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A Bulk Queueing System with Starting Failure and Repair, Multiple Vacation, Stand-by Server, Closedown and N-Policy

S. Karpagam[1,*] **and Govindan Ayyappan**[2]

[1] *Department of Mathematics, Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and Technology, Chennai*
e-mail : karpagam19sks@gmail.com

[2] *Department of Mathematics, Pondicherry Engineering College, Puducherry*
e-mail : ayyappanpec@hotmail.com

Abstract In this investigation, single server bulk queueing system with is analyzed with server's starting failure and repair, stand-by server, multiple vacation, closedown and N-policy. Numerical results of the proposed model have been derived.

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Keywords: multiple vacation; starting failure; stand-by server; closedown and N-policy

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1. Introduction

Many researchers have worked extensively assuming unreliable server and it has to be repaired immediately. However, few researchers have worked on the introduction of stand-by server towards uninterrupted service. Queueing systems with such unreliable stations are the topics of worth investigating from the performance prediction point of view. In many real life situations, during start-up, we come across sudden crashing of the hard disk in the case of a computer server or failure of a lime injecting pump to start in a waste water treatment plant or failure of a machine in an assembly line in manufacturing industry or failure of a boat to start in a boat club. These type of fault is attended to by a repairman if he is free or a standby machine is put into operation if available. During this interval queue builds up. Many Researchers have worked on random failure of machine in regular operation. However very few studies have been made on starting failure models of queuing system. Development of such model can be useful to many industries like manufacturing systems. An attempt has been made in this paper to study queuing models with starting failure and implemented it to bulk queuing models.

*Corresponding author.

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Application of starting failure models was found in many industries like manufacturing systems. Studying starting failure in bulk queuing models is therefore essential. The server must be switched on by an incoming batch of customers if the number of customers in the queue is greater than or equal to $'a'$ else maintenance will begin and the customers will have to join the queue.

Such models have been analysed by many researchers with diverse combinations. Mokaddis et al. [1] studied the feedback retrial queueing system with starting failure and single vacation. Recently, Yanga et al. [2] discussed a multi-server retrial queueing system with Bernoulli feedback and starting failure. Krishnakumar et al. [3] discussed a single server feedback retrial queue with starting failures.

Rajadurai et al. [4] studied a retrial queueing system with orbit search under single vacation and starting failures. Some of the authors like, Wang et al. [5], Ayyappan et al. [6] and Varalakshmi et al. [7] discussed feedback retrial queueing model with the concept of starting failures. However, no work has been done in the queueing literature with the combination of bulk queueing system with starting failure and repair, multiple vacation, stand-by server, closedown and N-policy.

Arivudainambi and Gowsalya [8] analysed an M/G/1 retrial queue with two types of service, Bernoulli vacation and starting failure. Analysis of a pre-emptive priority retrial queue with starting failure, negative customers and at most J vacations was discussed by Yuvarani and Saravanarajan [9]. Using the supplementary variable technique they obtained the PGF's for the system/orbit size in steady state. The stochastic decomposition and some important system measures were discussed. Ke et al. [10] studied retrial queues with starting failure and service interruption. They use the matrix-geometric method to calculate stationary probabilities and to build steady-state measurements of system performance.

Jeyakumar and Rameshkumar [11] analysed an bulk queue with closedown time and controllable arrivals during multiple adaptive vacations. They obtained various characteristics of the queueing systems. Performance analysis of an feedback retrial queue with non-persistent customers and multiple vacations with the N-policy was discussed by Jailaxmi et al. [12]. Sudhesh et al. [13] analysed a N-policy queue with disastrous breakdown. Ayyappan and Karpagam [14] analysed a bulk queue with stand-by server, unreliable server, Bernoulli schedule multiple vacation, immediate feedback and N-policy. Recently, Kolledath et al. [15] broughtout an excellent survey on stand-by.

The novel contribution of this paper is the incorporation of starting failure and repair, stand-by server, multiple vacation, closedown and N-Policy in the bulk queueing system which we commonly come across in our real life situations in a manufacturing industry or transport sector or networking. The results of this paper can be applied in scheduling in production line, ATMs, computer networks and satellite communication, etc. This study can be extended further by considering the concepts of delaying repair, working vacation policies, impatient customers. We consider a queueing model that has, apart from the main channel, a stand-by that is only used during the regular service channel's maintenance times. Stand-by may not be as effective as the regular service channel, but it can still do much to stop the queue from becoming out of control during the main service channel's failure times. Stand-by support is essential for achieving high performance reliability and availability of any queueing model operating in machine environment.

2. Mathematical Model with Real Life Example

We consider a batch arrival bulk service queue with starting failure, repair, stand-by server, multiple vacation, closedown and N-policy in which the arrival follows a compound Poisson process with rate λ. Both servers service time, main server vacation time and closedown times follows general(arbitrary) distributions. The probability if the regular server starting failure is 'p' and there is no failure is '$q(p+q=1)$' and the regular server's repair time follows exponential distributions with rate η. Let us assume that the regular server fail to start with probability 'p' and the stand-by server comes into service. When the regular servers return after repair or service completion with the number of clients in the queue is less than 'a' then it starts the closedown work. After that it leaves for sequence of vacation until 'N' clients in the queue and it begins serving a batch of 'b' clients without starting failure with probability 'q'.

A real life situation exists in Industrial Township where the domestic sewage generated is treated in a Municipal sewage Treatment plant and the treated water is used for agricultural purpose. In a township with more than 1.5 lakh population plant of 30 Million Litres per day is in operation. The sewage generated from various blocks of the township is brought by a network of pipelines and reaches storage well. A pump in the storage well pumps the sewage to primary and secondary filters to remove suspended solids and then aerated in Aeration tanks for further oxidation of waste material present in the sewage. When the pump fails or the line fails or more flow than anticipated, the well over flows and the operator opens the overflow valve and the sewage is diverted to a surge reservoir meant for this purpose and stored there. Otherwise flooding of the area will happen and create health hazard. Once the pump / line is repaired, the sewage from the surge reservoir is pumped back to the storage well until it reaches certain level and the pump starts pumping for treatment and disposal in the usual manner.

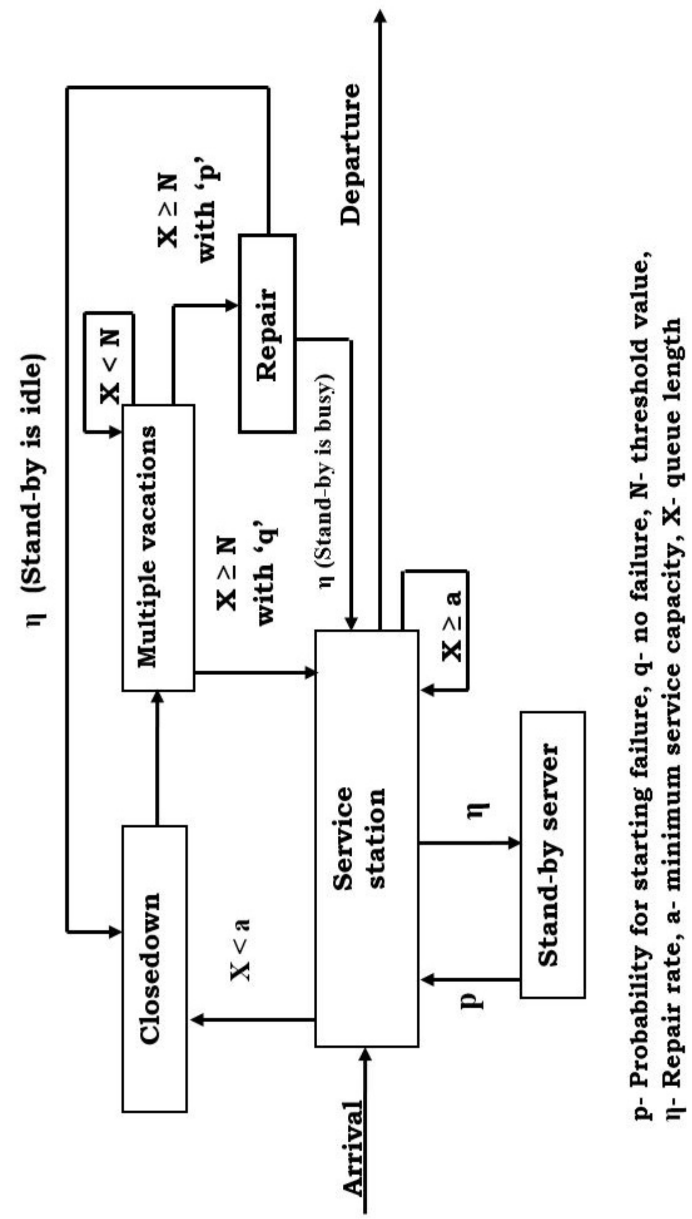

FIGURE 1. Pictorial representation of the model

2.1. Notations and Probabilities

- λ - Arrival rate.
- X - Group size random variable.
- $Pr(X = k) = g_k$.
- $X(z)$ - the Probability Generating Function (PGF) of X.
- $S_v(.)$, $S_u(.)$, $V(.)$ and $C(.)$ represent the Cumulative Distribution Functions (CDF) of service time of stand-by server, service time of regular server, vacation time and closedown time of regular server with corresponding probability density functions are $s_v(w)$, $s_u(w)$, $v(w)$ and $c(w)$ respectively.
- $S_u^0(t)$, $S_v^0(t)$, $V^{(0)}(t)$ and $C^{(0)}(t)$ represent the remaining service time of regular server, remaining service time of stand-by server, remaining vacation and closedown times of regular server at time 't' respectively.
- $\tilde{S}_u(\tau)$, $\tilde{S}_v(\tau)$, $\tilde{V}(\tau)$ and $\tilde{C}(\tau)$ represent the Laplace Stieltjes Transform (LST) of S_u, S_v, V and C respectively.

