Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 937–943

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Representable Autometrized Algebra and MV Algebra

B.V. Subba Rao[1], Akella Kanakam[2] and Phani Yedlapalli[3,*]

[1] *Department of Mathematics, P.G. Courses, P.R. Government College(Auto.), Kakinada Andhra Pradesh-533005, India*
e-mail : bvsrao.65@gmail.com

[2] *Department of Mathematics, Ideal Institute of Technology, Kakinada-533005, (A.P.), India*
e-mail : sreepada1970@gmail.com

[3] *Department of Mathematics, Shri Vishnu Engineering College for Women(Auto.), Vishnupur Bhimavaram-534202, (A.P.), India*
e-mail : phaniyedlapalli23@gmail.com

Abstract In this paper, we characterized the class of MV algebras among the class of bounded Representable Autometrized Algebraa $A = (A, +, \leq, *)$ satisfying the condition:

(T) $$1 * (1 * x) = x \text{ for all } x \in A.$$

We obtained the connection between the homomorphism of Representable Autometrized Algebras and homomorphism of induced MV Algebra.

MSC: 06F05

Keywords: commutative DRl-monoids; homomorphism of lattice ordered autometrized algebras: MV algebras; representable autometrized algebra

Submission date: 14.09.2018 / Acceptance date: 29.03.2021

1. Introduction

C.C. Chang ([1, 2]) introduced MV algebras as an algebraic counter part of Lukasicwicz infinite valued proportional logic. DRl-Semigroups were introduced and studied by KLN Swamy ([3–5]), Representable Autometrized Algebras were introduced and studied by B.V.Subba Rao ([6–12]). The class of $DRl-$ semigroups is a proper subclass of the class of Representable Autometrized Algebras. Rachneck ([13, 14]) obtained the relation between DRl-monoids (Swamy [3–5]), and MV algebras ([1, 2]). He gave a connection between the homomorphism of $DRl-$ monoids and homomorphism of MV-algebras.

In the next Section, we obtained a characterization of MV-algebras among the class of bounded Representable Autometrized Algebras $A = (A, +, \leq, *)$ satisfying the condition (T) $1 * (1 * x) = x$, for all $x \in A$.

*Corresponding author.

Published by The Mathematical Association of Thailand.

In Section 3 following Rachuneck ([13, 14]), we proved a theorem giving the relation between the homomorphism of bounded Representable Autometrized Algebras and the homomorphisms of induced MV-algebras.

2. Characterization of MV-Algebras among the Class of Bounded Representable Autometrized Algebras

We recall the following

Definition 2.1. (C.C.Chang ([1, 2]) An algebra $A = (A, \oplus, \neg, 0)$ of signature $< 2, 1, 0 >$ is called an MV-algebra, if and only if, A satifies the following identities.
$(MV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
$(MV2)$ $x \oplus y = y \oplus x$
$(MV3)$ $x \oplus 0 = x$
$(MV4)$ $\neg\neg x = x$
$(MV5)$ $x \oplus \neg 0 = \neg 0$
$(MV6)$ $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

Note 2.2. Rachunek ([13, 14]) obtained the connection between DRl-monoids introduced by Swamy ([3–5]) and the MV-algebras.

Infact, given an MV-algebra $A = (A, \oplus, \neg, 0)$ and setting $x \leq y \Leftrightarrow \neg(\neg x \oplus y) \oplus y = y$, he proved that $\leq$ is a lattice order on A, and for any $r, s, \in A$ there exists a least element $r \otimes s$ with property $s \oplus (r \otimes s) \geq r$ and $A = (a, \oplus, 0, \vee, \wedge, \otimes)$ is a DRl-semigroup with smallest element 0 and greatest element $1 = \neg 0$ satisfying the identity $1 \otimes (1 \otimes x) = x$.

Further, given a bounded DRl monoid $A = (A, +, 0, \vee, \wedge, -)$ with smallest element 0 and the greatest element 1 satisfying the identity $1 - (1 - x) = x$ then, Rachunek ([13, 14]) proved that $A = (A, \oplus, \leq, 0)$ is an MV-algebra, where $\neg x = 1 - x$ for any $x \in A$.

Remark 2.3. The class of all commutative DRl-monoids introduced and studied by Swamy ([3–5]) is a proper subclass of the class of all Representable Autometrizrd algebras introduced and studied by Subba Rao ([6–8]).

In the following theorem, we characterize the MV-algebras among the Representable Autometrized Algebras.

