k}} + \frac{1}{m^{1/k}}]. \tag{2.6}$$

Taking limit $n, m \to \infty$ on both sides and from (2.6), we get

$$\lim_{n,m\to\infty} S_b(a_n, a_n, a_m) \leq k[2\frac{1}{n^{1/k}} + \frac{1}{m^{1/k}}] = 0.$$

Thus, it follows that $\{a_n\}$ is an S_b-Cauchy sequence. As X is complete, there is a point $w \in X$ such as $\lim_{n\to\infty} S_b(a_n, a_n, w) = 0$ which implies $\lim_{n\to\infty} a_n = w$.
Moreover, since t is continuous, $w = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} ta_n = t(w)$.
Thus, we have $t(w) = w$. ∎

Example 2.3. Let $X = \left\{\frac{1}{x} : x \in \mathbb{N}\right\} \cup \{0\}$ be endowed with the usual metric S_b, defined as:

$$S_b(a, b, c) = |a - c| + |b - c|.$$

It can be seen easily that (X, S_b) is an S_b-metric space. Define the mapping $t : X \to X$ as

$$ta = \begin{cases} 0 \text{ for } a = 0 \\ \dfrac{1}{4x+2} \text{ for } a = \dfrac{1}{x} : x > 1 \\ 1 \text{ for } a = 1, \end{cases}$$

and the mapping $\alpha : X \times X \to [0, \infty)$ as:

$$\alpha(a, b) = \begin{cases} 1 \text{ for } a, b \in X - \{1\} \\ 0 \text{ otherwise.} \end{cases}$$

It is easy to find that t is (α,F)-contractive mapping for $F(a) = \ln a$ for every $a > 0$ with some $\tau > 0$. For some $a_0 = \frac{1}{2}$, we have $\alpha(a_0, ta_0) = \alpha(ta_0, a_0) = 1$. Furthermore for any sequence $\{a_n\}$ in X with $a_n \to a$ as $n \to \infty$ and $\alpha(a_{n-1}, a_n) = 1$ for all $n \in \mathbb{N}$ and also we have $\alpha(a_n, a) = 1$ for all $n \in \mathbb{N}$. So, it follows from Theorem 2.2 that t has a fixed point.
In the search of uniqueness fixed point, we assume the following condition:
(Ξ) For all $a, b \in Fix(t)$, there is a point $z \in X$ such that $\alpha(z, a) \geq 1$ and $\alpha(z, b) \geq 1$, where, $Fix(t)$ represents the set of all fixed points of t.
The following theorem guarantees the uniqueness of the fixed point.

Theorem 2.4. *If we add the new condition (Ξ) in the hypothesis of above Theorem 2.2, then we get the uniqueness of the fixed point of the self-mapping t.*

Proof. Contrarily, assume that there exists two different fixed points w and v of t. Condition (Ξ) implies that there exists $z \in X$ such that

$$\alpha(z, w) \geq 1 \textit{ and } \alpha(z, v) \geq 1. \tag{2.7}$$

Since t is α-admissible, therefore from inequality (2.7), we get

$$\alpha(t^n z, w) \geq 1 \text{ and } \alpha(t^n z, v) \geq 1, \text{ for all } n \in N \cup \{0\}. \tag{2.8}$$

Let us define a sequence $a_n \in X$ by $a_{n+1} = ta_n = t^n a_0$ for every $n \in N \cup \{0\}$ and $a_0 = a$.
From inequalities (2.8) and (2.1), we get

$$\tau + F(S_b(a_{n+1}, a_{n+1}, w)) \leq \tau + F(\alpha(a_n, w) S_b(ta_n, ta_n, tw)) \leq F(S_b(a_n, a_n, w)),$$
$$\text{for all } n \in N \cup \{0\}.$$

Inductively, we have

$$F(S_b(a_n, a_n, w)) \leq F(S_b(a_0, a_0, w)) - n\tau, \text{ for all } n \in N \cup \{0\}.$$
$$\Rightarrow \lim_{n\to\infty} F(S_b(a_n, a_n, w)) = -\infty. \tag{2.9}$$

It follows from the b. of Definition 1.3,

$$\lim_{n\to\infty} S_b(a_n, a_n, w) = 0. \tag{2.10}$$

Similarly

$$\lim_{n\to\infty} S_b(a_n, a_n, v) = 0, \tag{2.11}$$

and the limit of a sequence is always unique, hence $w = v$.
Hence, we have obtained that u is the required unique fixed point. ∎

Definition 2.5. [1] Let (X, S_b) be a S_b-metric space with a continuity of functional S_b and a pair (t, s) of self-mappings $t, s\colon X \to X$ is α-admissible pair if for any $a, b \in X$ with $\alpha(a, b) \geq 1$, we get $\alpha(ta, sb) \geq 1$ and $\alpha(sa, tb) \geq 1$.