Let us define the following probabilities for further refinement of the above queueing system:

$$\begin{aligned}
&S_n(t)\Delta t = Pr\{N_q(t) = n,\ \epsilon(t) = 5\},\ 0 \leq n \leq a-1,\\
&M_{r,e}(w,t)\Delta t = Pr\{N_s(t) = r,\ N_q(t) = e,\ w \leq S_u^0(t) \leq w + \Delta t,\ \epsilon(t) = 1\},\\
&\qquad\qquad\qquad\qquad a \leq r \leq b,\ e \geq 0,\\
&C_n(w,t)\Delta t = Pr\{N_q(t) = n,\ w \leq C^0(t) \leq w + \Delta t,\ \epsilon(t) = 2\},\ n \geq 0,\\
&V_{l,j}(w,t)\Delta t = Pr\{\phi(t) = l,\ N_q(t) = j,\ w \leq V^0(t) \leq w + \Delta t,\ \epsilon(t) = 3\},\\
&\qquad\qquad\qquad\qquad l \geq 1,\ j \geq 0.\\
&L_{r,e}(w,t)\Delta t = Pr\{N_s(t) = r,\ N_q(t) = e,\ w \leq S_v^0(t) \leq w + \Delta t,\ \epsilon(t)4 =\},\\
&\qquad\qquad\qquad\qquad a \leq r \leq b,\ e \geq 0.
\end{aligned}$$

where $\epsilon(t) = 1, 2, 3, 4$ and 5 denotes regular server's busy, in closedown, in vacation, stand-by server is busy and idle respectively.
$\phi(t) = j$, if the regular server is on j^{th} vacation.
$N_s(t)$ and $N_q(t)$=Number of clients in the service station and queue at time t respectively.

3. Queue Size Distribution

For the above queueing model, as time 't' increases by Δt, Δt reduces the available service time of the regular server, vacation time, and closed-down time.

Stand-by server idle

$$S_0(t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)S_0(t) + \sum_{r=a}^{b} L_{r,0}(0,t)\Delta t, \tag{3.1}$$

$$S_e(t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)S_e(t) + \sum_{r=a}^{b} L_{r,e}(0,t)\Delta t + \sum_{k=1}^{e} S_{e-k}(t)\lambda g_k \Delta t, \quad 1 \leq e \leq a-1. \tag{3.2}$$

The regular server busy

$$M_{i,0}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)M_{i,0}(w,t) + \eta\int_0^\infty L_{i,0}(y,t)\ dys_u(w)\Delta t + \sum_{r=a}^{b} M_{r,i}(0,t)s_u(w)\Delta t,\ a \le i \le b, \tag{3.3}$$

$$M_{i,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)M_{i,e}(w,t) + \eta\int_0^\infty L_{i,e}(y,t)\ dys_u(w)\Delta t + \sum_{k=1}^{e} M_{i,e-k}(w,t)\lambda g_k\Delta t,\ e \ge 1,\ a \le i \le b-1, \tag{3.4}$$

$$M_{b,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)M_{b,e}(w) + \eta\int_0^\infty L_{b,e}(y,t)\ dys_u(w)\Delta t + \sum_{r=a}^{b} M_{r,b+e}(0,t)s_u(w)\Delta t + \sum_{k=1}^{e} M_{b,e-k}(w,t)\lambda g_k\Delta t,\quad 1 \le e \le N-b-1, \tag{3.5}$$

$$M_{b,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)M_{b,e}(w,t) + \eta\int_0^\infty L_{b,e}(y,t)\ dys_u(w)\Delta t + \sum_{k=1}^{e} M_{b,e-k}(w,t)\lambda g_k\Delta t + \sum_{r=a}^{b} M_{r,b+e}(0,t)s_u(w)\Delta t + q\sum_{l=1}^{\infty} V_{l,b+e}(0,t)s_u(w)\Delta t,\ e \ge N-b. \tag{3.6}$$

The stand-by server busy

$$L_{i,0}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)L_{i,0}(w,t) + \sum_{r=a}^{b} L_{r,i}(0,t)s_v(w)\Delta t + \sum_{k=0}^{a-1} S_k(t)\lambda g_{i-k}s_v(w)\Delta t,\ a \le i \le b, \tag{3.7}$$

$$L_{i,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)L_{i,e}(w,t) + \sum_{k=1}^{e} L_{i,e-k}(w,t)\lambda g_k\Delta t,\quad e \ge 1,\ a \le i \le b-1, \tag{3.8}$$

$$L_{b,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)(1-\eta\Delta t)L_{b,e}(w,t) + \sum_{r=a}^{b} L_{r,b+e}(0,t)s_v(w)\Delta t + \sum_{k=1}^{e} L_{b,e-k}(w,t)\lambda g_k\Delta t + \sum_{k=0}^{a-1} S_k(t)\lambda g_{b+e-k}s_v(w)\Delta t,\quad 1 \le e \le N-b-1, \tag{3.9}$$

$$\begin{aligned}
L_{b,e}(w-\Delta t, t+\Delta t) &= (1-\lambda\Delta t)(1-\eta\Delta t)L_{b,e}(w,t) + \sum_{r=a}^{b} L_{r,b+e}(0,t)s_v(w)\Delta t \\
&+ \sum_{k=1}^{e} L_{b,e-k}(w,t)\lambda g_k \Delta t + \sum_{k=0}^{a-1} S_k(t)\lambda g_{b+e-k} s_v(w)\Delta t \\
&+ p\sum_{l=1}^{\infty} V_{l,b+e}(0,t)s_v(w)\Delta t,\ e \geq N-b. \qquad (3.10)
\end{aligned}$$

Closedown

$$C_0(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)C_0(w,t) + \sum_{r=a}^{b} M_{r,0}(0,t)c(w)\Delta t + \eta S_0(t)c(w)\Delta t, \qquad (3.11)$$

$$\begin{aligned}
C_e(w-\Delta t, t+\Delta t) &= (1-\lambda\Delta t)C_e(w,t) + \sum_{r=a}^{b} M_{r,e}(0,t)c(w)\Delta t + \eta S_e(t)c(w)\Delta t \\
&+ \sum_{k=1}^{e} C_{e-k}(w,t)\lambda g_k \Delta t,\ 1 \leq e \leq a-1, \qquad (3.12)
\end{aligned}$$

$$C_e(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)C_e(w,t) + \sum_{k=1}^{e} C_{e-k}(w,t)\lambda g_k \Delta t,\ e \geq a. \qquad (3.13)$$

Vacation

$$V_{1,0}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)V_{1,0}(w,t) + C_0(0,t)v(w)\Delta t, \qquad (3.14)$$

$$\begin{aligned}
V_{1,e}(w-\Delta t, t+\Delta t) &= (1-\lambda\Delta t)V_{1,e}(w,t) + \sum_{k=1}^{e} V_{1,e-k}(w,t)\lambda g_k \Delta t \\
&+ C_e(0,t)v(w)\Delta t,\ e \geq 1, \qquad (3.15)
\end{aligned}$$

$$V_{l,0}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)V_{l,0}(w,t) + V_{l-1,0}(0,t)v(w)\Delta t,\ l \geq 2, \qquad (3.16)$$

$$\begin{aligned}
V_{l,e}(w-\Delta t, t+\Delta t) &= (1-\lambda\Delta t)V_{l,e}(w,t) + \sum_{k=1}^{e} V_{l,e-k}(w,t)\lambda g_k \Delta t \\
&+ V_{l-1,e}(0,t)v(w)\Delta t,\ l \geq 2,\ 1 \leq e \leq N-1, \qquad (3.17)
\end{aligned}$$

$$V_{l,e}(w-\Delta t, t+\Delta t) = (1-\lambda\Delta t)V_{l,e}(w,t) + \sum_{k=1}^{e} V_{l,e-k}(w,t)\lambda g_k \Delta t,\ l \geq 2, e \geq N. \qquad (3.18)$$

The Kolmogorov backward equation governing the system in steady state for the proposed model is:

$$(\lambda+\eta)S_0=\sum_{r=a}^{b}L_{r,0}(0), \tag{3.19}$$

$$(\lambda+\eta)S_e=\sum_{r=a}^{b}L_{r,e}(0)+\sum_{k=1}^{e}S_{e-k}\lambda g_k,\ 1\le e\le a-1, \tag{3.20}$$

$$-M_{i,0}^{'}(w)=-\lambda M_{i,0}(w)+\sum_{r=a}^{b}M_{r,i}(0)s_u(w)+\eta\int_0^\infty L_{i,0}(y)\ dys_u(w), a\le i\le b, \tag{3.21}$$

$$-M_{i,e}^{'}(w)=-\lambda M_{i,e}(w)+\eta\int_0^\infty L_{i,e}(y)\ dys_u(w)+\sum_{k=1}^{e}M_{i,e-k}(w)\lambda g_k,\ e\ge 1,\quad a\le i\le b-1, \tag{3.22}$$

$$-M_{b,e}^{'}(w)=-\lambda M_{b,e}(w)+\eta\int_0^\infty L_{b,e}(y)\ dys_u(w)+\sum_{r=a}^{b}M_{r,b+e}(0)s_u(w)+\sum_{k=1}^{e}M_{b,e-k}(w)\lambda g_k,\ 1\le e\le N-b-1, \tag{3.23}$$

$$-M_{b,e}^{'}(w)=-\lambda M_{b,e}(w)+\eta\int_0^\infty L_{b,e}(y)\ dys_u(w)+\sum_{k=1}^{e}M_{b,e-k}(w)\lambda g_k+\sum_{r=a}^{b}M_{r,b+e}(0)s_u(w)+q\sum_{l=1}^{\infty}V_{l,b+e}(0)s_u(w),\ e\ge N-b, \tag{3.24}$$

$$-L_{i,0}^{'}(w)=-(\lambda+\eta)L_{i,0}(w)+\sum_{r=a}^{b}L_{r,i}(0)s_v(w)+\sum_{k=0}^{a-1}S_k\lambda g_{i-k}s_v(w),\quad a\le i\le b, \tag{3.25}$$

$$-L_{i,e}^{'}(w)=-(\lambda+\eta)L_{i,e}(w)+\sum_{k=1}^{e}L_{i,e-k}(w)\lambda g_k,\ e\ge 1,\ a\le i\le b-1, \tag{3.26}$$

$$-L_{b,e}^{'}(w)=-(\lambda+\eta)L_{b,e}(w)+\sum_{r=a}^{b}L_{r,b+e}(0)s_v(w)+\sum_{k=1}^{e}L_{b,e-k}(w)\lambda g_k+\sum_{k=0}^{a-1}S_k\lambda g_{b+e-k}s_v(w),\ 1\le e\le N-b-1, \tag{3.27}$$

$$-L_{b,e}^{'}(w)=-(\lambda+\eta)L_{b,e}(w)+\sum_{r=a}^{b}L_{r,b+e}(0)s_v(w)+\sum_{k=1}^{e}L_{b,e-k}(w)\lambda g_k+\sum_{k=0}^{a-1}S_k\lambda g_{b+e-k}s_v(w)+p\sum_{l=1}^{\infty}V_{l,b+e}(0)s_v(w),\ e\ge N-b, \tag{3.28}$$

