Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 877–887

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Cubic Intuitionistic Structure Applied to Commutative Ideals of BCK-Algebras

Tapan Senapati[1], Young Bae Jun[2], Aiyared Iampan[3,*] and Ronnason Chinram[4]

[1] *Department of Mathematics, Padima Janakalyan Banipith, Kukrakhupi, 721517, India*
e-mail : math.tapan@gmail.com

[2] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*
e-mail : skywine@gmail.com

[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*
e-mail : aiyared.ia@up.ac.th

[4] *Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*
e-mail : ronnason.c@psu.ac.th

Abstract In BCK-algebras, the concept of cubic intuitionistic (CI) commutative ideals is incorporated. The article discusses the link between a CI subalgebra, a CI ideal, and a CI commutative ideal. The circumstances under which a CI ideal is a CI commutative ideal are defined in detail. The characteristics of a CI commutative ideal are discussed in detail. A CI commutative ideal's CI extension property is demonstrated.

MSC: 06F35; 03G25; 94D05

Keywords: CIS; CI subalgebra; CI ideal; CI commutative ideal

Submission date: 05.02.2019 / Acceptance date: 08.06.2022

1. Introduction

Jun et al. [1] proposed cubic sets in 2012 and subsequently used this concept in a variety of algebras (see [2–6]). In 2017, Jun [7] pioneered the theory of a cubic intuitionistic set (CIS), which is an extended form of a cubic set. He established the concepts of (cross) external CIS, cross-right (left) internal CIS, (right, left) internal CIS, and double right (left) internal CIS, as well as studied the possibility of these concepts and their combinations. With this approach, he was able to apply it to subalgebras and ideals in a BCK/BCI-algebra and derive several interesting findings. The links between a CI subalgebra and a CI ideal were presented in the context of a BCK/BCI-algebra. According to Senapati et al. [8–11], the CIS concept was applied to different ideals (a-ideal, p-ideal, and q-ideal) of BCI-algebras, KU-algebras, and B-algebras. According

*Corresponding author.

Published by The Mathematical Association of Thailand.

to Senapati et al. [12], the conception of CIS may be extended to (positive) implicative ideals of a BCK-algebra, and links between both of them can be established.

In this study, CI commutative ideals in BCK-algebras are introduced, which is the first paper to do so. It is discussed in this paper what the link between a CI subalgebra, a CI ideal, and a CI commutative ideal is, as well as the circumstances under which a CI ideal is a CI commutative ideal. It is shown how to characterise a CI commutative ideal and how to examine the CI extension property for a CI commutative ideal.

2. Preliminaries

This section examines several foundational principles pertinent to this work.

We refer to a *BCI-algebra* as an algebra with a constant 0 and a binary operation "$*$" that satisfies the following axioms for every $\eta, \wp, \partial \in \hbar$:

(i) $((\eta * \wp) * (\eta * \partial)) * (\partial * \wp) = 0$
(ii) $(\eta * (\eta * \wp)) * \wp = 0$
(iii) $\eta * \eta = 0$
(iv) $\eta * \wp = 0$ and $\wp * \eta = 0$ imply $\eta = \wp$.

We can define a partial ordering "$\leq$" by $\eta \leq \wp$ if and only if $\eta * \wp = 0$.

If a BCI-algebra $\hbar$ fulfils $0 * \eta = 0$, for all $\eta \in \hbar$, then $\hbar$ is a *BCK-algebra*. Any BCK/BCI-algebra $\hbar$ fulfils the following axioms for any $\eta, \wp, \partial \in \hbar$:

$(a1)$ $(\eta * \wp) * \partial = (\eta * \partial) * \wp$
$(a2)$ $((\eta * \partial) * (\wp * \partial)) * (\eta * \wp) = 0$
$(a3)$ $\eta * 0 = \eta$
$(a4)$ $\eta * \wp \leq \eta$
$(a5)$ $\eta \leq \wp$ implies $\eta * \partial \leq \wp * \partial$ and $\partial * \wp \leq \partial * \eta$.

A BCK-algebra $\hbar$ is considered to be commutative if it fulfils the identity $\eta * (\eta * \wp) = \wp * (\wp * \eta)$ for every $\eta, \wp \in \hbar$.

A non-empty subset S of $\hbar$ is referred to as a *subalgebra* of $\hbar$ if $\eta * \wp \in S$ for any $\eta, \wp \in S$.

A nonempty subset I of $\hbar$ is said to be an *ideal* of $\hbar$ if it fulfils

(I_1) $0 \in I$ and
(I_2) $\eta * \wp \in I$ and $\wp \in I$ imply $\eta \in I$.

A non-empty subset I of $\hbar$ is an *commutative ideal* of $\hbar$ (see [13]) if it fulfils (I_1) and (I_3) $(\eta * \wp) * \partial \in I$ and $\partial \in I$ imply $\eta * (\wp * (\wp * \eta)) \in I$, for every $\eta, \wp, \partial \in \hbar$.

