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Strong Convergence for the Modified Split Monotone Variational Inclusion and Fixed Point Problem

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Abstract The purpose of this research is to modify the split monotone variational inclusion problem and prove a strong convergence theorem for finding a common element of the set of solutions of this problem and the set of fixed points of a nonexpansive mapping in Hilbert space. We also apply our main result involving a κ-strictly pseudo-contractive mapping. Moreover, we give the numerical example to support some of our results.

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1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in C : Tx = x\}$.
A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

It is well known that if $T : H \to H$ is a nonexpansive mapping, we have

$$\langle Ty - Tx, (I - T)x - (I - T)y \rangle \leq \frac{1}{2}\|(I - T)x - (I - T)y\|^2, \forall x, y \in H.$$

Moreover, we also know that

$$\langle y - Tx, (I - T)x \rangle \leq \frac{1}{2}\|(I - T)x\|^2,$$

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for all $x \in H$ and $y \in F(T)$.
A mapping $h : C \to C$ is said to be a *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|h(x) - h(y)\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \left\|Ax - Ay\right\|^2, \forall x, y \in C.$$

In 2006, Marino and Xu [1] introduced the general iterative method based on the viscosity approximation method proposed by Moudafi [2] in 2000 as folows:

$$\begin{cases} x_0 \in H_1 \text{ arbitrary chosen,} \\ x_{n+1} = (I - \alpha_n D)\, Tx_n + \alpha_n \xi h\left(x_n\right), \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where T is a nonexpansive mapping, h is a contractive mapping on H, D is a strongly positive bounded linear self-adjoint operator and $\{\alpha_n\}$ is a sequence in (0,1). They also proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.1).

Let $f : H \to H$ be a mapping and $M : H \to 2^H$ be a multi-valued mapping. The *variational inclusion problem* is to find $u \in H$ such that

$$\theta \in f(u) + Mu, \tag{1.2}$$

where θ is zero vector in H. The set of the solution of (1.2) is denoted by $VI(H, f, M)$. A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H$, $u \in Mx$ and $v \in My$ implies that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in \mathrm{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^{M_1}(u) = (I + \lambda M_1)^{-1}(u), \forall u \in H,$$

is called the *resolvent operator* associated with M where λ is a positive number and I is an identity mapping, see [3].

In 2008, Zhang et al. [3] proved a strong convergence theorem for finding a common element of the set of solutions of the variational inclusion problem and the set of fixed points of nonexpansive mappings in Hilbert space. They introduced the iterative scheme as follows:

$$\begin{aligned} &y_n = J_\lambda^{M_1}(x_n - \lambda A x_n), \\ &x_{n+1} = \alpha_n x + (1 - \alpha_n) S y_n, \forall n \in \mathbb{N}, \end{aligned}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and λ. In 2014, Khuangsatung and Kangtunyakarn [4] modified a variational inclusion problem as follows: Finding $u \in H$ such that

$$\theta \in \sum_{i=1}^{N} a_i f_i(u) + Mu, \tag{1.3}$$

where $f_i : H \to H$ is a single valued mapping, $M : H \to 2^H$ is a multi-valued mapping, $a_i \in (0,1)$ with $\sum_{i=1}^N a_i = 1$, and θ is a zero vector for all $i = 1, 2, ..., N$. Such a problem is called *the modified variational inclusion*. The set of solutions (1.3) is denoted by $VI(H, \sum_{i=1}^{N} a_i A_i, M)$. If $A_i \equiv A$ for all $i = 1, 2, ..., N$, then (1.3) reduces to (1.2).

Let H_1 and H_2 be real Hilbert spaces. Let $M_1 : H_1 \to 2^{H_1}$ be a multi-valued mapping on a Hilbert space H_1, $M_2 : H_2 \to 2^{H_2}$ be a multi-valued mapping on a Hilbert space H_2, $A : H_1 \to H_2$ be a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two given single-valued operators. In 2011, Moudafi [5] introduced the *split monotone variational inclusion problem (SMVIP)* as follows: Find $x^* \in H_1$ such that

$$\theta \in f(x^*) + M_1 x^*, \tag{1.4}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } \theta \in g(y^*) + M_2 y^*. \tag{1.5}$$