$$-C_0^{'}(w)=-\lambda C_0(w)+\sum_{r=a}^{b}M_{r,0}(0)c(w)+\eta S_0c(w), \tag{3.29}$$

$$-C_e^{'}(w) = -\lambda C_e(w) + \sum_{r=a}^{b} M_{r,e}(0)c(w) + \eta S_e c(w) + \sum_{k=1}^{e} C_{e-k}(w)\lambda g_k, \quad 1 \le e \le a-1, \tag{3.30}$$

$$-C_e^{'}(w) = -\lambda C_e(w) + \sum_{k=1}^{e} C_{e-k}(w)\lambda g_k, \; e \ge a, \tag{3.31}$$

$$-V_{1,0}^{'}(w) = -\lambda V_{1,0}(w) + C_0(0)v(w), \tag{3.32}$$

$$-V_{1,e}^{'}(w) = -\lambda V_{1,e}(w) + C_e(0)v(w) + \sum_{k=1}^{e} V_{1,e-k}(w)\lambda g_k, \; e \ge 1, \tag{3.33}$$

$$-V_{l,0}^{'}(w) = -\lambda V_{l,0}(w) + V_{l-1,0}(0)v(w), \; l \ge 2, \tag{3.34}$$

$$-V_{l,e}^{'}(w) = -\lambda V_{l,e}(w) + V_{l-1,e}(0)v(w) + \sum_{k=1}^{e} V_{l,e-k}(w)\lambda g_k, \; l \ge 2, 1 \le e \le N-1, \tag{3.35}$$

$$-V_{l,e}^{'}(w) = -\lambda V_{l,e}(w) + \sum_{k=1}^{e} V_{l,e-k}(w)\lambda g_k, \; l \ge 2, \; e \ge N. \tag{3.36}$$

Applying LST on both sides of equations (3.21) to (3.36), we get,

$$\tau\tilde{M}_{i,0}(\tau) - M_{i,0}(0) = \lambda\tilde{M}_{i,0}(\tau) - \sum_{r=a}^{b} M_{r,i}(0)\tilde{S}_u(\tau) - \eta\int_0^\infty L_{i,0}(y)\; dy\tilde{S}_u(\tau), \quad a \le i \le b, \tag{3.37}$$

$$\tau\tilde{M}_{i,e}(\tau) - M_{i,e}(0) = \lambda\tilde{M}_{i,e}(\tau) - \eta\int_0^\infty L_{i,e}(y)\; dy\tilde{S}_u(\tau) - \sum_{k=1}^{e} \tilde{M}_{i,e-k}(\tau)\lambda g_k, \quad e \ge 1, \; a \le i \le b-1, \tag{3.38}$$

$$\begin{aligned}\tau\tilde{M}_{b,e}(\tau) - M_{b,e}(0) &= \lambda\tilde{M}_{b,e}(\tau) - \eta\int_0^\infty L_{b,e}(y)\; dy\tilde{S}_u(\tau) - \sum_{r=a}^{b} M_{r,b+e}(0)\tilde{S}_u(\tau) \\ &\quad - \sum_{k=1}^{e} \tilde{M}_{b,e-k}(\tau)\lambda g_k, \; 1 \le e \le N-b-1,\end{aligned} \tag{3.39}$$

$$\begin{aligned}\tau\tilde{M}_{b,e}(\tau) - M_{b,e}(0) &= \lambda\tilde{M}_{b,e}(\tau) - \eta\int_0^\infty L_{b,e}(y)\; dy\tilde{S}_u(\tau) - \sum_{k=1}^{e} \tilde{M}_{b,e-k}(\tau)\lambda g_k \\ &\quad - \sum_{r=a}^{b} M_{r,b+e}(0)\tilde{S}_u(\tau) - q\sum_{l=1}^{\infty} V_{l,b+e}(0)\tilde{S}_u(\tau), \; e \ge N-b,\end{aligned} \tag{3.40}$$

$$\tau\tilde{L}_{i,0}(\tau) - L_{i,0}(0) = (\lambda+\eta)\tilde{L}_{i,0}(\tau) - \sum_{r=a}^{b} L_{r,i}(0)\tilde{S}_v(\tau) - \sum_{k=0}^{a-1} S_k\lambda g_{i-k}\tilde{S}_v(\tau), \quad a \le i \le b, \tag{3.41}$$

$$\tau\tilde{L}_{i,e}(\tau) - L_{i,e}(0) = (\lambda+\eta)\tilde{L}_{i,e}(\tau) - \sum_{k=1}^{e}\tilde{L}_{i,e-k}(\tau)\lambda g_k,\ e \geq 1,\ a \leq i \leq b-1, \tag{3.42}$$

$$\tau\tilde{L}_{b,e}(\tau) - L_{b,e}(0) = (\lambda+\eta)\tilde{L}_{b,e}(\tau) - \sum_{r=a}^{b} L_{r,b+e}(0)\tilde{S}_v(\tau) - \sum_{k=1}^{e}\tilde{L}_{b,e-k}(\tau)\lambda g_k - \sum_{k=0}^{a-1} S_k\lambda g_{b+e-k}\tilde{S}_v(\tau),\ 1 \leq e \leq N-b-1, \tag{3.43}$$

$$\tau\tilde{L}_{b,e}(\tau) - L_{b,e}(0) = (\lambda+\eta)\tilde{L}_{b,e}(\tau) - \sum_{r=a}^{b} L_{r,b+e}(0)\tilde{S}_v(\tau) - \sum_{k=1}^{e}\tilde{L}_{b,e-k}(\tau)\lambda g_k - \sum_{k=0}^{a-1} S_k\lambda g_{b+e-k}\tilde{S}_v(\tau) - p\sum_{l=1}^{\infty} V_{l,b+e}(0)\tilde{S}_v(\tau),\ e \geq N-b, \tag{3.44}$$

$$\tau\tilde{C}_0(\tau) - C_0(0) = \lambda\tilde{C}_0(\tau) - \sum_{r=a}^{b} M_{r,0}(0)\tilde{C}(\tau) - \eta S_0\tilde{C}(\tau), \tag{3.45}$$

$$\tau\tilde{C}_e(\tau) - C_e(0) = \lambda\tilde{C}_e(\tau) - \sum_{r=a}^{b} M_{r,e}(0)\tilde{C}(\tau) - \sum_{k=1}^{e}\tilde{C}_{e-k}(\tau)\lambda g_k - \eta S_e\tilde{C}(\tau),\ 1 \leq e \leq a-1, \tag{3.46}$$

$$\tau\tilde{C}_e(\tau) - C_e(0) = \lambda\tilde{C}_e(\tau) - \sum_{k=1}^{e}\tilde{C}_{e-k}(\tau)\lambda g_k,\ e \geq a, \tag{3.47}$$

$$\tau\tilde{V}_{1,0}(\tau) - V_{1,0}(0) = \lambda\tilde{V}_{1,0}(\tau) - C_0(0)\tilde{V}(\tau), \tag{3.48}$$

$$\tau\tilde{V}_{1,e}(\tau) - V_{1,e}(0) = \lambda\tilde{V}_{1,e}(\tau) - C_e(0)\tilde{V}(\tau) - \sum_{k=1}^{e}\tilde{V}_{1,e-k}(\tau)\lambda g_k,\ e \geq 1, \tag{3.49}$$

$$\tau\tilde{V}_{l,0}(\tau) - V_{l,0}(0) = \lambda\tilde{V}_{l,0}(\tau) - V_{l-1,0}(0)\tilde{V}(\tau),\ l \geq 2, \tag{3.50}$$

$$\tau\tilde{V}_{l,e}(\tau) - V_{l,e}(0) = \lambda\tilde{V}_{l,e}(\tau) - V_{l-1,e}(0)\tilde{V}(\tau) - \sum_{k=1}^{e}\tilde{V}_{l,e-k}(\tau)\lambda g_k,\ l \geq 2,\ 1 \leq e \leq N-1, \tag{3.51}$$

$$\tau\tilde{V}_{l,e}(\tau) - V_{l,e}(0) = \lambda\tilde{V}_{l,e}(\tau) - \sum_{k=1}^{e}\tilde{V}_{l,e-k}(\tau)\lambda g_k,\ l \geq 2,\ e \geq N. \tag{3.52}$$

The PGF of queue size is obtained by defining the following:

$$\tilde{M}_i(z,\tau) = \sum_{e=0}^{\infty}\tilde{M}_{i,e}(\tau)z^e,\ M_i(z,0) = \sum_{e=0}^{\infty} M_{i,e}(0)z^e,\ a \leq i \leq b,$$

$$\tilde{L}_i(z,\tau) = \sum_{e=0}^{\infty}\tilde{L}_{i,e}(\tau)z^e,\ L_i(z,0) = \sum_{e=0}^{\infty} L_{i,e}(0)z^e,\ a \leq i \leq b,$$

$$\tilde{C}(z,\tau)=\sum_{e=0}^{\infty}\tilde{C}_e(\tau)z^e,\ C(z,0)=\sum_{e=0}^{\infty}C_e(0)z^e,$$
$$\tilde{V}_l(z,\tau)=\sum_{e=0}^{\infty}\tilde{V}_{l,e}(\tau)z^e\ V_l(z,0)=\sum_{e=0}^{\infty}V_{l,e}(0)z^e,\ l\geq 1. \tag{3.53}$$