An *intuitionistic fuzzy set* in a set $\hbar$ (see [14]) is defined to be an object of the form

$$\psi = \{\langle \eta, \xi_\psi(\eta), \zeta_\psi(\eta)\rangle \mid \eta \in \hbar\}$$

where $\xi_\psi(\eta) \in [0,1]$ and $\zeta_\psi(\eta) \in [0,1]$ with $\xi_\psi(\eta) + \zeta_\psi(\eta) \leq 1$. The intuitionistic fuzzy set

$$\psi = \{\langle \eta, \xi_\psi(\eta), \zeta_\psi(\eta)\rangle \mid \eta \in \hbar\}$$

is simply denoted by $\psi(\eta) = (\xi_\psi(\eta), \zeta_\psi(\eta))$ for $\eta \in \hbar$ or $\psi = (\xi_\psi, \zeta_\psi)$.

An *interval-valued intuitionistic fuzzy set* A over a set $\hbar$ (see [15]) is an object of the form

$$A = \{\langle \eta, \Re_A(\eta), \Im_A(\eta)\rangle \mid \eta \in \hbar\}$$

where $\Re_A(\eta) \subseteq [0,1]$ and $\Im_A(\eta) \subseteq [0,1]$ are intervals and for each $\eta \in \hbar$,

$$\sup \Re_A(\eta) + \sup \Im_A(\eta) \leq 1$$

Given two closed subintervals $D_1 = [D_1^-, D_1^+]$ and $D_2 = [D_2^-, D_2^+]$ of $[0, 1]$, we define the order "$\ll$" in the following way:

$$D_1 \ll D_2 \ \Leftrightarrow \ D_1^- \leq D_2^- \text{ and } D_1^+ \leq D_2^+.$$

We also define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) in the following way:

$$\begin{aligned}\text{rmin}\{D_1, D_2\} &= \left[\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}\right], \\ \text{rmax}\{D_1, D_2\} &= \left[\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}\right].\end{aligned}$$

Denote by $D[0, 1]$ the set of all closed subintervals of $[0, 1]$. In this paper we use the interval-valued intuitionistic fuzzy set

$$A = \{\langle \eta, \Re_A(\eta), \Im_A(\eta)\rangle \mid \eta \in \hbar\}$$

over $\hbar$ in which $\Re_A(\eta)$ and $\Im_A(\eta)$ are closed subintervals of $[0, 1]$ for all $\eta \in \hbar$. Also, we use the notations $\Re_A^-(\eta)$ and $\Re_A^+(\eta)$ to mean the left end point and the right end point of the interval $\Re_A(\eta)$, respectively, and so we have $\Re_A(\eta) = [\Re_A^-(\eta), \Re_A^+(\eta)]$. The interval-valued intuitionistic fuzzy set

$$A = \{\langle \eta, \Re_A(\eta), \Im_A(\eta)\rangle \mid \eta \in \hbar\}$$

over $\hbar$ is simply denoted by $A(\eta) = \langle \Re_A(\eta), \Im_A(\eta)\rangle$ for $\eta \in \hbar$ or $A = \langle \Re_A, \Im_A\rangle$.

Let $\hbar$ be a nonempty set. By a *CIS* in $\hbar$ (see [7]) we mean a structure

$$\mathcal{A} = \{\langle \eta, A(\eta), \psi(\eta)\rangle \mid \eta \in \hbar\}$$

in which A is an interval-valued intuitionistic fuzzy set in $\hbar$ and ψ is an intuitionistic fuzzy set in $\hbar$.

A CIS $\mathcal{A} = \{\langle \eta, A(\eta), \psi(\eta)\rangle \mid \eta \in \hbar\}$ is simply denoted by $\mathcal{A} = \langle A, \psi\rangle$.

3. (Cubic Intuitionistic) Subalgebras and Ideals

Until otherwise stated, assume that $\hbar$ is a BCK-algebra in this section.

Definition 3.1 ([7]). A CIS $\mathcal{A} = \langle A, \psi\rangle$ in $\hbar$ is referred to as a *CI subalgebra* of $\hbar$ if the following conditions are valid.

$$(\forall \eta, \wp \in \hbar) \begin{pmatrix} \Re_A(\eta * \wp) \gg \text{rmin}\{\Re_A(\eta), \Re_A(\wp)\} \\ \Im_A(\eta * \wp) \ll \text{rmax}\{\Im_A(\eta), \Im_A(\wp)\} \end{pmatrix}, \tag{3.1}$$

$$(\forall \eta, \wp \in \hbar) \begin{pmatrix} \xi_\psi(\eta * \wp) \leq \max\{\xi_\psi(\eta), \xi_\psi(\wp)\} \\ \zeta_\psi(\eta * \wp) \geq \min\{\zeta_\psi(\eta), \zeta_\psi(\wp)\} \end{pmatrix}. \tag{3.2}$$