The set of all solutions of (1.4) and (1.5) is denoted by $\Theta = \{x^* \in H_1 : x^* \in VI(H_1, f, M_1)$ and $Ax^* \in VI(H_2, g, M_2)\}$. In order to solve the SMVIP, he introduced the following iterative algorithm:

$$x_{n+1} = J_\lambda^{M_1}(I - \lambda f)(x_n + \gamma A^*(J_\lambda^{M_2}(I - \lambda g) - I)Ax_n), \forall n \in \mathbb{N},$$

where $J_\lambda^{M_1}$ and $J_\lambda^{M_2}$ are the resolvents of M_1 and M_2, respectively, $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A, and f, g are α_1 and α_2 inverse strongly monotone operators, respectively. He also proved that the sequence generated by the proposed algorithm weakly converges to a solution of SMVIP under suitable conditions. Many research papers have increasingly investigated SMVIP, see, for instance, [6, 7], and the references therein. We know that spacial cases of SMVIP include the split feasibility problem, the proximal split feasibility problem, the split common fixed point problem, the split variational inclusion problem, the split variational inequality problem, and so on, see for instance, [8–12], and the references therein. The split feasibility problem can be applied to solving important real world problems in medical fields such as intensity-modulated radiation therapy (IMRT) (see, [13])

In this paper, motivated by [1], [4], and [5], we introduce the modified split monotone variational inclusion problem (MSMVIP) which is to find $x^* \in H_1$ such that

$$\theta \in \sum_{i=1}^{N} a_i f_i(x^*) + M_1 x^*, \tag{1.6}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } \theta \in \sum_{i=1}^{N} b_i g_i(y^*) + M_2 y^*, \tag{1.7}$$

where $f_i : H_1 \to H_1$ is a single valued mapping, $g_i : H_2 \to H_2$ is a single valued mapping, for all $i = 1, 2, ..., N$, $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, $b_i \in (0, 1)$ with $\sum_{i=1}^{N} b_i = 1$, $M_j : H_j \to 2^{H_j}$ be a multi-valued mapping on a Hilbert space H_j, for all $j = 1, 2$, $A : H_1 \to H_2$ is a bounded linear operator, and θ is a zero vector. The set of all solutions of (1.6) and (1.7) is denoted by $\Omega = \{x^* \in H_1 : x^* \in VI(H_1, \sum_{i=1}^{N} a_i f_i, M_1)$ and $Ax^* \in VI(H_2, \sum_{i=1}^{N} b_i g_i, M_2)\}$, where $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$, $b_i \in (0, 1)$ with $\sum_{i=1}^{N} b_i = 1$.

The purpose of this article is to prove a strong convergence theorem for finding a common element of the set of solutions of the modified split monotone variational inclusion problem (MSMVIP) and the set of fixed points of a nonexpansive mapping in Hilbert space. Moreover, we also apply our main result involving a κ-strictly pseudo-contractive mapping. In the last section, we give the numerical example to support some of our results.

2. Preliminaries

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subset of H_1 and H_2, respectively. Recall that H_1 satisfies *Opial's condition* [14], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in H_1$ with $y \neq x$.
For a proof of the our main results, we will use the following lemmas.

Lemma 2.1 ([15]). *Given $x \in H_1$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

Lemma 2.2 ([16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1): $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2): $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([3]). *$u \in H_1$ is a solution of variational inclusion (1.2) if and only if $u = J_\lambda^{M_1}(u - \lambda f(u)), \forall \lambda > 0$, i.e.,*

$$VI(H_1, f, M_1) = F(J_\lambda^{M_1}(I - \lambda f)), \forall \lambda > 0.$$

Further, if $\lambda \in (0, 2\alpha]$, then $VI(H_1, f, M_1)$ is closed convex subset in H_1.

Lemma 2.4 ([4]). *Let H_1 be a real Hilbert space and let $M_1 : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,2,...,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N VI(H_1, f_i, M_1) \neq \emptyset$. Then*

$$VI(H_1, \sum_{i=1}^N a_i f_i, M_1) = \bigcap_{i=1}^N VI(H_1, f_i, M_1),$$

where $\sum_{i=1}^N a_i = 1$, and $0 < a_i < 1$ for every $i = 1, 2, \ldots, N$. Moreover, we have $J_\lambda^{M_1}(I - \lambda \sum_{i=1}^N a_i f_i)$ is a nonexpansive mapping, for all $0 < \lambda < 2\eta$.