Equations (3.37) to (3.52) are multiplied by z^n, summing over n ($n=0$ to ∞), and using equation (3.53), we get,

$$(\tau-f(z))\tilde{M}_i(z,\tau)=M_i(z,0)-\tilde{S}_u(\tau)\Big[\sum_{r=a}^{b}M_{r,i}(0)+\eta\tilde{L}_i(z,0)\Big],\ a\leq i\leq b-1, \tag{3.54}$$

$$\begin{aligned}z^b(\tau-f(z))\tilde{M}_b(z,\tau)&=(z^b-\tilde{S}_u(\tau))M_b(z,0)-\tilde{S}_u(\tau)\Big[z^b\eta\tilde{L}_b(z,0)\\&+\sum_{r=a}^{b-1}M_r(z,0)-\sum_{e=0}^{b-1}\sum_{r=a}^{b}M_{r,e}(0)z^e\\&+q\sum_{l=1}^{\infty}\Big(V_l(z,0)-\sum_{n=0}^{N-1}V_{l,n}(0)z^n\Big)\Big],\end{aligned} \tag{3.55}$$

$$(\tau-g(z))\tilde{L}_i(z,\tau)=L_i(z,0)-\tilde{S}_v(\tau)\Big[\sum_{r=a}^{b}L_{r,i}(0)+\sum_{k=0}^{a-1}S_k\lambda g_{i-k}\Big],\ a\leq i\leq b-1, \tag{3.56}$$

$$\begin{aligned}z^b(\tau-g(z))\tilde{L}_b(z,\tau)&=L_b(z,0)(z^b-\tilde{S}_v(\tau))-\tilde{S}_v(\tau)\Big[\sum_{r=a}^{b-1}L_r(z,0)\\&-\sum_{e=0}^{b-1}\sum_{r=a}^{b}L_{r,e}(0)z^e+\lambda\sum_{k=0}^{a-1}S_kz^k\sum_{e=b}^{\infty}g_{e-k}z^{e-k}\\&+p\sum_{l=1}^{\infty}\Big(V_l(z,0)-\sum_{n=0}^{N-1}V_{l,n}(0)z^n\Big)\Big],\end{aligned} \tag{3.57}$$

$$(\tau-f(z))\tilde{C}(z,\tau)=C(z,0)-\tilde{C}(\tau)\Big(\sum_{n=0}^{a-1}\sum_{r=a}^{b}M_{r,n}(0)z^n+\eta\sum_{n=0}^{a-1}S_nz^n\Big), \tag{3.58}$$

$$(\tau-f(z))\tilde{V}_1(z,\tau)=V_1(z,0)-\tilde{V}(\tau)C(z,0), \tag{3.59}$$

$$(\tau-f(z))\tilde{V}_l(z,\tau)=V_l(z,0)-\tilde{V}(\tau)\sum_{n=0}^{N-1}V_{l-1,n}(0)z^n,\ l\geq 2, \tag{3.60}$$

where $f(z)=(1-X(z))\lambda,\ g(z)=\eta+(1-X(z))\lambda.$

Substitute $\tau = f(z)$ in (3.54) and (3.55), we get,

$$M_i(z,0) = \tilde{S}_u(f(z))\Big[\sum_{r=a}^{b} M_{r,i}(0) + \eta\tilde{L}_i(z,0)\Big],\ a \leq i \leq b-1, \tag{3.61}$$

$$M_b(z,0) = \frac{\tilde{S}_u(f(z))}{(z^b - \tilde{S}_u(f(z)))}\Big[z^b\eta\tilde{L}_b(z,0) + \sum_{r=a}^{b-1} M_r(z,0) - \sum_{e=0}^{b-1}\sum_{r=a}^{b} M_{r,e}(0)z^e + q\sum_{l=1}^{\infty}\Big(V_l(z,0) - \sum_{n=0}^{N-1} V_{l,n}(0)z^n\Big)\Big]. \tag{3.62}$$

Substitute $\tau = g(z)$ in (3.56) and (3.57), we get,

$$L_i(z,0) = \tilde{S}_v(g(z))\Big[\sum_{r=a}^{b} L_{r,i}(0) + \sum_{k=0}^{a-1} S_k\lambda g_{i-k}\Big],\ a \leq i \leq b-1, \tag{3.63}$$

$$L_b(z,0) = \frac{\tilde{S}_v(g(z))}{(z^b - \tilde{S}_v(g(z)))}\Big[\lambda\sum_{k=0}^{a-1} S_k z^k \sum_{e=b}^{\infty} g_{e-k}z^{e-k} + \sum_{r=a}^{b-1} L_r(z,0) - \sum_{e=0}^{b-1}\sum_{r=a}^{b} L_{r,e}(0)z^e + p\sum_{l=1}^{\infty}\Big(V_l(z,0) - \sum_{n=0}^{N-1} V_{l,n}(0)z^n\Big)\Big]. \tag{3.64}$$

Substitute $\tau = f(z)$ in (3.58) to (3.60), we get,

$$C(z,0) = \tilde{C}(f(z))\Big(\sum_{n=0}^{a-1}\sum_{r=a}^{b} M_{r,n}(0)z^n + \eta\sum_{n=0}^{a-1} S_n z^n\Big), \tag{3.65}$$

$$V_1(z,0) = \tilde{V}(f(z))C(z,0), \tag{3.66}$$

$$V_l(z,0) = \tilde{V}(f(z))\sum_{n=0}^{N-1} V_{l-1,n}(0)z^n,\ l \geq 2. \tag{3.67}$$

Substitute equations (3.61) to (3.67) in (3.54) to (3.60), we get,

$$\tilde{M}_i(z,\tau) = \frac{(\tilde{S}_u(f(z)) - \tilde{S}_u(\tau))}{(\tau - f(z))}\Big[\sum_{r=a}^{b} M_{r,i}(0) + \eta\tilde{L}_i(z,0)\Big],\ a \leq i \leq b-1, \tag{3.68}$$

$$\tilde{M}_b(z,\tau) = \frac{(\tilde{S}_u(f(z)) - \tilde{S}_u(\tau))}{(\tau - f(z))(z^b - \tilde{S}_u(f(z)))}\Big[z^b\eta\tilde{L}_b(z,0) + \sum_{r=a}^{b-1} M_r(z,0) - \sum_{e=0}^{b-1}\sum_{r=a}^{b} M_{r,e}(0)z^e + q\sum_{l=1}^{\infty}\Big(V_l(z,0) - \sum_{n=0}^{N-1} V_{l,n}(0)z^n\Big)\Big], \tag{3.69}$$

$$\tilde{L}_i(z,\tau) = \frac{(\tilde{S}_v(g(z)) - \tilde{S}_v(\tau))}{(\tau - g(z))}\Big[\sum_{r=a}^{b} L_{r,i}(0) + \sum_{k=0}^{a-1} S_k\lambda g_{i-k}\Big],\ a \leq i \leq b-1, \tag{3.70}$$

$$\tilde{L}_b(z,\tau) = \frac{(\tilde{S}_v(g(z)) - \tilde{S}_v(\tau))}{(\tau - g(z))(z^b - \tilde{S}_v(g(z)))}\Big[\sum_{r=a}^{b-1} L_r(z,0) - \sum_{e=0}^{b-1}\sum_{r=a}^{b} L_{r,e}(0)z^e + \lambda\sum_{k=0}^{a-1} S_k z^k \sum_{e=b}^{\infty} g_{e-k} z^{e-k} + p\sum_{l=1}^{\infty}\Big(V_l(z,0) - \sum_{n=0}^{N-1} V_{l,n}(0)z^n\Big)\Big], \quad (3.71)$$

$$\tilde{C}(z,\tau) = \frac{(\tilde{C}(f(z)) - \tilde{C}(\tau))\Big(\sum_{e=0}^{a-1}\sum_{r=a}^{b} M_{r,e}(0)z^e + \eta\sum_{e=0}^{a-1} S_e z^e\Big)}{(\tau - f(z))}, \quad (3.72)$$

$$\tilde{V}_1(z,\tau) = \frac{(\tilde{V}(f(z)) - \tilde{V}(\tau))C(z,0)}{(\tau - f(z))}, \quad (3.73)$$

$$\tilde{V}_l(z,\tau) = \frac{(\tilde{V}(f(z)) - \tilde{V}(\tau))\sum_{e=0}^{N-1} V_{l-1,e}(0)z^e}{(\tau - f(z))},\ l \geq 2. \quad (3.74)$$

4. Probability Generating Function of Queue Size

4.1. PGF of Queue Size at an Arbitrary Time Epoch

The PGF of the queue size at an arbitrary time epoch is obtained as

$$P(z) = \sum_{i=a}^{b} \tilde{M}_i(z,0) + \sum_{i=a}^{b} \tilde{L}_i(z,0) + \tilde{C}(z,0) + \sum_{l=1}^{\infty} \tilde{V}_l(z,0) + S(z). \quad (4.1)$$

By substituting $\tau = 0$ in the equations (3.68) to (3.74), then the equation (4.1) becomes

$$P(z) = \frac{\begin{aligned}\Big[&A_1(z)\sum_{i=a}^{b-1}(z^b - z^i)m_i + A_2(z)\sum_{i=a}^{b-1}(z^b - z^i)(q_i + \sum_{k=0}^{a-1} S_k\lambda g_{i-k}) \\ &+ (A_3(z) - A_1(z))\sum_{n=0}^{a-1} m_n z^n + (\eta A_3(z) - g(z)A_2(z))\sum_{n=0}^{a-1} S_n z^n \\ &+ (1 - \tilde{V}(f(z)))[g(z)A_4(z) - qA_1(z) - pA_2(z)]\sum_{n=0}^{N-1} v_n z^n\Big]\end{aligned}}{f(z)g(z)A_4(z)} \quad (4.2)$$

where $m_i = \sum_{r=a}^{b} M_{r,i}(0),\ v_i = \sum_{l=1}^{\infty} V_{l,i}(0),\ q_i = \sum_{r=a}^{b} L_{r,i}(0)$ and the expressions for $A_1(z)$, $A_2(z)$, $A_3(z)$, and $A_4(z)$ are defined in Appendix-I.

4.2. Steady State Condition

Any PGF of queue size should satisfy $P(1) = 1$. In order to satisfy this condition applying "L' Hopital's rule" and evaluating $\lim_{z\to 1} P(z)$, then equating the expression to 1, we have, $H = K_1$. Since m_i, q_i, v_i and S_i are probabilities of 'i' clients being in the queue, it follows that H must be positive. Thus $P(1) = 1$ is satisfied iff $K_1 > 0$. If

$$\rho = \frac{\lambda E(X)E(S)}{b} < 1$$

is the condition for the existence of steady state for the model under consideration. Equation (4.2) has '$2b+N$' unknowns $m_0, m_1, ..., m_{b-1}, q_a, ..., q_{b-1}$, S_0, S_1, ..., S_{a-1} and $v_0, v_1, ..., v_{N-1}$. We can express v_i interms of m_i and S_i in such a way that equation (76) has only '$2b$' unknowns. By Rouche's theorem, "Rouche's theorem is a direct consequence of the argument principle and a powerful tool for determining regions of the complex plane in which there may be zeros of a given analytic function. The scope of application of Rouche's theorem goes well beyond the field of queueing theory. While the verification of the conditions needed to apply Rouche's theorem can become rather difficult, in queueing theory this is usually straightforward. For most queueing applications, the region of interest is typically the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, and the ingredient that makes Rouche's theorem work is oftentimes the stability condition. This is why Rouche's theorem is a popular and standardized tool in queueing theory. However, the standard way in which Rouche's theorem is applied requires the analytic continuation of the function of interest outside the unit disk. This can be done for many functions, but definitely not for all." it can be proved that "$A_4(z)$ has $2b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives $2b$ equations in $2b$ unknowns". We can solve these equations by any suitable numerical technique.