Definition 3.2 ([7]). A CIS $\mathcal{A} = \langle A, \psi\rangle$ is referred to as a *CI ideal* of $\hbar$ if the following conditions hold true.

$$\begin{cases} \Re_A(0) \text{ is an upper bound of } \{\Re_A(\eta) \mid \eta \in \hbar\} \\ \Im_A(0) \text{ is a lower bound of } \{\Im_A(\eta) \mid \eta \in \hbar\} \end{cases} \tag{3.3}$$

$$\begin{cases} \xi_\psi(0) \text{ is a lower bound of } \{\xi_\psi(\eta) \mid \eta \in \hbar\} \\ \zeta_\psi(0) \text{ is an upper bound of } \{\zeta_\psi(\eta) \mid \eta \in \hbar\} \end{cases} \tag{3.4}$$

$$(\forall \eta, \wp \in \hbar) \begin{pmatrix} \Re_A(\eta) \gg \text{rmin}\{\Re_A(\eta * \wp), \Re_A(\wp)\} \\ \Im_A(\eta) \ll \text{rmax}\{\Im_A(\eta * \wp), \Im_A(\wp)\} \end{pmatrix} \tag{3.5}$$

$$(\forall \eta, \wp \in \hbar) \begin{pmatrix} \xi_\psi(\eta) \leq \max\{\xi_\psi(\eta * \wp), \xi_\psi(\wp)\} \\ \zeta_\psi(\eta) \geq \min\{\zeta_\psi(\eta * \wp), \zeta_\psi(\wp)\} \end{pmatrix}. \tag{3.6}$$

Theorem 3.3 ([7]). *In a BCK-algebra $\hbar$, every CI ideal is a CI subalgebra.*

Lemma 3.4 ([7]). *Every CI ideal $\mathcal{A} = \langle A, \psi\rangle$ in $\hbar$ satisfies the following condition*

$$(\forall \eta, \wp \in \hbar) \left(\eta \leq \wp \Rightarrow \begin{cases} \Re_A(\eta) \gg \Re_A(\wp), \Im_A(\eta) \ll \Im_A(\wp) \\ \xi_\psi(\eta) \leq \xi_\psi(\wp), \quad \zeta_\psi(\eta) \geq \zeta_\psi(\wp) \end{cases} \right).$$

Proposition 3.5 ([7]). *Let $\mathcal{A} = \langle A, \psi\rangle$ be a CI ideal of $\hbar$. If the inequality $\eta * \wp \leq \partial$ holds in $\hbar$, then $\Re_A(\eta) \gg \text{rmin}\{\Re_A(\wp), \Re_A(\partial)\}$, $\Im_A(\eta) \ll \text{rmax}\{\Im_A(\wp), \Im_A(\partial)\}$, $\xi_\psi(\eta) \leq \max\{\xi_\psi(\wp), \xi_\psi(\partial)\}$ and $\zeta_\psi(\eta) \geq \min\{\zeta_\psi(\wp), \zeta_\psi(\partial)\}$.*

4. Cubic Intuitionistic Commutative Ideals

Definition 4.1. A CIS $\mathcal{A} = \langle A, \psi\rangle$ is called a *CI commutative ideal* of $\hbar$ if it satisfies conditions (3.3), (3.4) and

$$(\forall \eta, \wp, \partial \in \hbar) \begin{pmatrix} \Re_A(\eta * (\wp * (\wp * \eta))) \gg \text{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\} \\ \Im_A(\eta * (\wp * (\wp * \eta))) \ll \text{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\} \end{pmatrix} \tag{4.1}$$

$$(\forall \eta, \wp, \partial \in \hbar) \begin{pmatrix} \xi_\psi(\eta * (\wp * (\wp * \eta))) \leq \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\} \\ \zeta_\psi(\eta * (\wp * (\wp * \eta))) \geq \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\} \end{pmatrix}. \tag{4.2}$$

Example 4.2. Consider a BCK-algebra $\hbar = \{0, \theta, \phi, \varrho\}$ with the following Cayley table:

$*$	0	θ	ϕ	ϱ
0	0	0	0	0
θ	θ	0	0	θ
ϕ	ϕ	θ	0	ϕ
ϱ	ϱ	ϱ	ϱ	0

Consider a CIS $\mathcal{A} = \langle A, \psi\rangle$ in $\hbar$ in the following way:

$\hbar$	$A = \langle \Re_A, \Im_A \rangle$	$\psi = (\xi_\psi, \zeta_\psi)$
0	$\langle [0.8, 0.9], [0.0, 0.1] \rangle$	$(0.2, 0.7)$
θ	$\langle [0.6, 0.7], [0.2, 0.3] \rangle$	$(0.4, 0.5)$
ϕ	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$	$(0.6, 0.3)$
ϱ	$\langle [0.3, 0.5], [0.4, 0.5] \rangle$	$(0.6, 0.3)$

Simulations of a repetitive nature reveal that $\mathcal{A} = \langle A, \psi \rangle$ is the CI commutative ideal of $\hbar$.