Theorem 2.4. *Let $A = (A, +, \leq, *)$ be a Representable Autometrized Algebra with* 0 *(additive identity) as the least element and* 1 *as the greatest element, satisfying:*
(T) $1 * (1 * x) = x$, *for all $x \in A$ Then, $A = (A, +, \neg, *)$ is an MV-algebra, if and only if, A satisfies the following condition.*
(V) $1 * [(1 * x) + y] + y = 1 * [(x + (1 * y))] + x$ *for all $x, y \in A$, where $\neg x = 1 * x$ for all $x \in A$.*

Proof. Assume that $A = (A, +, \leq, *)$ is a Representable Autometrized Algebra with 0 (the additive identity) as the least element and 1 as the greatest element, satisfying
(T) $1 * (1 * x) = x$, for all $x \in A$
let us assume that A satisfies the condition (V) mentioned in the theorem. Let us define $\neg x = 1 * x$ for all $x \in A$, clearly $A = (A, +, \leq, 0)$ is an algebra of species $< 2, 1, 0 >$ and we have the following properties
$(MV1)$ $x + (y + z) = (x + y) + z$, for all $x, y, z \in A$. (since + is Associative in A)

$(MV2)$ $x + y = y + x$ for all $x, y \in A$. (since $+$ is Commutative in A)
$(MV3)$ $x + 0 = x$ for all $x \in A$. (since 0 is the identity in A w.r.to $+$)
$(MV4)$ $\neg(\neg x) = 1 * (1 * x) = x$ by (T) (Hypothesis)
we have $1 + 1 \leq 1$ (since 1 is the greatest element)
$1 + 1 \geq 1 + 0 = 1$(since, 0 is the least element)
$1 + 1 \geq 1$ (since 1 is the greatest element)
Therefore $1 = 1$
$(MV5)$ $x + \neg 0 = x + (1 * 0) = x + 1 = 1 = 1 * 0 = \neg 0$
(since $x + 1 \geq 0 + 1 = 1 = 1 + 1 \geq x + 1$ and 1 is the greatest element)
$(MV6)$ $\neg(\neg x + y) + y = \neg(x + \neg y) + x$ for all $x, y \in A$
(by (V) since $\neg a = 1 * a$ for all $a \in A$) Therefore, $(A = (A, +\neg, 0)$ satisfies all the conditions $(MV1\ to\ MV6)$ Hence, $A = (A, +, \neg, 0)$ is an MV-algebra.

Conversly, assume that $A = (A, +, \neg, 0)$ is an MV algebra. Therefore A satisfies the axiom $(MV6)$.
Therefore from the $(MV6)$ above (By definition of MV-algebra)
we have $1 * [(1 * x) + y] + y = 1 * [x + (1 * y)] + x$ for all $x, y \in A$. (since $\neg x = 1 * x$)
Thus, A satisfies the condition (V)

$$1 * [(1 * x) + y] + y = 1 * [x + (1 * y)] + x \text{ for all } x, y \in A.$$

Hence the Theorem. ∎

Remark 2.5. If $A = (A, +, \leq, *)$ is a Representable Autometrized algebra with 0 (additive identity) as the least element, and 1 as the greatest element, then A need not satisfy the condition(T) $1 * (1 * x) = x$ for all $x \in A$ nor the condition (V) mentioned in the above theorem.

For example, consider the following example given by Subba Rao (Page 436, [6]).

Example 2.6. Let $A = \{0, a, b, c, d, e\}$, Define $0 < a < b < c < d < e$. Define $+$ and $*$ by the following Tables.

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	b	b	e	e	e
b	b	b	b	e	e	e
c	c	e	e	e	e	e
d	d	e	e	e	e	e
e	e	e	e	e	e	e

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	a	c	c	c
b	b	a	0	c	c	c
c	c	c	c	0	a	a
d	d	c	c	a	0	a
e	e	c	c	a	a	0

It is routine to verify that $A = (A, +, \leq, *)$ is a Representable Autometrized Algebra with 0 as the least element and e as the greatest element. But, however this Representable Autometrized algebra A does not satisfy the condition (T) $1 * (1 * x) = x$, for all $x \in A$. In fact, e is the greatest element in A and $b \in A$. But $e * (e * b) = e * c = a \neq b$, and $e * (e * d) = e * a = c \neq d$.
Thus, this example shows that there exists a bounded Representable Autometrized Algebra $A = (A, +, \leq, *)$ not satisfying the condition (T) mentioned in the above theorem.
Further, let us take $x = a$ and $y = b$ of A in the condition
(V) $1 * ((1 * x) + y) + y = 1 * (x + (1 * y)) + x$, for all $x, y \in A$

$$\begin{aligned}
\text{L.H.S of } (V) &= \neg(\neg a + b) + b \\
&= \neg(e * a + b) + b \\
&= \neg(c + b) + b \\
&= \neg e + b \\
&= (e * e) + b \\
&= 0 + b \\
&= b
\end{aligned}$$

where as

$$\begin{aligned}
\text{R.H.S of } (V) &= \neg(a + \neg b) + a \\
&= \neg(a + (e * b)) + a \\
&= \neg(a + c) + a \\
&= \neg e + a \\
&= (e * e) + a \\
&= 0 + a \\
&= a
\end{aligned}$$

but $b \neq a$.