4.3. Result

The probability v_i ("the regular server's vacation completion epoch, there are 'i' ($0 \leq i \leq N-1$) clients in the queue"), can be expressed as sum of the probabilities of 'i' ($0 \leq i \leq a-1$) S_i ("clients in the queue during stand-by server's idle") and m_i("regular server's busy period").

Case 1: For n=0, 1, 2,..., a-1

$$v_n = \sum_{i=0}^{n} K_i(m_{n-i} + \eta S_{n-i}), n = 0, 1, 2, ..., a-1, \text{where} \tag{4.3}$$

$$K_n = \frac{\psi_n + \sum_{i=1}^{n} \gamma_i K_{n-i}}{1-\gamma_0}, n = 1, 2, ..., a-1, \text{with } K_0 = \frac{\gamma_0 \beta_0}{1-\gamma_0}, \psi_n = \sum_{i=0}^{n} \gamma_i \beta_{n-i}$$

where the probabilities of the 'i' clients arrive during vacation (γ_i's) and closedown time (β_i's).

Case 2: For n = a, a+1,..., N-1

$$v_n = \sum_{k=0}^{a-1} (m_k + \eta S_k)(\sum_{i=0}^{n-k} \gamma_i \beta_{n-k-i}) + \sum_{k=0}^{n-1} \gamma_{n-k} q_k,\ n = a, a+1, ..., N-1. \tag{4.4}$$

4.4. Particular Case

Case 1: By assuming no failures, closedown and N-policy the equation (4.2) becomes

$$P(z) = \frac{\left[(1-\tilde{S}_u(f(z))) \sum_{i=a}^{b-1}(z^b - z^i)m_i + (z^b-1)(1-\tilde{V}(f(z))) \sum_{n=0}^{a-1}(c_n + v_n)z^n\right]}{f(z)(z^b - \tilde{S}_u(f(z)))} \tag{4.5}$$

which coincide with results of Haridass et al.[16].
Case 2: By assuming no failures and N-policy the equation (4.2) becomes

$$P(z) = \frac{\Big[(1-\tilde{S}_u(f(z)))\sum_{i=a}^{b-1}(z^b-z^i)m_i + (z^b-1)(1-\tilde{V}(f(z)))\sum_{n=0}^{a-1}v_n z^n + (z^b-1)(1-\tilde{V}(f(z))\tilde{C}(f(z)))\sum_{n=0}^{a-1}m_n z^n\Big]}{f(z)(z^b-\tilde{S}_u(f(z)))}, \tag{4.6}$$

which coincide with results of Jeyakumar et al.[17].

5. PGF of Queue Size at Various Epochs

5.1. Main Server's Service:

$$N(z) = \frac{\begin{aligned}&\Big[g(z)(z^b-\tilde{S}_v(g(z)))\Big(\sum_{i=a}^{b-1}(z^b-z^i)m_i - \sum_{n=0}^{a-1}m_n z^n\Big)\\ &+ z^b\eta(1-\tilde{S}_v(g(z)))\Big(\sum_{i=a}^{b-1}(z^b-z^i)[q_i+\sum_{k=0}^{a-1}S_k\lambda g_{i-k}] - g(z)S(z)\Big)\\ &+ [pz^b\eta(1-\tilde{S}_v(g(z))) + qg(z)(z^b-\tilde{S}_v(g(z)))]\times\\ &\Big[\tilde{V}(f(z))\tilde{C}(f(z))[\sum_{n=0}^{a-1}m_n z^n + \eta S(z)] + (\tilde{V}(f(z))-1)\sum_{n=0}^{N-1}v_n z^n\Big]\Big]\end{aligned}}{f(z)g(z)(z^b-\tilde{S}_u(f(z)))(z^b-\tilde{S}_v(g(z)))}. \tag{5.1}$$

5.2. Stand-by Server's Service:

$$B(z) = \frac{\begin{aligned}&\Big[(1-\tilde{S}_v(g(z)))\Big[\Big(\sum_{i=a}^{b-1}(z^b-z^i)[q_i+\sum_{k=0}^{a-1}S_k\lambda g_{i-k}] - g(z)S(z)\Big) + p\\ &\Big(\tilde{V}(f(z))\tilde{C}(f(z))[\sum_{n=0}^{a-1}m_n z^n + \eta S(z)] + (\tilde{V}(f(z))-1)\sum_{n=0}^{N-1}v_n z^n\Big)\Big]\Big]\end{aligned}}{g(z)(z^b-\tilde{S}_v(g(z)))}. \tag{5.2}$$

5.3. Vacation:

$$V(z) = \frac{\Big[(1-\tilde{V}(f(z)))\Big[\tilde{C}(f(z))\sum_{n=0}^{a-1}[m_n z^n + \eta S(z)] + \sum_{n=0}^{N-1}v_n z^n\Big]\Big]}{f(z)}. \tag{5.3}$$

5.4. Closedown:

$$C(z) = \frac{\Big[(1-\tilde{C}(f(z)))\Big(\sum_{n=0}^{a-1}\sum_{r=a}^{b} P_{r,n}(0)z^n + \eta\sum_{n=0}^{a-1} S_n z^n\Big)\Big]}{f(z)}. \tag{5.4}$$

6. Some Performance Measures

6.1. Expected Queue Length

The mean queue length E(Q) at an arbitrary time epoch is given by

$$E(Q) = \frac{\begin{aligned}&\Big[f_1(S_u,S_v)\Big[\sum_{i=a}^{b-1}[b(b-1)-i(i-1)](m_i+q_i+\sum_{k=0}^{a-1}S_k\lambda g_{i-k})\\ &+f_2(X,S_u,S_v)\sum_{i=a}^{b-1}(b-i)m_i+f_3(X,S_u,S_v)\sum_{i=a}^{b-1}(b-i)(q_i+\sum_{k=0}^{a-1}S_k\lambda g_{i-k})\\ &+f_4(X,S_u,S_v,V,C)\sum_{n=0}^{a-1}m_n+f_5(X,S_u,S_v,V,C)\sum_{n=0}^{a-1}S_n\\ &+f_6(X,S_u,S_v,V)\sum_{n=0}^{N-1}v_n+f_7(X,S_u,S_v,V,C)\sum_{n=0}^{a-1}nm_n\\ &+f_8(X,S_u,S_v,V,C)\sum_{n=0}^{a-1}nS_n+f_9(X,S_u,S_v,V)\sum_{n=0}^{N-1}nv_n\Big]\end{aligned}}{3(J_1)^2}, \tag{6.1}$$

the expressions for f_i' s are defined in Appendix-II.

6.2. Main Server's Expected Length of Idle Period-E(I)

E(I) = $E(I_1)$ + E(C) where I_1 is the idle period random variable due to multiple vacation process and E(C) is the mean length of closedown time. Define a random variable B as,

$$B = \begin{cases} 0 & \text{after the first vacation if the server finds atleast `}N\text{' clients} \\ 1 & \text{after the first vacation if the server finds less than `}N\text{' clients} \end{cases}$$

Now

$$\begin{aligned} E(I_1) &= E(I_1/(B=0))P(B=0) + E(I_1/(B=1))P(B=1) \\ &= E(V)P(B=0) + (E(V)+E(I_1))P(B=1). \end{aligned}$$

$$E(I) = \frac{E(V)}{P(B=0)} + E(C). \tag{6.2}$$

where

$$P(B=0)=\sum_{n=0}^{a-1}\sum_{i=0}^{n}\sum_{j=0}^{n-i}\gamma_j\beta_{n-i-j}[m_i+\eta S_i]+\sum_{n=a}^{N-1}\sum_{i=0}^{a-1}\sum_{j=0}^{n-i}\gamma_j\beta_{n-i-j}[m_i+\eta S_i] \quad (6.3)$$

6.3. Expected Waiting Time-E(W)

It is obtained using the "Little's formula"

$$E(W)=E(Q)/\lambda E(X) \quad (6.4)$$

7. Numerical Example

This section deals with the numerical illustration of the proposed queueing model through variations in the parameters using MATLAB software. We consider vacation, closedown time of regular server follows exponential distribution, the service time of both servers to follow Erlang-2 distribution in Table 1,3 and both servers to follow Exponential distribution in Table 2,4 and 5.

Let us consider the service rate for regular sever as μ_1 and that of stand-by server as μ_2. Vacation and closedown rate of regular server be γ and ζ respectively.
The results have been analysed in tabular forms and two dimensional graphs. The arbitrary chosen values satisfy the stability condition.

Let $a=5, b=8, N=10,\ \mu_1=10, \mu_2=7, \alpha=1, \eta=2,\ \gamma=10$ and $\zeta=8$.

λ	ρ	$E(Q)$	$E(W)$	$E(I)$
5.0	0.1250	19.6328	1.96328	3.50110
5.1	0.1275	20.7612	2.03541	3.45603
5.2	0.1300	21.9238	2.10806	3.41111
5.3	0.1325	23.1149	2.18065	3.36660
5.4	0.1350	24.3408	2.25378	3.32233
5.5	0.1375	25.6028	2.32753	3.27833
5.6	0.1400	26.6977	2.38372	3.23472
5.7	0.1425	27.8108	2.43954	3.19122
5.8	0.1450	28.9427	2.49506	3.14777
5.9	0.1475	30.0941	2.55034	3.10507
6.0	0.1500	31.2654	2.60545	3.06227
6.1	0.1525	32.4571	2.66042	3.02019
6.2	0.1550	33.6700	2.71532	2.97817
6.3	0.1575	34.9043	2.77018	2.93625
6.4	0.1600	36.1606	2.82505	2.89485
6.5	0.1625	37.4395	2.87996	2.85351