Next we establish a relationship between a CI commutative ideal and a CI ideal.

Theorem 4.3. *Each CI commutative ideal of $\hbar$ is a CI ideal of $\hbar$.*

Proof. Assume that $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$. By replacing 0 for $\wp$ in (4.1) and (4.2), we obtain

$$\begin{aligned}
\Re_A(\eta * (0 * (0 * \eta))) &\gg \text{rmin}\{\Re_A((\eta * 0) * \partial), \Re_A(\partial)\} = \text{rmin}\{\Re_A(\eta * \partial), \Re_A(\partial)\}, \\
\Im_A(\eta * (0 * (0 * \eta))) &\ll \text{rmax}\{\Im_A((\eta * 0) * \partial), \Im_A(\partial)\} = \text{rmax}\{\Im_A(\eta * \partial), \Im_A(\partial)\}, \\
\xi_\psi(\eta * (0 * (0 * \eta))) &\leq \max\{\xi_\psi((\eta * 0) * \partial), \xi_\psi(\partial)\} = \max\{\xi_\psi(\eta * \partial), \xi_\psi(\partial)\}, \\
\zeta_\psi(\eta * (0 * (0 * \eta))) &\geq \min\{\zeta_\psi((\eta * 0) * \partial), \zeta_\psi(\partial)\} = \min\{\zeta_\psi(\eta * \partial), \zeta_\psi(\partial)\}.
\end{aligned}$$

Using (a3) and $0 * \eta = 0$, we get

$$\begin{aligned}
\Re_A(\eta) = \Re_A(\eta * (0 * (0 * \eta))) &\gg \text{rmin}\{\Re_A(\eta * \partial), \Re_A(\partial)\}, \\
\Im_A(\eta) = \Im_A(\eta * (0 * (0 * \eta))) &\ll \text{rmax}\{\Im_A(\eta * \partial), \Im_A(\partial)\}, \\
\xi_\psi(\eta) = \xi_\psi(\eta * (0 * (0 * \eta))) &\leq \max\{\xi_\psi(\eta * \partial), \xi_\psi(\partial)\}, \\
\zeta_\psi(\eta) = \xi_\psi(\eta * (0 * (0 * \eta))) &\geq \min\{\zeta_\psi(\eta * \partial), \zeta_\psi(\partial)\}.
\end{aligned}$$

This shows that $\mathcal{A} = \langle A, \psi \rangle$ satisfies (3.5) and (3.6). Combining (3.3) and (3.4), we get $\mathcal{A} = \langle A, \psi \rangle$ is CI ideal of $\hbar$. ■

The converse of Theorem 4.3 may not be true as shown in the following example.

Example 4.4. Consider a BCK-algebra $\hbar = \{0, \theta, \phi, \varrho, \tau\}$ with the following Cayley table:

$*$	0	θ	ϕ	ϱ	τ
0	0	0	0	0	0
θ	θ	0	θ	0	0
ϕ	ϕ	ϕ	0	0	0
ϱ	ϱ	ϱ	ϱ	0	0
τ	τ	τ	τ	ϱ	0

Consider a CIS $\mathcal{A} = \langle A, \psi \rangle$ in $\hbar$ in the following way:

$\hbar$	$A = \langle \Re_A, \Im_A \rangle$	$\psi = (\xi_\psi, \zeta_\psi)$
0	$\langle [0.5, 0.6], [0.3, 0.4] \rangle$	$(0.1, 0.8)$
θ	$\langle [0.4, 0.5], [0.4, 0.5] \rangle$	$(0.2, 0.7)$
ϕ	$\langle [0.2, 0.3], [0.5, 0.6] \rangle$	$(0.4, 0.5)$
ϱ	$\langle [0.2, 0.3], [0.5, 0.6] \rangle$	$(0.4, 0.5)$
τ	$\langle [0.2, 0.3], [0.5, 0.6] \rangle$	$(0.4, 0.5)$

It is easy to check that $\mathcal{A} = \langle A, \psi\rangle$ is a CI ideal of $\hbar$, but it is not a CI commutative ideal of $\hbar$ because $\Re_A(\phi*(\varrho*(\varrho*\phi))) \gg \text{rmin}\{\Re_A((\phi*\varrho)*0), \Re_A(0)\}$ and $\Im_A(\phi*(\varrho*(\varrho*\phi))) \ll \text{rmax}\{\Im_A((\phi*\varrho)*0), \Im_A(0)\}$ does not hold, and $\xi_\psi(\phi*(\varrho*(\varrho*\phi))) \nleq \max\{\xi_\psi((\phi*\varrho)*0), \xi_\psi(0)\}$, $\zeta_\psi(\phi*(\varrho*(\varrho*\phi))) \ngeq \max\{\zeta_\psi((\phi*\varrho)*0), \zeta_\psi(0)\}$.

To make sure that a CI ideal is also a CI commutative ideal, we set up a rule.