Definition 2.6. Let (X, S_b) be a S_b-metric space with a continuity of functional S_b and a pair (t, s) of self-mappings $t, s\colon X \to X$ is (α,F, S_b)-contractive if there exists a function $\alpha\colon X \times X \to [0, \infty)$ and $\tau > 0$ such that

$$\tau + F(\alpha(a, b) \max\{S_b(ta, ta, sb), S_b(sa, sa, tb)\}) \leq F(S_b(a, a, b)), \tag{2.12}$$

for each $a, b \in X$ with $\max\{\alpha(a, b) \max\{S_b(ta, ta, sb), S_b(sa, sa, tb)\}, S_b(a, a, b)\} > 0$.

Theorem 2.7. *Let (X, S_b) be an S_b-metric space with S_b a continuous functional. Consider an (α,F, S_b)-contractive pair (t, s) satisfying the following conditions:*

i.: *(t, s) is α-admissible,*
ii.: *there is a point $a_0 \in X$ such as $\alpha(a_0, ta_0) \geq 1$ and $\alpha(ta_0, a_0) \geq 1$,*
iii.: *for any sequence $\{a_n\}$ in X with $a_n \to 0$ as $n \to \infty$ and $\alpha(a_n, a_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$, then $\alpha(a_n, a) \geq 1$ for all $n \in N \cup \{0\}$.*

Then t and s have a common fixed point.

Proof. For any $a_0 \in X$ and by hypothesis **ii.**, we have $\alpha(a_0, ta_0) \geq 1$ and $\alpha(ta_0, a_0) \geq 1$. Since (t, s) is an α-admissible pair, a sequence $\{a_n\}$ in X such that $ta_{2n} = a_{2n+1}$ and $sa_{2n+1} = a_{2n+2}$, we have $\alpha(a_n, a_{n+1}) \geq 1$ and $\alpha(a_{n+1}, a_n) \geq 1$ for every $n \in N \cup \{0\}$. As (t, s) is an (α,F, S_b)-contractive pair, we have

$$\begin{aligned}\tau + F(S_b(a_{2n+1}, a_{2n+1}, a_{2n+2})) =& \tau + F(S_b(ta_{2n}, ta_{2n}, sa_{2n+1}))\\ &\leq \tau + F(\alpha(a_{2n}, a_{2n+1})\times\\ &\max\{S_b(ta_{2n}, ta_{2n}, sa_{2n+1}), S_b(sa_{2n}, sa_{2n}, ta_{2n+1})\}\\ &\leq F(S_b(a_{2n}, a_{2n}, a_{2n+1})).\end{aligned}$$

This implies that

$$\tau + F(S_b(a_{2n+1}, a_{2n+1}, a_{2n+2})) \leq F(S_b(a_{2n}, a_{2n}, a_{2n+1})). \tag{2.13}$$

Similarly, we can get that

$$\begin{aligned}\tau + F(S_b(a_{2n+2}, a_{2n+2}, a_{2n+3})) =& \tau + F(S_b(sa_{2n+1}, sa_{2n+1}, ta_{2n+2})) \\ &\leq \tau + F(\alpha(a_{2n+1}, a_{2n+2})\times \\ &\max\{S_b(ta_{2n+1}, ta_{2n+1}, sa_{2n+2}), S_b(sa_{2n+1}, sa_{2n+1}, ta_{2n+2})\} \\ &\leq F(S_b(a_{2n+1}, a_{2n+1}, a_{2n+2})).\end{aligned}$$

Similarly, it implies that

$$\tau + F(S_b(a_{n+1}, a_{n+1}, a_{n+2})) \leq F(S_b(a_n, a_n, a_{n+1})). \tag{2.14}$$