Table 1. Arrival rate (vs) Performance measures

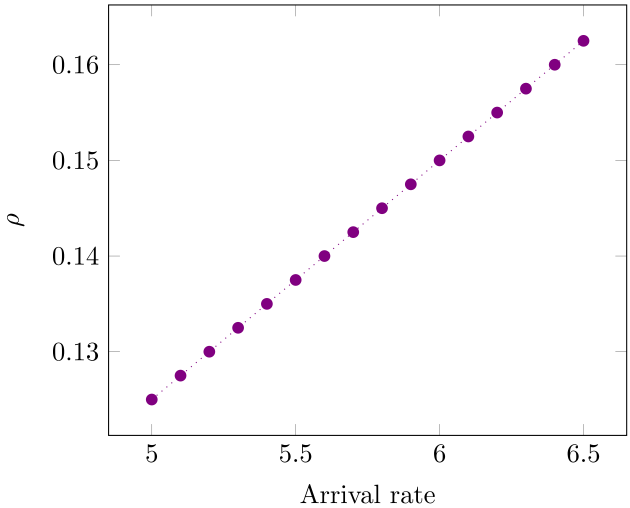

FIGURE 2. Arrival rate (vs) ρ

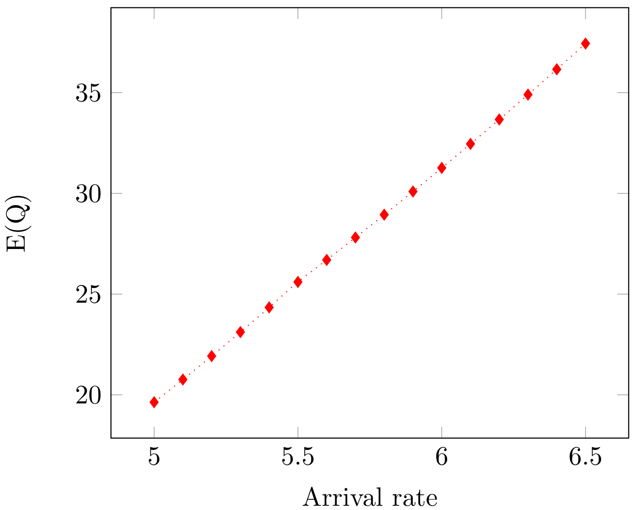

FIGURE 3. Arrival rate (vs) E(Q)

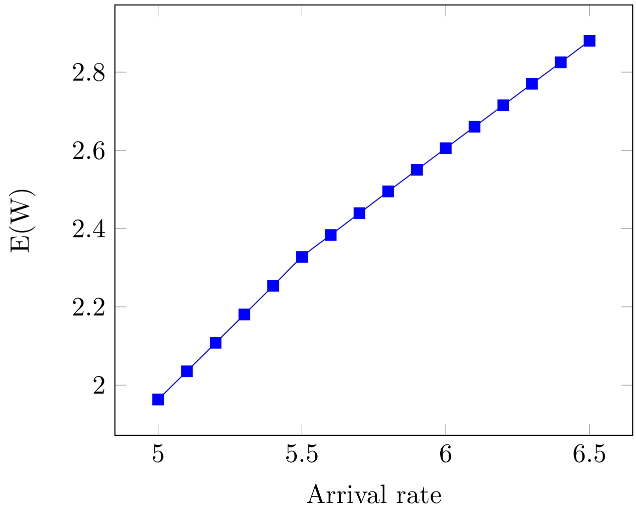

Figure 4. Arrival rate (vs) E(W)

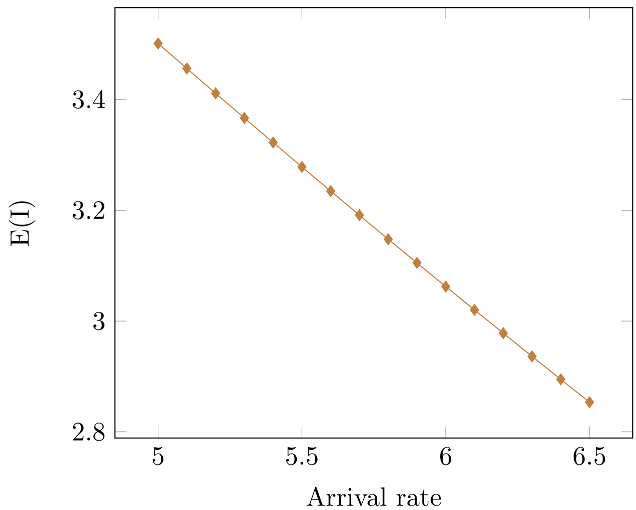

Figure 5. Arrival rate (vs) E(I)

Take $a = 5,\ b = 8,\ N = 10,\ \mu_1 = 15,\ \mu_2 = 12,\ \alpha = 1,\ \eta = 2,\ \gamma = 4$ and $\zeta = 3$

λ	ρ	$E(Q)$	$E(W)$	$E(I)$
5.0	0.0833	203.514	20.3514	21.3026
5.5	0.0917	235.484	21.4076	19.8302
6.0	0.1000	270.561	22.5467	18.4244
6.5	0.1083	308.939	23.7645	17.0869
7.0	0.1167	350.847	25.0605	15.8113
7.5	0.1250	396.461	26.4307	14.6014
8.0	0.1333	446.088	27.8805	13.4478
8.5	0.1417	499.588	29.3875	12.3518
9.0	0.1500	557.366	30.9648	11.3112
9.5	0.1583	619.391	32.5995	10.3250
10.0	0.1667	685.986	34.2993	9.39074

TABLE 2. Arrival rate (vs) Performance measures

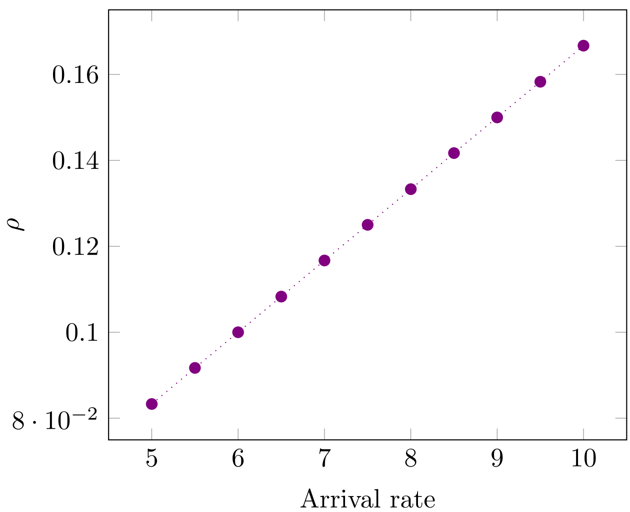

FIGURE 6. Arrival rate (vs) ρ

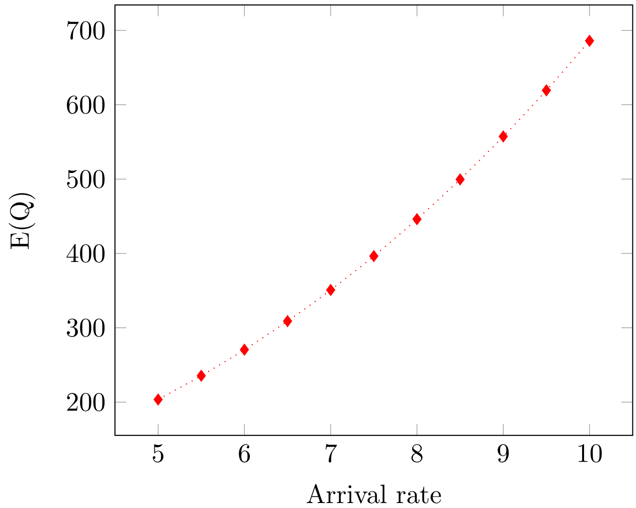

FIGURE 7. Arrival rate (vs) E(Q)

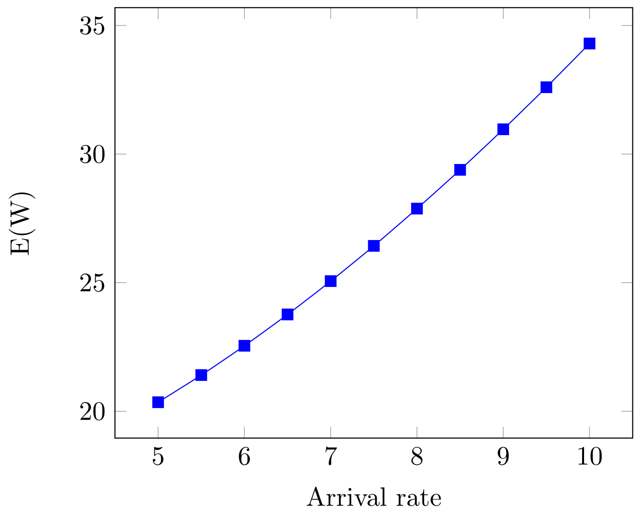

FIGURE 8. Arrival rate (vs) E(W)

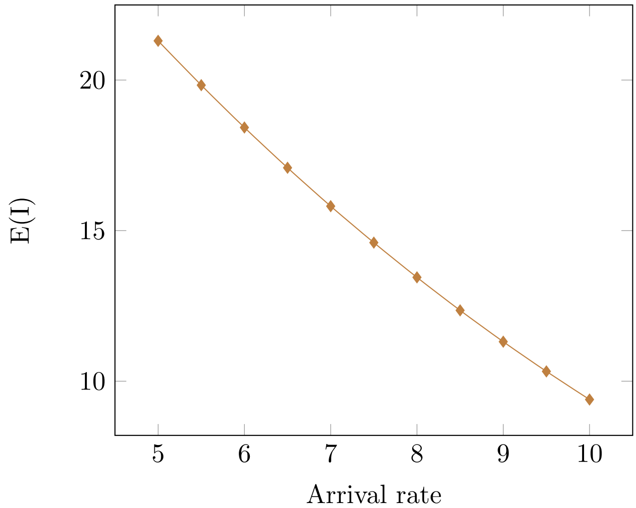

FIGURE 9. Arrival rate (vs) E(I)

The effect of increasing arrival rate λ are shown in Table 1,2 and Figure 2-9 Thus, if the λ increases, then ρ, E(Q) and E(W) are increasing and expected idle E(I) is decreasing.