Theorem 4.5. *Let $\mathcal{A} = \langle A, \psi\rangle$ be a CI ideal of $\hbar$. Then $\mathcal{A} = \langle A, \psi\rangle$ is a CI commutative ideal of $\hbar$ if and only if it fulfills the conditions $\Re_A(\eta*(\wp*(\wp*\eta))) \gg \Re_A(\eta*\wp)$, $\Im_A(\eta*(\wp*(\wp*\eta))) \ll \Im_A(\eta*\wp)$, $\xi_\psi(\eta*(\wp*(\wp*\eta))) \leq \xi_\psi(\eta*\wp)$ and $\zeta_\psi(\eta*(\wp*(\wp*\eta))) \geq \zeta_\psi(\eta*\wp)$ for all $\eta, \wp \in \hbar$.*

Proof. Assume that $\mathcal{A} = \langle A, \psi\rangle$ is a CI commutative ideal of $\hbar$. Taking $\partial = 0$ in (4.1) and (4.2), and using (3.3), (3.4) and (a3), we get the conditions.

Conversely, suppose $\mathcal{A} = \langle A, \psi\rangle$ satisfies the above four conditions. As $\mathcal{A} = \langle A, \psi\rangle$ is a CI ideal, hence

$$\begin{aligned}
\Re_A(\eta*\wp) &\gg \text{rmin}\{\Re_A((\eta*\wp)*\partial), \Re_A(\partial)\},\\
\Im_A(\eta*\wp) &\ll \text{rmax}\{\Im_A((\eta*\wp)*\partial), \Im_A(\partial)\},\\
\xi_\psi(\eta*\wp) &\leq \max\{\xi_\psi((\eta*\wp)*\partial), \xi_\psi(\partial)\},\\
\zeta_\psi(\eta*\wp) &\geq \min\{\zeta_\psi((\eta*\wp)*\partial), \zeta_\psi(\partial)\},
\end{aligned}$$

for all $\eta, \wp, \partial \in \hbar$. Hence, combining with the given four conditions, we obtain (4.1) and (4.2). The proof is complete. ∎

Theorem 4.6. *In a commutative BCK-algebra $\hbar$, every CI ideal is a CI commutative ideal.*

Proof. Let $\mathcal{A} = \langle A, \psi\rangle$ be a CI ideal of a commutative BCK-algebra $\hbar$. It is sufficient to show that $\mathcal{A} = \langle A, \psi\rangle$ satisfies condition (4.1) and (4.2). Now

$$\begin{aligned}
((\eta*(\wp*(\wp*\eta)))*((\eta*\wp)*\partial))*\partial &= ((\eta*(\wp*(\wp*\eta)))*\partial)*((\eta*\wp)*\partial)\\
&\leq (\eta*(\wp*(\wp*\eta)))*(\eta*\wp)\\
&= (\eta*(\eta*\wp))*(\wp*(\wp*\eta))\\
&= 0,
\end{aligned}$$

for all $\eta, \wp, \partial \in \hbar$. Thus $(\eta*(\wp*(\wp*\eta)))*((\eta*\wp)*\partial) \leq \partial$. It follows from Proposition 3.5 that

$$\begin{aligned}
\Re_A(\eta*(\wp*(\wp*\eta))) &\gg \text{rmin}\{\Re_A((\eta*\wp)*\partial), \Re_A(\partial)\},\\
\Im_A(\eta*(\wp*(\wp*\eta))) &\ll \text{rmax}\{\Im_A((\eta*\wp)*\partial), \Im_A(\partial)\},\\
\xi_\psi(\eta*(\wp*(\wp*\eta))) &\leq \max\{\xi_\psi((\eta*\wp)*\partial), \xi_\psi(\partial)\},\\
\zeta_\psi(\eta*(\wp*(\wp*\eta))) &\geq \min\{\zeta_\psi((\eta*\wp)*\partial), \zeta_\psi(\partial)\}.
\end{aligned}$$

Hence $\mathcal{A} = \langle A, \psi\rangle$ is a CI commutative ideal of $\hbar$. ∎

Let $\mathcal{A} = \langle A, \psi \rangle$ be a CIS in a nonempty set $\hbar$. Given $(\varepsilon, \delta) \in [0,1] \times [0,1]$ and $([s_{\Re}, t_{\Re}], [s_{\Im}, t_{\Im}]) \in D[0,1] \times D[0,1]$, we consider the sets

$$\begin{aligned} &\Re_A[s_{\Re}, t_{\Re}] := \{\eta \in \hbar \mid \Re_A(\eta) \gg [s_{\Re}, t_{\Re}]\}, \\ &\Im_A[s_{\Im}, t_{\Im}] := \{\eta \in \hbar \mid \Im_A(\eta) \ll [s_{\Im}, t_{\Im}]\}, \\ &\xi_\psi[\varepsilon] := \{\eta \in \hbar \mid \xi_\psi(\eta) \leq \varepsilon\}, \\ &\zeta_\psi[\delta] := \{\eta \in \hbar \mid \zeta_\psi(\eta) \geq \delta\}. \end{aligned}$$