Lemma 2.5 ([1]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.6. *For every $j = 1, 2$, $M_j : H_j \to 2^{H_j}$ be a multi-valued maximal monotone mapping on a Hilbert space H_j and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Assume that Ω is a nonempty. Then the following are equivalent:*

(1) $x^* \in \Omega$,

(2) $x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax^*)$,

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A, $a_i \in (0, 1)$ with $\sum_{i=1}^N a_i = 1$, and $b_i \in (0, 1)$ with $\sum_{i=1}^N b_i = 1$, for all $i = 1, 2, ..., N$.

Proof. Let the condition holds. Put $G_1 = \sum_{i=1}^N a_i f_i$, and $G_2 = \sum_{i=1}^N b_i g_i$.
(1) $\Longrightarrow$ (2) Let $x^* \in \Omega$, we have $x^* \in VI(H_1, \sum_{i=1}^N a_i f_i, M_1) = VI(H_1, G_1, M_1)$ and $Ax^* \in VI(H_2, \sum_{i=1}^N b_i g_i, M_2) = VI(H_2, G_2, M_2)$. From Lemma 2.3, we have $x^* \in F(J_{\lambda_1}^{M_1}(I - \lambda_1 G_1))$ and $Ax^* \in F(J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))$. This implies that

$$x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*).$$

(2) $\Longrightarrow$ (1) Let $x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*)$ and let $z \in \Omega$. From Lemma 2.4, we have the mapping $J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)$ and $J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)$ are nonexpansive mappings. Since $z \in \Omega$ and (1) $\Longrightarrow$ (2), we have

$$z = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(z + \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az).$$

Since $J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)$ and $J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)$ are nonexpansive mappings, we have

$$\begin{aligned}
\|x^* - z\|^2 &= \|J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*) \\
&\quad - J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az)\|^2 \\
&\leq \|(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*) \\
&\quad - \ (z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az)\|^2 \\
&= \|(x^* - z) - \gamma\big(A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^* \\
&\quad - A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Az\big)\|^2 \\
&= \|x^* - z\|^2 - 2\gamma\langle x^* - z, A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad + \ \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 - 2\gamma\langle Ax^* - Az, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad + \ \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 + 2\gamma\langle Az - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^* + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^* \\
&\quad - Ax^*, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle + \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2 \\
&= \|x^* - z\|^2 + 2\gamma\big(\langle Az - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax^*, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\rangle \\
&\quad - \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2\big) + \gamma^2\|A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2
\end{aligned}$$

$$\leq \|x^* - z\|^2 + 2\gamma\Big(\frac{1}{2}\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2$$
$$- \ \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2\Big) + \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2$$
$$= \|x^* - z\|^2 - \gamma\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2$$
$$+ \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2$$
$$= \|x^* - z\|^2 - \gamma(1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*\|^2. \tag{2.1}$$

Applying (2.1), we have

$$Ax^* \in F(J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)). \tag{2.2}$$

From Lemma (2.3), we have

$$Ax^* \in VI(H_2, G_2, M_2). \tag{2.3}$$

From the definition of x^* and (2.2), we have

$$x^* = J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x^* - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax^*)$$
$$= J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)x^*.$$

Then $x^* \in F(J_{\lambda_1}^{M_1}(I - \lambda_1 G_1))$. From Lemma (2.3), we have

$$x^* \in VI(H_1, G_1, M_1). \tag{2.4}$$

From (2.3) and (2.4), we have $x^* \in \Omega$. ■

3. Main Theorem

In this section, we prove a strong convergence theorem for the modified split monotone variational inclusion and the set of fixed point of a nonexpansive mapping in Hilbert space.