Take $a = 5,\ b = 8,\ N = 10,\ \lambda = 10,\ \mu_2 = 11,\ \alpha = 1,\ \eta = 2,\ \gamma = 4$ and $\zeta = 3$

μ_1	ρ	$E(Q)$	$E(W)$	$E(I)$
12	0.1042	291.096	29.1096	6.8132
13	0.0962	287.604	28.7604	6.8621
14	0.0893	283.641	28.3641	6.9189
15	0.0833	279.199	27.9199	6.9845
16	0.0781	274.086	27.4086	7.0625
17	0.0735	268.143	26.8143	7.1568
18	0.0694	261.332	26.1332	7.2701
19	0.0658	253.163	25.3163	7.4136
20	0.0625	243.417	24.3417	7.5972

TABLE 3. Regular server's service rate (vs) Performance measures

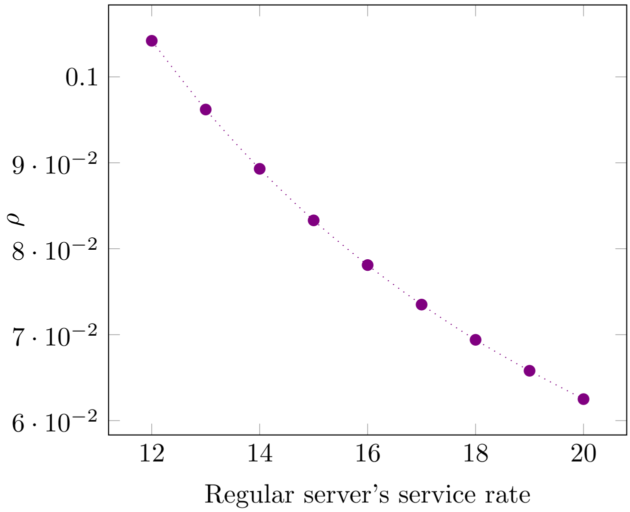

FIGURE 10. Regular server's service rate (vs) ρ

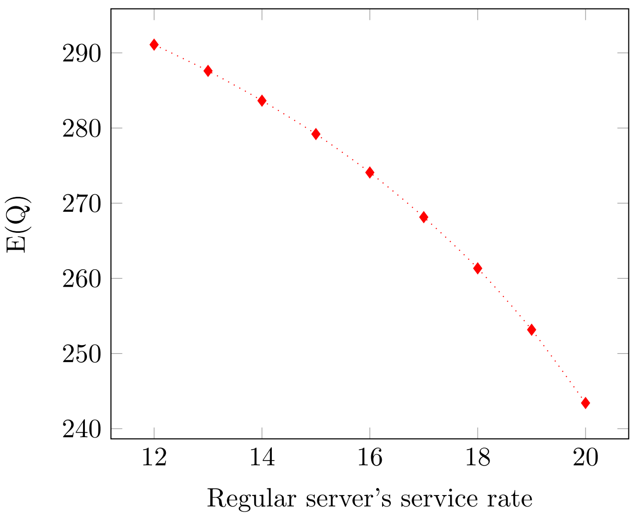

FIGURE 11. Regular server's service rate (vs) E(Q)

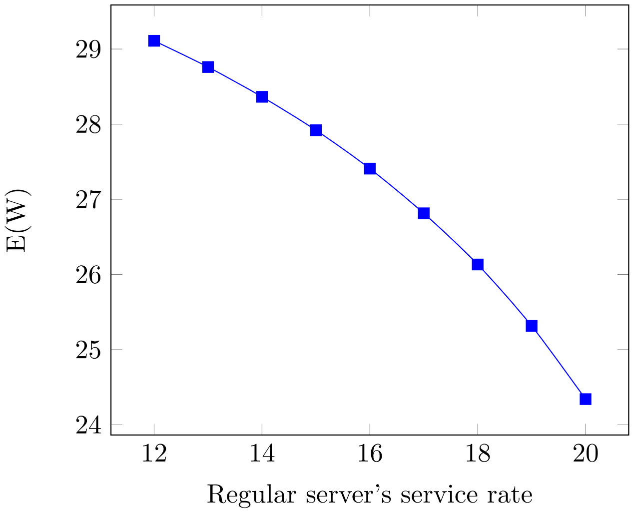

FIGURE 12. Regular server's service rate (vs) E(W)

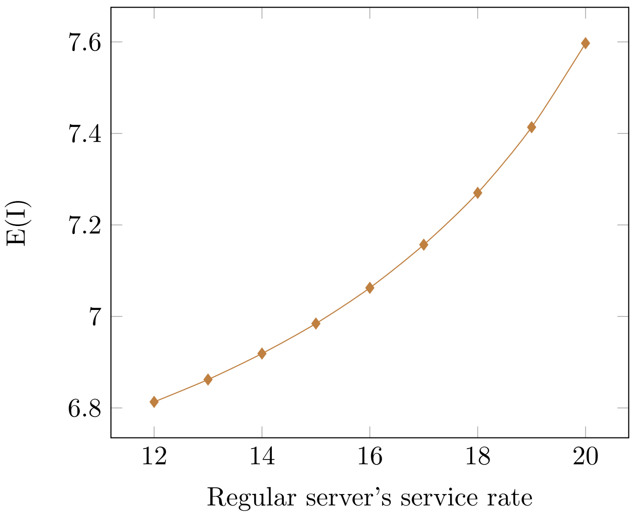

FIGURE 13. Regular server's service rate (vs) E(I)

Take $a = 5,\ b = 8,\ N = 10,\ \lambda = 5,\ \mu_2 = 10,\ \alpha = 1,\ \eta = 2,\ \gamma = 4$ and $\zeta = 3$

μ_1	ρ	$E(Q)$	$E(W)$	$E(I)$
11.0	0.1136	151.618	15.1618	9.47479
11.5	0.1087	151.452	15.1452	9.49037
12.0	0.1042	151.242	15.1242	9.50841
12.5	0.1000	150.973	15.0973	9.52953
13.0	0.0962	150.633	15.0633	9.55433
13.5	0.0926	150.287	15.0287	9.58017
14.0	0.0893	149.823	14.9823	9.61161
14.5	0.0862	149.187	14.9187	9.65056
15.0	0.0833	148.139	14.8139	9.70586

TABLE 4. Regular server's service rate (vs) Performance measures

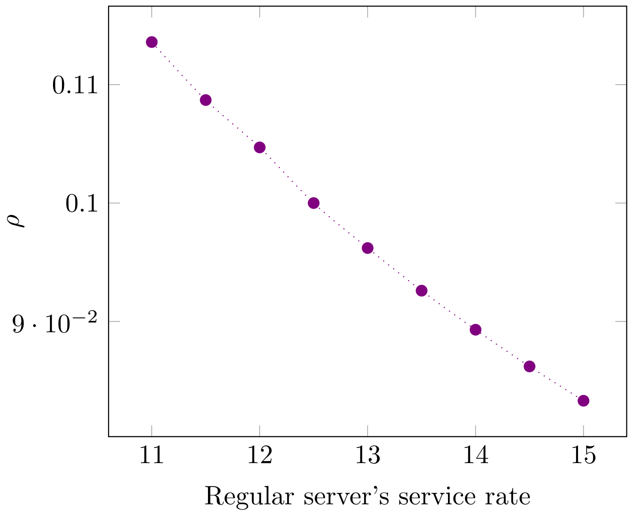

FIGURE 14. Regular server's service rate (vs) ρ

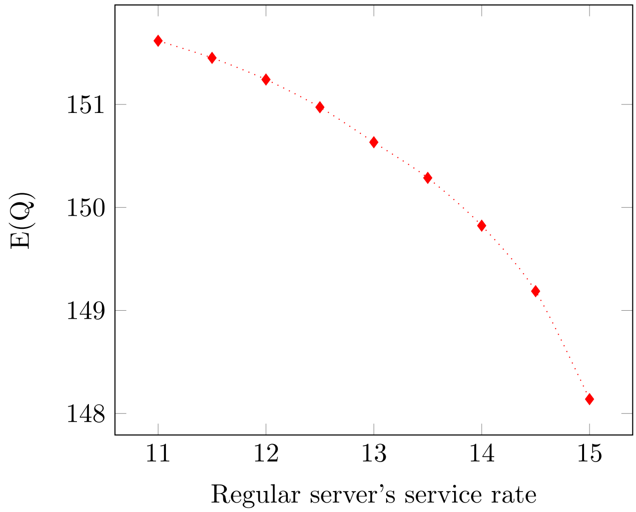

Figure 15. Regular server's service rate (vs) E(Q)

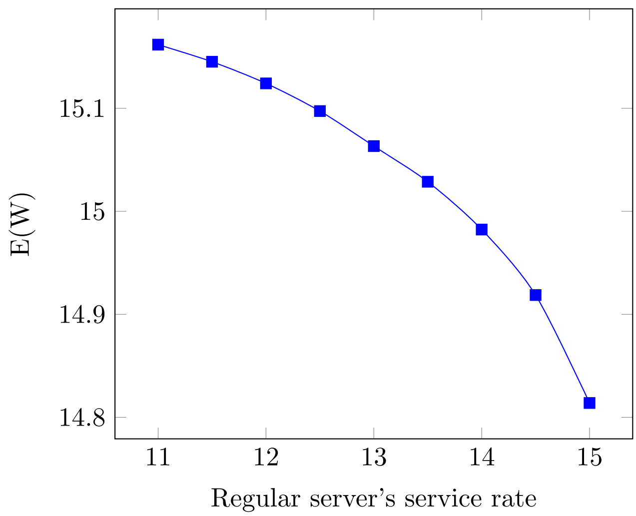

Figure 16. Regular server's service rate (vs) E(W)

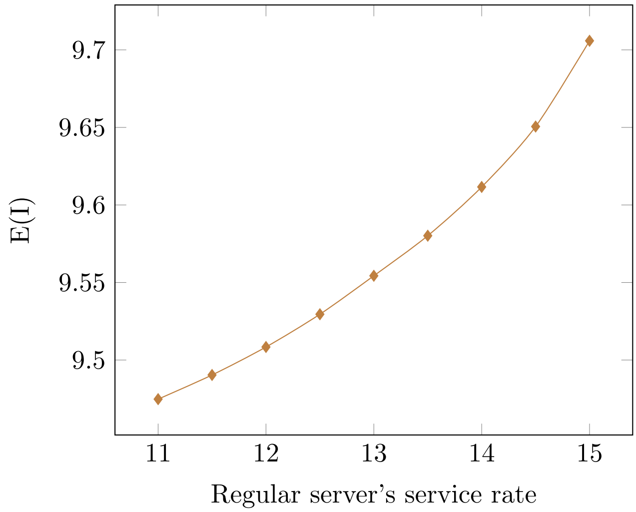

FIGURE 17. Regular server's service rate (vs) E(I)

The effect of increasing regular server's service rate μ_1 are shown in Table 3, 4 and Figure 10-17 Thus, if the μ_1 increases, then ρ, E(Q) and E(W) are decreasing and expected idle E(I) is increasing.