Then from (2.12) and (2.14), we have

$$F(S_b(a_n, a_n, a_{n+1})) \leq F(S_b(a_0, a_0, a_1)) - n\tau \text{ for all } n \in N \cup \{0\}. \tag{2.15}$$

$$\Rightarrow \lim_{n\to\infty} F(S_b(a_n, a_n, a_{n+1})) \leq \lim_{n\to\infty}[F(S_b(a_0, a_0, a_1)) - n\tau].$$

$$\Rightarrow \lim_{n\to\infty} F(S_b(a_n, a_n, a_{n+1})) = -\infty.$$

$$\Rightarrow \lim_{n\to\infty} S_b(a_n, a_n, a_{n+1}) = 0. \tag{2.16}$$

Let $S_{b,n} = S_b(a_n, a_n, a_{n+1})$ and by using $c.$ of Definition 1.3 there exists $k \in (0, 1)$ such that equation (2.16) becomes

$$\lim_{n\to\infty} S_{b,n}^k F(S_{b,n}) = 0. \tag{2.17}$$

By $c.$ of Definition 1.3 and taking limit to inequality (2.15), we have:

$$\begin{aligned}&F(S_{b,n}) - F(S_{b,0}) \leq -n\tau \\ &\Rightarrow S_{b,n}^k F(S_{b,n}) - S_{b,n}^k F(S_{b,0}) \leq -nS_{b,n}^k \tau \\ &\Rightarrow \lim_{n\to\infty}[S_{b,n}^k F(S_{b,n}) - S_{b,n}^k F(S_{b,0})] \leq \lim_{n\to\infty} -nS_{b,n}^k \tau \\ &\Rightarrow \lim_{n\to\infty} nS_{b,n}^k = 0 \quad as \quad \tau > 0.\end{aligned}$$

So, there exists $n_0 \in N$ such that $nS_{b,n}^k \leq 1$ for all $n \geq n_0$

$$\Rightarrow S_{b,n} \leq \frac{1}{n^{1/k}}. \tag{2.18}$$

For $m > n$, we show that $\{x_n\}$ is an S_b-Cauchy sequence, consider

$$\begin{aligned}S_b(a_n, a_n, a_m) \leq& k[S_b(a_n, a_n, a_t) + S_b(a_n, a_n, a_t) + S_b(a_m, a_m, a_t)] \\ &\leq k[2S_{b,n} + S_{b,m}] \\ &\leq k[2\frac{1}{n^{1/k}} + \frac{1}{m^{1/k}}].\end{aligned} \tag{2.19}$$

Taking limit $n, m \to \infty$ on both sides and from (2.19), we get

$$\lim_{n,m\to\infty} S_b(a_n, a_n, a_m) \leq k[2\frac{1}{n^{1/k}} + \frac{1}{m^{1/k}}] = 0.$$

Therefore, $\{a_n\}$ is an S_b-Cauchy sequence.
Thus, there exists $w \in X$ such that $\lim_{n\to\infty} S_b(a_n, a_n, w) = 0$.

So, this implies $\lim_{n\to\infty} ta_{2n} = \lim_{n\to\infty} sa_{2n+1} = w$.
From equation (2.12) and from assumption **iii.** of the theorem, we get

$$\begin{aligned} S_b(a_n, a_n, tw) &\leq k[S_b(a_n, a_n, a_{2n+2}) + S_b(a_n, a_n, a_{2n+2}) + S_b(a_{2n+2}, a_{2n+2}, tw)] \\ &= k[S_b(a_n, a_n, a_{2n+2}) + S_b(a_n, a_n, a_{2n+2}) + S_b(sa_{2n+1}, sa_{2n+1}, tw)] \\ &\leq k[S_b(a_n, a_n, a_{2n+2}) + S_b(a_n, a_n, a_{2n+2}) + \alpha(a_{2n+1}, w)\times \\ &\max\{S_b(ta_{2n+1}, ta_{2n+1}, sw), S_b(sa_{2n+1}, sa_{2n+1}, tw)\}))] \\ &< k[S_b(a_n, a_n, a_{2n+2}) + S_b(a_n, a_n, a_{2n+2}) + S_b(a_{2n+1}, a_{2n+1}, w)] \\ &< k[2S_{b,n} + S_{b,2n+1}] \\ &< k[2\frac{1}{n^{1/k}} + \frac{1}{(2n+1)^{1/k}}]. \end{aligned}$$

Taking limit on the both sides of above inequality, we have
$\lim_{n\to\infty} S_b(a_n, a_n, tw) = 0$ and $\lim_{n\to\infty} S_b(a_n, a_n, w) = 0$.
Thus, we have $tw = w$. Analogously, we can find that $sw = w$.
Hence, $tw = sw = w$. ■

Remark 2.8. Note that the Theorem 2.7 also holds if we replace condition **ii.** by as given below:
There exists $a_0 \in X$ such that $\alpha(a_0, sa_0) \geq 1$ and $\alpha(sa_0, a_0) \geq 1$.