Theorem 3.1. *For every $j = 1, 2$, $M_j : H_j \to 2^{H_j}$ be a multi-valued maximal monotone mapping on a Hilbert space H_j and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1, 2, ..., N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Let Ω be a solution of (1.6) and (1.7) and $\Omega \neq \emptyset$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $\Psi = F(T) \cap \Omega$ is nonempty. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0, 1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0, 1)$ with $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} u_n = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax_n), \\ x_{n+1} = \alpha_n \xi h(x_n) + (I - \alpha_n D)\left(\beta_n T x_n + (1 - \beta_n)u_n\right), \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii): $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii): *$a_i \in (0,1)$ with $\sum_{i=1}^N a_i = 1$, and $b_i \in (0,1)$ with $\sum_{i=1}^N b_i = 1$, for all $i = 1, 2, ..., N$.*

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi h)(z)$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n \le \|D\|^{-1}$, for all $n \in \mathbb{N}$.
We divide the proof into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded. Let $z \in \Psi$. From Lemma 2.6, we have

$$z = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(z - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Az).$$

Put $y_n = \beta_n T x_n + (1-\beta_n)u_n$ and applying (2.1) in Lemma 2.6, we have

$$\begin{aligned} \|y_n - z\| &= \|\beta_n T x_n + (1-\beta_n)u_n - z\| \\ &\le \beta_n \|Tx_n - z\| + (1-\beta_n)\|u_n - z\| \\ &\le \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{3.2}$$

From the definition of x_n and (3.2), we have

$$\begin{aligned} \|x_{n+1} - z\| =& \ \|\alpha_n \xi h(x_n) + (I - \alpha_n D)y_n - z\| \\ =& \ \|\alpha_n(\xi h(x_n) - Dz) + (I - \alpha_n D)(y_n - z)\| \\ \le& \ \alpha_n \|\xi h(x_n) - Dz\| + \|I - \alpha_n D\| \, \|y_n - z\| \\ \le& \ \alpha_n (\xi \|h(x_n) - h(z)\| + \|\xi h(z) - Dz\|) + (1 - \alpha_n \bar{\xi})\|x_n - z\| \\ \le& \ (1 - \alpha_n(\bar{\xi} - \xi\alpha)) \|x_n - z\| + \alpha_n \|\xi h(z) - Dz\| \\ \le& \ \max\left\{ \|x_1 - z\|, \frac{\|\xi h(z) - Dz\|}{\bar{\xi} - \xi\alpha} \right\}. \end{aligned}$$

By mathematical induction, we have $\|x_n - z\| \le K, \forall n \in \mathbb{N}$. It implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Put $G_1 = \sum_{i=1}^N a_i f_i$, and $G_2 = \sum_{i=1}^N b_i g_i$. From the definition of u_n, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 =& \ \|J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n) \\ & - J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(x_{n-1} - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1})\|^2 \\ \le& \ \|(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n) \\ & - (x_{n-1} - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1})\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|(x_n - x_{n-1}) - \gamma A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\langle x_n - x_{n-1}, A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\rangle \\
&\quad + \ \gamma^2\|A^*\big((I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\big)\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\Big\langle Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n \\
&\quad - \ Ax_{n-1} + J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1} - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, \\
&\quad (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n) - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1})\Big\rangle \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad - \ 2\gamma\langle(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1} \\
&\quad + \ J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\gamma\big(\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \langle J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1}, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad + \ 2\gamma\big(- \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \langle J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_{n-1} - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2)Ax_n, (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n \\
&\quad - \ (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\rangle\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 \\
&\quad + \ 2\gamma\big(- \|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\quad + \ \frac{1}{2}\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2\big) \\
&\quad + \ \gamma^2 L\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&= \|x_n - x_{n-1}\|^2 \\
&\quad - \ \gamma(1 - \gamma L)\|(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_n - (I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))Ax_{n-1}\|^2 \\
&\leq \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.3}
$$

From the definition of y_n and (3.3), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\beta_n Tx_n + (1-\beta_n)u_n - \beta_{n-1}Tx_{n-1} - (1-\beta_{n-1})u_{n-1}\| \\
&= \|\beta_n(Tx_n - Tx_{n-1}) + (\beta_n - \beta_{n-1})Tx_{n-1} + (1-\beta_n)(u_n - u_{n-1}) \\
&\quad + (\beta_{n-1} - \beta_n)u_{n-1}\| \\
&\le \beta_n\|Tx_n - Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + (1-\beta_n)\|u_n - u_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|u_{n-1}\| \\
&\le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|u_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| + |\beta_n - \beta_{n-1}|\|u_{n-1}\|. \qquad (3.4)
\end{aligned}$$