This shows that A does not satisfy the condition (V) mentioned in the above theorem. Thus, there exists a bounded Representable Autometrized Algebra which neither satisfies the condition (T) nor condition (V).

3. Extension of Rachunk's Theorem to DRl-Monoids-MV-Algebras to Bounded Representable Autometrized Algebras

We recall the following definition of homomorphism of Lattice ordered Autometrized Algebras introduced by Subba Rao [6].

Definition 3.1. Let $S = (S, +, \leq, *)$ and $T = (T, +, \leq, *)$ be lattice ordered autometrized algebras. We say that amapping $f : S \longrightarrow T$ is a homomorphism from S to T, if and only if,
$(H1)$ $f(a + b) = f(a) + f(b)$ for any a,b in S,
$(H2)$ $f(a \vee b) = f(a) \vee b$ for any a,b in S,
$(H3)$ $f(a \wedge b) = f(a) \wedge b$ for any a,b in S and
$(H4)$ $f(a * b) = f(a) * b$ for any a,b in S.

A homomorphism $f : S \longrightarrow T$ is called
(i) An epimorphism, if and only if f is onto
(ii) Monomorphism, if and only if, f is $1 - 1$ and $f^{-1} : T \longrightarrow S$ is also a homomorphism.

Note 3.2. The above definition holds good for Representable Autometrized Algebra also, since every Representable Autometrized Algebra is by definition, a lattice ordered Autometrized Algebra satisfying some natural conditions.

We recall the following definition of a homomorphism of MV-algebras from C.C.Chang [1].

Definition 3.3. Let $A = (A, \oplus, \neg, 0)$ and $B = (B, \oplus, \neg, 0)$ be MV-algebras. A mapping $f : A \longrightarrow B$ is said to be a homomorphism from the MV-algebra A into MV-algebra B, if and onlt if,
(i) $f(0) = 0$,
(ii) $f(a \oplus b) = f(a) \oplus f(b)$,
(iii) $f(\neg a) = \neg f(a)$ for all $a, b \in A$.

If f is onto, f is called an epimorphism, and if f is one-one and onto, then f is called an isomorphism from A onto B, and in this case we say that the MV-algebras A and B are Isomorphic.

Note 3.4. If there exists a homomorphism g between two DRl- semigroups A and B such that $g(1) = 1^1$, where 1 and 1^1 are the greatest elements of A and B respectively, then Rachunek ([13, 14]) proved that g becomes a homomorphism of MV-algebras between the induced MV-algebras A and B in the following.

Theorem 3.5. (Rachunek [13, 14]) *Let $(A, +, 0, \vee, \wedge, -)$ and $(B, +, 0^1, \vee, \wedge, -)$ be DRl-semigroups with the least elements 0 and $0^1$and the greatest elements 1 and 1^1 respectively. satisfying the conditions*
(i) $1 - (1 - x) = x$ for all $x \in A$ and $1^1 - (1^1 - x) = x$ for $x \in B$.
(ii) $x + (y - x) = y + (x - y)$ for all $x, y \in A$ and $x + (y - x) = y + (x - y)$ for all $x, y \in B$ set $\neg x = 1 - x$ for any $x \in A$. Then $(A, +, \neg, 0)$ is an MV-algebra, and let $g : A \longrightarrow B$ be a homomorphism of DRl-semigroups such that $g(1) = 1^1$. Then g is a homomorphism of the induced MV algrbras.

In the following, we extended the above theorem to Representable Autometrized Algebras.

Theorem 3.6. *Let $A = (A, +, \leq, *)$ and $B = (B, +, \leq, *)$ be Representable Autometrized Algebras with 0 and 0^1 (additive identities) as the least elements and 1, 1^1 as the greatest elements respectively, satisfying the conditions*
(T) $1 * (1 * x) = x$ for all $x \in A$, and $1^1 * (1^1 * x) = x$ for all $x \in B$
(V) $1^1 * [(1^1 * x) + y] = 1^1 * [(x + (1^1 * y))] + x$ for all $x, y \in A$, and
$1^1 * [(1^1 * x) + y] = 1^1 * [(x + (1^1 * y))] + x$ for all $x, y \in B$
Let $g : A \longrightarrow B$ be a homomorphism of Representable Autometrized Algebras such that $g(1) = 1^1$. Then g is an MV-algebra homomorphism from the MV-algebra $A = (A, +, \neg, 0)$ into the MV-algebra $B = (B, +, \neg, 0^1)$, where the MV-algebras A and B are the induced MV-algebras of the Representable Autometrized Algebras A and B respectively (As per Theorem 2.4 above).