Theorem 4.7. *Let $\mathcal{A} = \langle A, \psi \rangle$ be a CI commutative ideal of $\hbar$, then the sets $\Re_A[s_{\Re}, t_{\Re}]$, $\Im_A[s_{\Im}, t_{\Im}]$, $\xi_\psi[\varepsilon]$ and $\zeta_\psi[\delta]$ are commutative ideals of $\hbar$ for all $([s_{\Re}, t_{\Re}], [s_{\Im}, t_{\Im}]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$.*

Proof. Assume that $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$. For any $([s_{\Re}, t_{\Re}], [s_{\Im}, t_{\Im}]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$, let $\eta \in \hbar$ be such that $\eta \in \Re_A[s_{\Re}, t_{\Re}] \cap \Im_A[s_{\Im}, t_{\Im}] \cap \xi_\psi[\varepsilon] \cap \zeta_\psi[\delta]$. Then $\Re_A(\eta) \gg [s_{\Re}, t_{\Re}]$, $\Im_A(\eta) \ll [s_{\Im}, t_{\Im}]$, $\xi_\psi(\eta) \leq \varepsilon$ and $\zeta_\psi(\eta) \geq \delta$. Now, by using (3.3) and (3.4), we get $\Re_A(0) \gg \Re_A(\eta) \gg [s_{\Re}, t_{\Re}]$, $\Im_A(0) \ll \Im_A(\eta) \ll [s_{\Im}, t_{\Im}]$, $\xi_\psi(0) \leq \xi_\psi(\eta) \leq \varepsilon$ and $\zeta_\psi(0) \geq \zeta_\psi(\eta) \geq \delta$. Thus $0 \in \Re_A[s_{\Re}, t_{\Re}] \cap \Im_A[s_{\Im}, t_{\Im}] \cap \xi_\psi[\varepsilon] \cap \zeta_\psi[\delta]$.

Suppose $(\eta * \wp) * \partial, \partial \in \Re_A[s_{\Re}, t_{\Re}] \cap \Im_A[s_{\Im}, t_{\Im}] \cap \xi_\psi[\varepsilon] \cap \zeta_\psi[\delta]$. Then $\Re_A((\eta * \wp) * \partial) \gg [s_{\Re}, t_{\Re}]$, $\Re_A(\partial) \gg [s_{\Re}, t_{\Re}]$, $\Im_A((\eta * \wp) * \partial) \ll [s_{\Im}, t_{\Im}]$, $\Im_A(\partial) \ll [s_{\Im}, t_{\Im}]$, $\xi_\psi((\eta * \wp) * \partial) \leq \varepsilon$, $\xi_\psi(\partial) \leq \varepsilon$, $\zeta_\psi((\eta * \wp) * \partial) \geq \delta$ and $\zeta_\psi(\partial) \geq \delta$. This implies

$$\begin{aligned} \Re_A(\eta * (\wp * (\wp * \eta))) &\gg \text{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\} \gg [s_{\Re}, t_{\Re}], \\ \Im_A(\eta * (\wp * (\wp * \eta))) &\ll \text{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\} \ll [s_{\Im}, t_{\Im}], \\ \xi_\psi(\eta * (\wp * (\wp * \eta))) &\leq \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\} \leq \varepsilon, \\ \zeta_\psi(\eta * (\wp * (\wp * \eta))) &\geq \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\} \geq \delta. \end{aligned}$$

Therefore, $\eta * (\wp * (\wp * \eta)) \in \Re_A[s_{\Re}, t_{\Re}] \cap \Im_A[s_{\Im}, t_{\Im}] \cap \xi_\psi[\varepsilon] \cap \zeta_\psi[\delta]$. Hence $\Re_A[s_{\Re}, t_{\Re}]$, $\Im_A[s_{\Im}, t_{\Im}]$, $\xi_\psi[\varepsilon]$ and $\zeta_\psi[\delta]$ are commutative ideals of $\hbar$. ■

Theorem 4.8. *Let $\mathcal{A} = \langle A, \psi \rangle$ be a CIS in $\hbar$ such that the non-empty sets $\Re_A[s_{\Re}, t_{\Re}]$, $\Im_A[s_{\Im}, t_{\Im}]$, $\xi_\psi[\varepsilon]$ and $\zeta_\psi[\delta]$ are commutative ideals of $\hbar$ for all $([s_{\Re}, t_{\Re}], [s_{\Im}, t_{\Im}]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$. Then $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$.*