From the definition of x_n and (3.4), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n\xi h(x_n) + (I - \alpha_n D)y_n - \alpha_{n-1}\xi h(x_{n-1}) - (I - \alpha_{n-1}D)y_{n-1}\| \\
&\le \alpha_n\xi\|h(x_n) - h(x_{n-1})\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&\quad + \|(I - \alpha_n D)\|\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
&\le \alpha_n\xi\alpha\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&\quad + (1 - \alpha_n\bar{\xi})\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
&\le \alpha_n\xi\alpha\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| \\
&\quad + (1 - \alpha_n\bar{\xi})\big(\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|u_{n-1}\|\big) \\
&\le (1 - \alpha_n(\bar{\xi} - \xi\alpha))\|x_n - x_{n-1}\| + \xi|\alpha_n - \alpha_{n-1}|\|h(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Dy_{n-1}\| + |\beta_n - \beta_{n-1}|\|Tx_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|u_{n-1}\|. \qquad (3.5)
\end{aligned}$$

Applying Lemma 2.2, (3.5) and the conditions (i), (ii), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.6)$$

Step 3. We show that $\lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|\alpha_n\xi h(x_n) + (I - \alpha_n D)y_n - y_n\| \\
&\le \alpha_n\|\xi h(x_n) - \alpha_n Dy_n\|.
\end{aligned}$$

Based on the above equation and the condition (i), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \qquad (3.7)$$

Observe that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|. \qquad (3.8)$$

From (3.6), (3.7), and (3.8), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \qquad (3.9)$$

From the definition of y_n, we have

$$\begin{aligned}\|y_n - z\|^2 &= \|\beta_n Tx_n + (1-\beta_n)u_n - z\|^2 \\ &= \beta_n\|Tx_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 - \beta_n(1-\beta_n)\|Tx_n - u_n\|^2 \\ &\le \|x_n - z\|^2 - \beta_n(1-\beta_n)\|Tx_n - u_n\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}\beta_n(1-\beta_n)\|Tx_n - u_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\ &= (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.\end{aligned}$$

From the condition (ii) and (3.9), we have

$$\lim_{n\to\infty} \|Tx_n - y_n\| = 0. \tag{3.10}$$

Observe that

$$\|Tx_n - x_n\| \le \|Tx_n - y_n\| + \|y_n - x_n\|. \tag{3.11}$$

From (3.9), (3.10), and (3.11), we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.12}$$

Since

$$y_n - Tx_n = \beta_n Tx_n + (1-\beta_n)u_n - Tx_n,$$

then

$$y_n - Tx_n = (1-\beta_n)(u_n - Tx_n).$$

From the equation above, (3.10), and the condition (ii), we have

$$\lim_{n\to\infty} \|Tx_n - u_n\| = 0. \tag{3.13}$$

Observe that

$$\|u_n - x_n\| \le \|u_n - Tx_n\| + \|Tx_n - x_n\|. \tag{3.14}$$

From (3.12), (3.13), and (3.14), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.15}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle(\xi h - D)z, x_n - z\rangle \le 0$, where $z = P_\Psi(I - D + \xi h)z$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle(\xi h - D)z, x_n - z\rangle = \lim_{k\to\infty} \langle(\xi h - D)z, x_{n_k} - z\rangle. \tag{3.16}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (3.15), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.

Next, we will show that $\omega \in \Omega$. Assume that $\omega \notin \Omega$. By Lemma 2.6, we have $\omega \neq J_{\lambda_1}^{M_1}(I - \lambda_1 G_1)(\omega - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 G_2))A\omega)$. By the Opial's condition and

(3.15), we obtain

$$\begin{aligned}
&\liminf_{k\to\infty}\|x_{n_k}-\omega\| \\
&\quad< \liminf_{k\to\infty}\|x_{n_k}-J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(\omega-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))A\omega)\| \\
&\quad\le \liminf_{k\to\infty}(\|x_{n_k}-J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(x_{n_k}-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))Ax_{n_k})\| \\
&\qquad+\ \|J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(x_{n_k}-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))Ax_{n_k}) \\
&\qquad-\ J_{\lambda_1}^{M_1}(I-\lambda_1G_1)(\omega-\gamma A^*(I-J_{\lambda_2}^{M_2}(I-\lambda_2G_2))A\omega)\|) \\
&\quad\le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in\Omega.$$