Proof. Let $A = (A, +, \leq, *)$ and $B = (B, +, \leq, *)$ be Representable Autometrized Algebras with the additive identities 0 and 0^1 as the least elements and 1, 1^1 as the greatest elements, respectively.

Let $A = (A, +, \neg, 0)$ and $B = (B, +, \neg, 0^1)$ be corresponding MV-algebras of the Representable Autometrized Algebras $A = (A, +, \leq, *)$ and $B = (B, +, \leq, *)$ respectively obtained as per the theorem(2.4) above.
Now let $g : A \longrightarrow B$ be a homomorphism of Representable Autometrized Algebras from A into B such that $g(1) = 1^1$.
Therefore,

$$g(x+y) = g(x) + g(y), \text{for all } x, y \in A,$$
$$g(x \vee y) = g(x) \vee g(y), \text{for all } x, y \in A,$$
$$g(x \wedge y) = g(x) \wedge g(y), \text{for all } x, y \in A,$$
$$g(x * y) = g(x) * g(y), \text{for all } x, y \in A, \text{ and}$$
$$g(0) = 0^1,$$
further $g(1) = 1^1$ (by hypothesis).

Since, $g(x+y) = g(x) + g(y)$ for all $x, y \in A$ and $g(0) = 0^1$, in order to show that g is an MV-algebra homomorphism, it is enough, if we prove that $g(\neg x) = \neg(g(x))$ for all $x \in A$.
Let $x \in A$, we have

$$\begin{aligned} g(\neg x) &= g(1 * x) \\ &= g(1) * g(x) \\ &= 1^1 * g(x) \\ &= \neg(g(x)). \end{aligned}$$

Hence, $g : (A, +, \neg, 0) \longrightarrow (B, +, \neg, 0^1)$ is an MV-algebra homomorphism.
Hence the theorem. ∎

Acknowledgements

We are very much indebted to **Dr.K.L.N.Swamy**, Emeritus Professor of Mathematics and Rector of Andhra University, Waltair, A.P-INDIA, for his continuous inspiring encouragement throughout the preparation of this paper.

References

[1] C.C. Chang, Algebra analysis of many valued logics, Trans. Amer. Math. Soc. 88 (2) (1958) 467–490.

[2] C.C. Chang, A new proof of the completeness of the Lukasiewicz axioms, Trans. Amer. Math. Soc. 93 (1959) 74–80.

[3] K.L.N. Swamy, Dually residuated lattice ordered semigroups, Math. Annalen 159 (1965) 105–114.

[4] K.L.N. Swamy, Dually residuated lattice ordered semigroups II, Math. Annalen 160 (1965) 64–71.

[5] K.L.N. Swamy, Dually residuated lattice ordered semigroups III, Math. Annalen 167 (1966) 71–74.

[6] B.V.S. Rao, Lattice ordered autometrized algebras, Math Seminar Notes 6 (1978) 429–448.

[7] B.V.S. Rao, Lattice ordered autometrized algebras II, Math Seminar Notes 7 (1979) 193–210.

[8] B.V.S. Rao, Lattice ordered autometrized algebras III, Math Seminar Notes 7 (1979) 441–455.

[9] B.V.S. Rao, P. Yedlapalli, A. Kanakam, Order topology and uniformity on A-metric space, International Research Journal of Pure Algebra 4 (4) (2014) 488–494.

[10] B.V.S. Rao, P. Yedlapalli, Metric spaces with distances in a representable autometrized algebras, Southeast Asian Bulletin of Mathematics 42 (3) (2018) 453–462.

[11] B.V.S. Rao, A. Kanakam, P. Yedlapalli, A note on representable autometrized algebras, Thai Journal of Mathematics 17 (1) (2019) 277–281.

[12] B.V.S. Rao, A. Kanakam, P. Yedlapalli, Representable Autometrized Semialgebra,Thai Journal of Mathematics 19 (4) (2021) 1267–1272.

[13] J. Rachunek, DRl-Semigroups and MV-algebras, Czechoslovak Math. J. 48 (123) (1998) 365–372.

[14] J. Rachunek, MV-Algebras are categorically equivalent to a class of DRl-semigroups, Math. Bohemica 123 (4) (1998) 437–441.