Example 2.9. Let (X, S_b) be an S_b-metric space where $X = \left\{\frac{1}{x} : x \in \mathbb{N}\right\} \cup \{0\}$ with metric defined as:

$$S_b(a, b, c) = |a - c| + |b - c|.$$

We can easily find that (X, S_b) is an S_b-metric space. Assume $t : X \to X$ as defined by:

$$ta = \begin{cases} 0 \text{ for } a = 0 \\ \dfrac{1}{3x+1} \text{ for } a = \dfrac{1}{x} : x > 1 \\ 1 \text{ for } a = 1, \end{cases}$$

and $s : X \to X$ is defined by:

$$sa = \begin{cases} 0 \text{ for } a = 0 \\ \dfrac{1}{4x+1} \text{ for } a = \dfrac{1}{x} : x > 1 \\ 1 \text{ for } a = 1, \end{cases}$$

and the mapping $\alpha : X \times X \to [0, \infty)$ is as defined below:

$$\alpha(a, b) = \begin{cases} 1 \text{ for } a, b \in X - \{1\} \\ 0 \text{ otherwise.} \end{cases}$$

Then we have that t and s are (α,F)-contractive mapping for $F(a) = \ln a$ for every $a > 0$ with some $\tau > 0$. Here for some $a_0 = \frac{1}{2}$, we get that $\alpha(a_0, ta_0) = \alpha(ta_0, a_0) = 1$. Furthermore for any sequence $\{a_n\}$ in X, we obtain $a_n \to a$ for $n \to \infty$ and also we get $\alpha(a_n, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. We have also $\alpha(a_n, a) \geq 1$ for all $n \in \mathbb{N}$. Thus, by Theorem 2.7, the pair (t, s) of self-mappings t and s has a common fixed point.

To find out the unique common fixed point for the pair of self-mappings, we apply the condition given below:

> (Ω)**:** For all $a, b \in CFix(t, s)$, we have $\alpha(a, b) \geq 1$, where as $CFix(t, s)$ represents the set of all common fixed points for the pair of self-mappings t and s.

Theorem 2.10. *By including condition (Ω) in the statement of the Theorem 2.7, we get the uniqueness of the common fixed point for mappings t and s.*

Proof. We consider on contrary that $a, b \in X$ are two different common fixed points of t and s. From the condition **i.** of Theorem 2.7, we have

$$\tau + F(S_b(a, a, b)) \leq F(\alpha(a, b) \max\{S_b(ta, ta, sb), S_b(sa, sa, tb)\}) \leq F(S_b(a, a, b)),$$

which is not possible for $S_b(a, a, b) > 0$ and as a result, we get $S_b(a, a, b) = 0$. Likewise, we can obtain that $S_b(b, b, a) = 0$. Therefore, we get $a = b$ which is contrary to our assumption. Hence t and s have a unique common fixed point. ∎

The next results obviously follow by assuming $\alpha(a, b) = 1$ for all $a, b \in X$.

Corollary 2.11. *Let (X, S) be an S-metric space with a continuity of S. Also consider $t \colon X \to X$ is a mapping and F-functions such that*

$$\tau + F(S(ta, ta, tb)) \leq F(S(a, a, b)) \;\forall\, a, b \in X \textit{ where } S(a, a, b) > 0.$$

Then t has a unique fixed point.

Corollary 2.12. *Let (X, S) be an S-metric space with a continuity of S. Also consider $t, s \colon X \to X$ are mappings and F-functions such that*

$$\tau + F(\max S(ta, ta, sb), S(sa, sa, tb)) \leq F(S(a, a, b)) \;\forall\, a, b \in X$$

$$\textit{where we have } \max\{\max\{S(ta, ta, sb), S(sa, sa, tb)\}, S(a, a, b)\} > 0.$$

Then t and s have a unique fixed point.

Corollary 2.13. *Let (X, S_b) be an S_b-metric space with a continuity of S_b. Also consider $t \colon X \to X$ is a mapping and F-functions such that*

$$\tau + F(S_b(ta, ta, tb)) \leq F(S_b(a, a, b)) \;\forall\, a, b \in X \textit{ where } S_b(a, a, b) > 0.$$

Then t has a unique fixed point.

Corollary 2.14. *Let (X, S_b) be an S_b-metric space with a continuity of S_b. Also consider $t, s \colon X \to X$ are mappings and F-functions such that*

$$\tau + F(\max S_b(ta, ta, sb), S_b(sa, sa, tb)) \leq F(S_b(a, a, b)) \;\forall\, a, b \in X$$

$$\textit{where we have } \max\{\max\{S_b(ta, ta, sb), S_b(sa, sa, tb)\}, S_b(a, a, b)\} > 0.$$

Then t and s have a unique fixed point.

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