Next, we will show that $\omega\in F(T)$. Assume that $\omega\notin F(T)$. Then $\omega\neq T\omega$. By the nonexpansiveness of T, the Opial's condition, and (3.12), we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\|x_{n_k}-T\omega\| \\
&\le \liminf_{k\to\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_K}-T\omega\|) \\
&\le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in F(T).$$

Therefore $\omega\in\Psi=\Omega\cap F(T)$.
Since $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ and $\omega\in\Psi$. By (3.16) and Lemma 2.1, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\xi h-D)z,x_n-z\rangle &= \lim_{k\to\infty}\langle(\xi h-D)z,x_{n_k}-z\rangle \\
&=\langle(\xi h-D)z,\omega-z\rangle \\
&\le 0. \qquad (3.17)
\end{aligned}$$

Step 5. Finally, we show that $\lim_{n\to\infty}x_n=z$, where $z=P_\Psi(I-D+\xi h)z$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n(\xi h(x_n)-Dz)+(I-\alpha_nD)(y_n-z)\|^2 \\
&\le \|(I-\alpha_nD)(y_n-z)\|^2+2\alpha_n\langle\xi h(x_n)-Dz,x_{n+1}-z\rangle \\
&\le (1-\alpha_n\bar{\xi})^2\|y_n-z\|^2+2\alpha_n\langle\xi h(x_n)-\xi h(z),x_{n+1}-x_0\rangle \\
&\quad+\ 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&\le (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+2\alpha_n\xi\|h(x_n)-h(z)\|\|x_{n+1}-z\| \\
&\quad+\ 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&\le (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+2\alpha_n\xi\alpha\|x_n-z\|\|x_{n+1}-z\| \\
&\quad+\ 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle \\
&= (1-\alpha_n\bar{\xi})^2\|x_n-z\|^2+\alpha_n\xi\alpha\|x_n-z\|^2+\alpha_n\xi\alpha\|x_{n+1}-z\|^2 \\
&\quad+\ 2\alpha_n\langle\xi h(z)-Dz,x_{n+1}-x_0\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq& \frac{1-2\alpha_n\bar{\xi}+(\alpha_n\bar{\xi})^2+\alpha_n\xi\alpha}{1-\alpha_n\xi\alpha}\|x_n-z\|^2\\ &+\frac{2\alpha_n}{1-\alpha_n\xi\alpha}\langle \xi h(z)-Dz, x_{n+1}-z\rangle\\ =&\left(1-\frac{2\alpha_n(\bar{\xi}-\xi\alpha)}{1-\alpha_n\xi\alpha}\right)\|x_n-z\|^2\\ &+\frac{2\alpha_n(\bar{\xi}-\xi\alpha)}{1-\alpha_n\xi\alpha}\left(\frac{\alpha_n\bar{\xi}^2}{2(\bar{\xi}-\xi\alpha)}\|x_n-z\|^2\right.\\ &+\left.\frac{1}{\bar{\xi}-\xi\alpha}\langle \xi h(z)-Dz, x_{n+1}-z\rangle\right).\end{aligned}$$

From the condition (i), (3.17) and Lemma 2.2, we can conclude that the sequence $\{x_n\}$ converges strongly to $z=P_\Psi(I-D+\xi h)z$. This completes the proof. ∎

As direct proof of Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces, $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings and let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $g : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let Θ be a solution of (1.4) and (1.5) and $\Theta \neq \emptyset$. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $\Psi = F(T)\cap\Theta$ is nonempty. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0,1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ with $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} u_n = J^{M_1}_{\lambda_1}(I-\lambda_1 f)(x_n-\gamma A^*(I-J^{M_2}_{\lambda_2}(I-\lambda_2 g))Ax_n),\\ x_{n+1} = \alpha_n\xi h(x_n)+(I-\alpha_n D)\left(\beta_n Tx_n+(1-\beta_n)u_n\right), \forall n\in\mathbb{N}, \end{cases} \tag{3.18}$$

where $0<\lambda_1<2\mu$, $0<\lambda_2<2\nu$, and $\gamma\in(0,\frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\}\subset(0,1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;

(ii): $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$;

Then the sequence $\{x_n\}$ converges strongly to $z=P_\Psi(I-D+\xi h)(z)$.