Proof. Assume that $\Re_A[s_{\Re}, t_{\Re}]$, $\Im_A[s_{\Im}, t_{\Im}]$, $\xi_\psi[\varepsilon]$ and $\zeta_\psi[\delta]$ are non-empty commutative ideals of $\hbar$, for all $([s_{\Re}, t_{\Re}], [s_{\Im}, t_{\Im}]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$. Let $\Re_A(\eta) = [s_{\Re}, t_{\Re}]$, $\Im_A(\wp) = [s_{\Im}, t_{\Im}]$, $\xi_\psi(a) = \varepsilon$ and $\zeta_\psi(b) = \delta$ for any $\eta, \wp, a, b \in \hbar$. Since $0 \in \Re_A[s_{\Re}, t_{\Re}]$, $0 \in \Im_A[s_{\Im}, t_{\Im}]$, $0 \in \xi_\psi[\varepsilon]$ and $0 \in \zeta_\psi[\delta]$, we have $\Re_A(0) \gg [s_{\Re}, t_{\Re}] = \Re_A(\eta)$, $\Im_A(0) \ll [s_{\Im}, t_{\Im}] = \Im_A(\eta)$, $\xi_\psi(0) \leq \varepsilon = \xi_\psi(\eta)$, and $\zeta_\psi(0) \geq \delta = \zeta_\psi(\eta)$ for all $\eta \in \hbar$. Hence

$$\begin{aligned} &\Re_A(0) \text{ is an upper bound of } \{\Re_A(\eta) \mid \eta \in \hbar\}, \\ &\Im_A(0) \text{ is a lower bound of } \{\Im_A(\eta) \mid \eta \in \hbar\}, \\ &\xi_\psi(0) \text{ is a lower bound of } \{\xi_\psi(\eta) \mid \eta \in \hbar\}, \\ &\zeta_\psi(0) \text{ is an upper bound of } \{\zeta_\psi(\eta) \mid \eta \in \hbar\}. \end{aligned}$$

For any $\eta, \wp, \partial \in \hbar$, let

$$\begin{aligned}
&\text{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\} := [s_\Re, t_\Re],\\
&\text{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\} := [s_\Im, t_\Im],\\
&\max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\} := \varepsilon,\\
&\min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\} := \delta.
\end{aligned}$$

Then $\Re_A((\eta * \wp) * \partial) \gg [s_\Re, t_\Re]$, $\Re_A(\partial) \gg [s_\Re, t_\Re]$, $\Im_A((\eta * \wp) * \partial) \ll [s_\Im, t_\Im]$, $\Im_A(\partial) \ll [s_\Im, t_\Im]$, $\xi_\psi((\eta * \wp) * \partial) \leq \varepsilon$, $\xi_\psi(\partial) \leq \varepsilon$, $\zeta_\psi((\eta * \wp) * \partial) \geq \delta$, and $\zeta_\psi(\partial) \geq \delta$, that is, $(\eta * \wp) * \partial \in \Re_A[s_\Re, t_\Re]$, $\partial \in \Re_A[s_\Re, t_\Re]$, $(\eta * \wp) * \partial \in \Im_A[s_\Im, t_\Im]$, $\partial \in \Im_A[s_\Im, t_\Im]$, $(\eta * \wp) * \partial \in \xi_\psi[\varepsilon]$, $\partial \in \xi_\psi[\varepsilon]$, $(\eta * \wp) * \partial \in \zeta_\psi[\delta]$, and $\partial \in \zeta_\psi[\delta]$. It follows from hypothesis that $\eta * (\wp * (\wp * \eta)) \in \Re_A[s_\Re, t_\Re]$, $\eta * (\wp * (\wp * \eta)) \in \Im_A[s_\Im, t_\Im]$, $\eta * (\wp * (\wp * \eta)) \in \xi_\psi[\varepsilon]$, and $\eta * (\wp * (\wp * \eta)) \in \zeta_\psi[\delta]$. Thus

$$\begin{aligned}
&\Re_A(\eta * (\wp * (\wp * \eta))) \gg [s_\Re, t_\Re] = \text{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\},\\
&\Im_A(\eta * (\wp * (\wp * \eta))) \ll [s_\Im, t_\Im] = \text{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\},\\
&\xi_\psi(\eta * (\wp * (\wp * \eta))) \leq \varepsilon = \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\},\\
&\zeta_\psi(\eta * (\wp * (\wp * \eta))) \geq \delta = \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\}.
\end{aligned}$$

Therefore $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$. ∎

The sets $\{\eta \in \hbar : \Re_A(\eta) = \Re_A(0)\}$, $\{\eta \in \hbar : \Im_A(\eta) = \Im_A(0)\}$, $\{\eta \in \hbar : \xi_\psi(\eta) = \xi_\psi(0)\}$ and $\{\eta \in \hbar : \zeta_\psi(\eta) = \zeta_\psi(0)\}$ are denoted by $T_{\Re_A}$, $T_{\Im_A}$, T_{ξ_ψ} and T_{ζ_ψ} respectively. These four sets are also commutative ideals of $\hbar$.