Take $a = 5,\ b = 8,\ N = 10,\ \mu_1 = 12,\ \alpha = 1,\ \eta = 2,\ \gamma = 4$ and $\zeta = 3$

μ_2	$E(Q)$	$E(W)$	$E(I)$
6.0	296.023	29.6023	9.17672
6.5	280.304	28.0304	9.1826
7.0	264.933	26.4933	9.1893
7.5	249.919	24.9919	9.19678
8.0	235.265	23.5265	9.20503
8.5	220.979	22.0979	9.21406
9.0	207.062	20.7062	9.22388
9.5	193.519	19.3519	9.23455
10.0	180.352	18.0352	9.2461

TABLE 5. Stand-by server service rate (vs) Performance measures

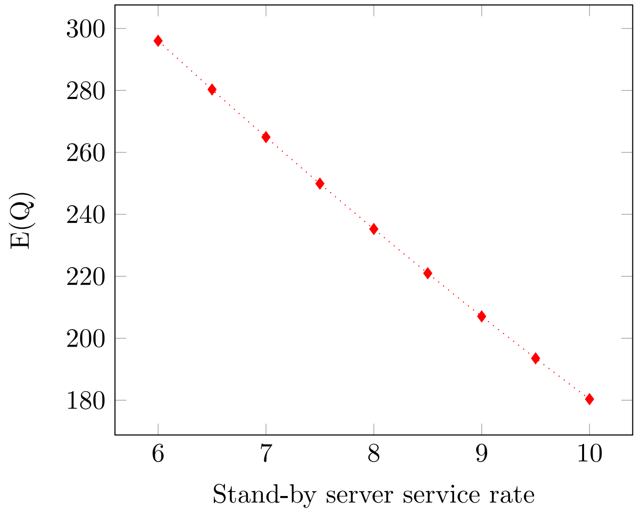

Figure 18. Stand-by server service rate(vs) E(Q)

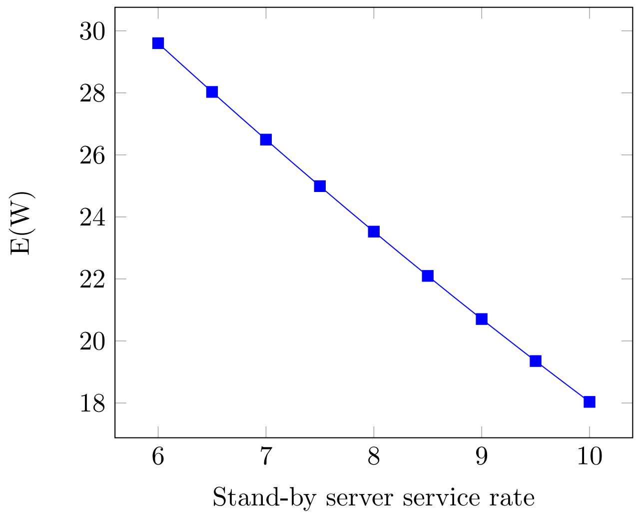

Figure 19. Stand-by server service rate (vs) E(W)

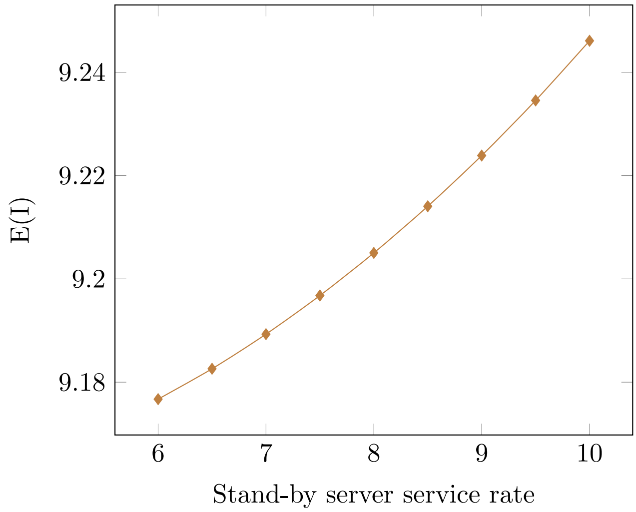

FIGURE 20. Stand-by server service rate (vs) E(I)

The effect of increasing Stand-by server service rate μ_2 are shown in Table 5 and Figure 18-20 Thus, if the μ_2 increases, then E(Q) and E(W) are decreasing and expected idle E(I) is increasing.

8. Conclusion

In this study we have analysed bulk queue system with starting failure, stand-by server, closedown, N-policy and multiple vacation. The steady state solution of the above system during idle, busy and vacation mode were estimated. Further, the important performance measures such as mean waiting time of a client and mean number of clients in the queue are derived. The analytical findings are validated with the help of numerical examples and it can find application in real life situations for example in transportation and networking sectors. The novel contribution in this paper is the incorporation of starting failure and repair, stand-by server, closedown and N-Policy in the case of bulk queuing system which we commonly come across in our real life situtions in a manufacturing industry or transport sector or networking. The results of this paper can be applied in production line, ATMs, computer networks and satellite communication, etc. Further work can be done in this area with additional parameters such as delaying repair, working vacation policies and impatient customers.

9. Appendix

9.1. Appendix-I

The expressions used in equations (4.2) are given below:

$$A_1(z) = g(z)T_2T_3$$
$$A_2(z) = T_4[f(z)T_1 + z^b\eta T_3],$$

$$
\begin{aligned}
A_3(z) &= g(z)T_2[T_1(1-\tilde{V}(f(z))\tilde{C}(f(z))) + q\tilde{V}(f(z))\tilde{C}(f(z))T_3] \\
&\quad + p\tilde{V}(f(z))\tilde{C}(f(z))A_2(z), \\
A_4(z) &= T_1T_2,
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= (z^b - \tilde{S}_u(f(z))),\ T_2 = (z^b - \tilde{S}_v(g(z))), \\
T_3 &= (1 - \tilde{S}_u(f(z))),\ T_4 = (1 - \tilde{S}_v(g(z))).
\end{aligned}
$$

9.2. Appendix-II

The expressions for f_i's in (6.1) are given below:

$$
\begin{aligned}
f_1(S_u,S_v) &= 3Y_1K_1,\ f_2(X,S_u,S_v) = 3(Y_3 - 2b\eta S_{b1})K_1 - 2Y_1K_2, \\
f_3(X,S_u,S_v) &= 3(Y_3 + 2bY_5)K_1 - 2Y_1K_2, \\
f_4(X,S_u,S_v,V,C) &= (Y_6 - (1-q)Y_7 + pY_8)K_1 - 2b(Y_9 + S_{u1})K_2, \\
f_5(X,S_u,S_v,V,C) &= \Big[\eta[Y_6 + qY_7 - (1-p)Y_8] + 3[\lambda X_2Y_1 + \lambda X_1(Y_3 + 2bY_5)]\Big]K_1 \\
&\quad - 2[b\eta(Y_9 - Y_5) + \lambda X_1Y_1]K_2, \\
f_6(X,S_u,S_v,V) &= 3K_1\Big[(\tilde{S}_v(\eta)-1)[V_1(\eta b(b-1) - 2b\lambda X_1) + b\eta V_2] \\
&\quad - 2b[\eta V_1(b - S_{u1} - S_{v1}) + \eta qS_{u1}V_1 - pY_5V_1]\Big], \\
f_7(X,S_u,S_v,V,C) &= 2bK_1(Y_9 + \eta S_{u1}), \\
f_8(X,S_u,S_v,V,C) &= 6K_1[b\eta(Y_9 - Y_5) + \lambda X_1Y_1], \\
f_9(S_v,V) &= 2b\eta V_1(\tilde{S}_v(\eta)-1)[3K_1 - K_2],
\end{aligned}
\tag{9.1}
$$

where

$$
\begin{aligned}
K_1 &= 2\eta\lambda X_1(b - S_{u1})(\tilde{S}_v(\eta)-1), \\
K_2 &= 3(\tilde{S}_v(\eta)-1)[\eta\lambda X_2(b - S_{u1}) + \lambda X_1(\eta b(b-1) - 2b\lambda X_1)] - 3\lambda X_1 \\
&\quad - 6b\eta\lambda X_1(b - S_{u1} - S_{v1}), \\
Y_1 &= \eta S_{u1}(1 - \tilde{S}_v(\eta)),\ Y_2 = \eta S_{u2} - 2\lambda X_1S_{u1}, \\
Y_3 &= (\tilde{S}_v(\eta)-1)Y_2 + 2\eta S_{u1}S_{v1}, \\
Y_4 &= (\tilde{S}_v(\eta)-1)[\eta S_{u3} - 3\lambda X_1S_{u2} - 3\lambda X_2S_{u1}] + 3S_{v1}Y_2 + 3\eta S_{u1}S_{v2}, \\
Y_5 &= (\tilde{S}_v(\eta)-1)(\lambda X_1 + \eta S_{u1}), \\
Y_6 &= 3b\Big[2(V_1 + C_1)[p(Y_5 + \eta S_{u1}) - \eta(b - S_{v1})] + \\
&\quad (\tilde{S}_v(\eta)-1)[\eta(V_2 + C_2 + 2V_1C_1) + (V_1 + C_1)(\eta(b-1) - 2\lambda X_1)]\Big], \\
Y_7 &= Y_4 - 3b(Y_2 + \eta(b-1)S_{u1}), \\
Y_8 &= Y_4 + 3b[(b-1)Y_5 + (\tilde{S}_v(\eta)-1)(\eta S_{u2} + \lambda X_2) + 2S_{v1}(\eta S_{u1} + \lambda X_1)], \\
Y_9 &= pY_5 + \eta(V_1 + C_1 - qS_{u1}),
\end{aligned}
$$

$$H = \begin{cases} \Big(Y_1\Big[\sum_{i=a}^{b-1} p_i + \sum_{i=a}^{b-1}(q_i + \sum_{k=0}^{a-1} S_k\lambda g_{i-k})\Big] \\ +b[pY_5 + \eta(V_1 + C_1 + (1-q)S_{u1})]\sum_{n=0}^{a-1} m_n \\ +\Big[b\eta[(V_1 + C_1 - qS_{u1}) - (1-p)Y_5] + \lambda X_1 Y_1\Big]\sum_{n=0}^{a-1} S_n \\ +b\eta V_1(\tilde{S}_v(\eta) - 1)\sum_{n=0}^{N-1} n v_n, \end{cases}$$

with $S_{u1} = \lambda X_1 E(S_u),\ S_{u2} = E(S_u^2)(\lambda X_1)^2 + \lambda X_2 E(S_u)$,

$S_{v1} = -\lambda X_1 \tilde{S}_v^{'}(\eta),\ S_{v2} = \tilde{S}_v^{''}(\eta)(\lambda X_1)^2 - \lambda X_2 \tilde{S}_v^{'}(\eta)$,

$V_1 = \lambda X_1 E(V),\ V_2 = \lambda X_2 E(V) + \lambda^2 X_1^2 E(V^2)$,

$C_1 = \lambda X_1 E(C),\ C_2 = \lambda X_2 E(C) + \lambda^2 X_1^2 E(C^2),\ X_1 = E(X)$ and $X_2 = E(X^2)$.

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