Proof. Put $f_i\equiv f$ and $g_i\equiv g$ for all $i=1,2,...,N$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. ∎

4. Application

In this section, we utilize our main theorem to prove a strong convergence theorem for finding a common element of the set of solutions of the modified split monotone variational inclusion problem (MSMVIP) and the set of fixed points of a κ-strictly pseudo-contractive mapping.

A mapping $T : C \to C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \kappa \|(I-T)x - (I-T)y\|^2,$$

for all $x, y \in C$. Note that the class of strictly pseudo-contractions strictly includes the class of nonexpansive mapping.

Lemma 4.1 (See [17]). *Let $T : C \to H_1$ be a κ-strict pseudo-contraction. Define $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces, $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings and let $A : H_1 \to H_2$ be a bounded linear operator. For every $i = 1,2,...,N$, let $f_i : H_1 \to H_1$ be μ_i-inverse strongly monotone mapping with $\mu = min_{i=1,2,...,N}\{\mu_i\}$ and $g_i : H_2 \to H_2$ be ν_i-inverse strongly monotone mapping with $\nu = min_{i=1,2,...,N}\{\nu_i\}$. Let Ω be a solution of (1.6) and (1.7) and $\Omega \neq \emptyset$. Let $T : H_1 \to H_1$ be a κ-strict pseudo-contraction with $\Psi = F(T) \cap \Omega$ is nonempty. Define the mapping $S = H_1 \to H_1$ by $Sx = \sigma x + (1-\sigma)Tx$ for every $x \in H_1$ and $\sigma \in (k,1)$. Let $h : H_1 \to H_1$ be a contractive mapping with $\alpha \in (0,1)$ and let D be a strongly positive bounded linear operator with coefficient $\bar{\xi} \in (0,1)$ and $0 < \xi < \frac{\bar{\xi}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by $x_1 \in H_1$ and*

$$\begin{cases} u_n = J_{\lambda_1}^{M_1}(I - \lambda_1 \sum_{i=1}^N a_i f_i)(x_n - \gamma A^*(I - J_{\lambda_2}^{M_2}(I - \lambda_2 \sum_{i=1}^N b_i g_i))Ax_n) \\ x_{n+1} = \alpha_n \xi h(x_n) + (I - \alpha_n D)(\beta_n S x_n + (1-\beta_n)u_n), \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $0 < \lambda_1 < 2\mu$, $0 < \lambda_2 < 2\nu$, and $\gamma \in (0, \frac{1}{L})$ with L is the spectral radius of the operator A^*A. Suppose $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ satisfying the following conditions:

(i): $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii): $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii): $a_i \in (0,1)$ *with* $\sum_{i=1}^N a_i = 1$, *and* $b_i \in (0,1)$ *with* $\sum_{i=1}^N b_i = 1$, *for all* $i = 1,2,...,N$.

Then the sequence $\{x_n\}$ converges strongly to $z = P_\Psi(I - D + \xi h)(z)$.

Proof. From Lemma 4.1 and Theorem 3.1, we obtain the desired result. ■

5. Numerical Result

The purpose of this section, we give a numerical example to support our main result. The following example is given to support Theorem 3.1.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers and let $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2$, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and $H_1 = H_2 = \mathbb{R}^2$. For every $i = 1,2,\ldots,N$, let $D, f_i, g_i, h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$D\mathbf{x} = \left(\frac{x_1}{2}, \frac{x_2}{2}\right), \quad f_i\mathbf{x} = \left(\frac{ix_1}{12}, \frac{ix_2}{12}\right), \quad g_i\mathbf{x} = \left(\frac{ix_1}{9}, \frac{ix_2}{9}\right), \quad h(\mathbf{x}) = \left(\frac{x_1}{3}, \frac{x_2}{3}\right),$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\mathbf{x} = \left(\frac{x_1}{5}, \frac{x_2}{5}\right),$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Observe that, for every $i = 1, 2, ..., N$, f_i and g_i are inverse strongly monotone mappings, D is a strongly positive bounded linear operator, h is a contractive mapping, and also we have T is a nonexpansive mapping. We also define $A : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$A\mathbf{x} = (5x_1 - 3x_2, 3x_1 + 5x_2),$$

and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$A^*\mathbf{x} = (5x_1 + 3x_2, 5x_2 - 3x_1).$$

Then A is a bounded linear operator. Moreover, the spectral radius of the operator A^*A is 34 and also we have $\gamma \in (0, \frac{1}{34})$.

For every $i = 1, 2, \ldots, N$. Suppose that $J_{\lambda_1}^{M_1} = J_{\lambda_2}^{M_2} = I$, $a_i = \frac{14}{15^i} + \frac{1}{N15^N}$, $b_i = \frac{15}{16^i} + \frac{1}{N16^N}$. Let $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$ be generated by (3.1), where $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{n}{5n+6}$ for every $n \in \mathbb{N}$. Put $\lambda_1 = \lambda_2 = \frac{1}{N}$ and $\xi = \frac{1}{2}$. It is easy to see that all sequences satisfy conditions of Theorem 3.1. For every $n \in \mathbb{N}$, we rewrite (3.1) as follows:

$$\begin{aligned} \mathbf{u}_n &= \left(I - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{14}{15^i} + \frac{1}{N15^N}\right) f_i\right) \\ &\quad \times \left(\mathbf{x}_n - \gamma A^*\left(I - \left(I - \frac{1}{N}\left(\sum_{i=1}^{N} \frac{15}{16^i} + \frac{1}{N15^N}\right) g_i\right)\right) A\mathbf{x}_n\right), \\ \mathbf{x}_{n+1} &= \frac{1}{2n}\left(\frac{1}{2}\right) h(\mathbf{x}_n) + (I - \frac{1}{2n}D)\left(\frac{n}{5n+6}T\mathbf{x}_n + (1 - \frac{n}{5n+6})\mathbf{u}_n\right), \end{aligned} \tag{5.1}$$

where $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$. Then the sequences $\mathbf{x}_n = \left(x_n^1, x_n^2\right)$ and $\mathbf{u}_n = \left(u_n^1, u_n^2\right)$ generated by (5.1) converge strongly to $\mathbf{0}$, where $\mathbf{0} = (0, 0)$.
Using the algorithm (5.1) and choosing $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$, the numerical results for the sequences x_n and u_n are shown the following table and figure.

n	u_n^1	u_n^2	x_n^1	x_n^2
1	4.9915097	-4.9915097	5.0000000	-5.0000000
2	3.8815482	-3.8815482	3.8881505	-3.8881505
3	3.2134041	-3.2134041	3.2188699	-3.2188699
4	2.6939520	-2.6939520	2.6985343	-2.6985343
5	2.2672346	-2.2672346	2.2710910	-2.2710910
⋮	⋮	⋮	⋮	⋮
50	0.0009374	-0.0009374	0.0009390	-0.0009390
⋮	⋮	⋮	⋮	⋮
96	0.0000003	-0.0000003	0.0000003	-0.0000003
97	0.0000003	-0.0000003	0.0000003	-0.0000003
98	0.0000002	-0.0000002	0.0000002	-0.0000002
99	0.0000002	-0.0000002	0.0000002	-0.0000002
100	0.0000001	-0.0000001	0.0000001	-0.0000001

Table 1. The values of the sequences $\{\mathbf{u}_n\}$ and $\{\mathbf{x}_n\}$ with initial values $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$.

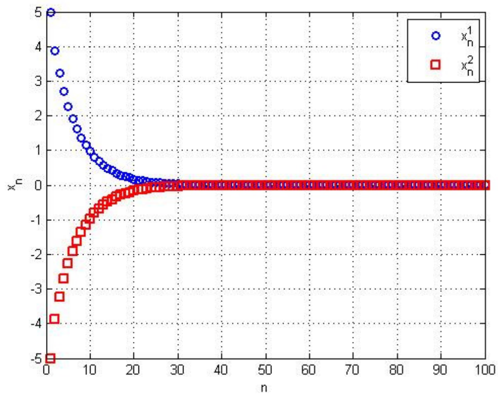

FIGURE 1. The behavior of $\{\mathbf{x}_n\}$ with initial values $\mathbf{x}_1 = (5, -5)$ and $n = N = 100$.

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