Theorem 4.9. *Let $\mathcal{A} = \langle A, \psi \rangle$ be a CI commutative ideal of $\hbar$. Then the sets $T_{\Re_A}$, $T_{\Im_A}$, T_{ξ_ψ} and T_{ζ_ψ} are commutative ideals of $\hbar$.*

Proof. Assume that $\mathcal{A} = \langle A, \psi \rangle$ be a CI commutative ideal of $\hbar$. Then it is self-evident that $0 \in T_{\Re_A} \cap T_{\Im_A} \cap T_{\xi_\psi} \cap T_{\zeta_\psi}$. Let $\eta, \wp, \partial \in \hbar$ be such that $(\eta * \wp) * \partial \in T_{\Re_A} \cap T_{\Im_A} \cap T_{\xi_\psi} \cap T_{\zeta_\psi}$ and $\partial \in T_{\Re_A} \cap T_{\Im_A} \cap T_{\xi_\psi} \cap T_{\zeta_\psi}$. Then $\Re_A((\eta * \wp) * \partial) = \Re_A(0) = \Re_A(\partial)$, $\Im_A((\eta * \wp) * \partial) = \Im_A(0) = \Im_A(\partial)$, $\xi_\psi((\eta * \wp) * \partial) = \xi_\psi(0) = \xi_\psi(\partial)$ and $\zeta_\psi((\eta * \wp) * \partial) = \zeta_\psi(0) = \zeta_\psi(\partial)$. Thus

$$\begin{aligned}
\Re_A(\eta * (\wp * (\wp * \eta))) &\gg \text{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\} = \text{rmin}\{\Re_A(0), \Re_A(0)\} = \Re_A(0),\\
\Im_A(\eta * (\wp * (\wp * \eta))) &\ll \text{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\} = \text{rmax}\{\Im_A(0), \Im_A(0)\} = \Im_A(0),\\
\xi_\psi(\eta * (\wp * (\wp * \eta))) &\leq \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\} = \max\{\xi_\psi(0), \xi_\psi(0)\} = \xi_\psi(0),\\
\zeta_\psi(\eta * (\wp * (\wp * \eta))) &\geq \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\} = \min\{\zeta_\psi(0), \zeta_\psi(0)\} = \zeta_\psi(0).
\end{aligned}$$

It follows from (3.3) and (3.4) that $\Re_A(\eta * (\wp * (\wp * \eta))) = \Re_A(0)$, $\Im_A(\eta * (\wp * (\wp * \eta))) = \Im_A(0)$, $\xi_\psi(\eta * (\wp * (\wp * \eta))) = \xi_\psi(0)$ and $\zeta_\psi(\eta * (\wp * (\wp * \eta))) = \zeta_\psi(0)$ i.e., $\eta * (\wp * (\wp * \eta)) \in T_{\Re_A} \cap T_{\Im_A} \cap T_{\xi_\psi} \cap T_{\zeta_\psi}$. Hence, the sets $T_{\Re_A}$, $T_{\Im_A}$, T_{ξ_ψ} and T_{ζ_ψ} are commutative ideals of $\hbar$. ∎

Theorem 4.10. *If P is a commutative ideal of $\hbar$, then there is a CI commutative ideal $\mathcal{A} = \langle A, \psi \rangle$ of $\hbar$ such that $\Re_A[s_\Re, t_\Re] = \Im_A[s_\Im, t_\Im] = \xi_\psi[\varepsilon] = \zeta_\psi[\delta] = P$ for some $([s_\Re, t_\Re], [s_\Im, t_\Im]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$.*

Proof. Assume that $\mathcal{A} = \langle A, \psi \rangle$ is a CIS in $\hbar$ denoted by

$$\Re_A(\eta) =: \begin{cases} [s_\Re, t_\Re], & \text{if } \eta \in P \\ 0, & \text{if } \eta \notin P, \end{cases} \quad \Im_A(\eta) =: \begin{cases} [s_\Im, t_\Im], & \text{if } \eta \in P \\ 0, & \text{if } \eta \notin P, \end{cases}$$

$$\xi_\psi(\eta) =: \begin{cases} \varepsilon, & \text{if } \eta \in P \\ 0, & \text{if } \eta \notin P, \end{cases} \quad \text{and} \quad \zeta_\psi(\eta) =: \begin{cases} \delta, & \text{if } \eta \in P \\ 0, & \text{if } \eta \notin P, \end{cases}$$

for fixed $([s_\Re, t_\Re], [s_\Im, t_\Im]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$. Let $\eta, \wp, \partial \in \hbar$. We shall divide the cases into the following categories in order to demonstrate that $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$.

Case I : If $(\eta * \wp) * \partial \in P$ and $\partial \in P$, then $\eta * (\wp * (\wp * \eta)) \in P$ by (I_3); hence,

$$\begin{aligned} \Re_A((\eta * \wp) * \partial) &= \Re_A(\partial) = \Re_A(\eta * (\wp * (\wp * \eta))) = [s_\Re, t_\Re], \\ \Im_A((\eta * \wp) * \partial) &= \Im_A(\partial) = \Im_A(\eta * (\wp * (\wp * \eta))) = [s_\Im, t_\Im], \\ \xi_\psi((\eta * \wp) * \partial) &= \xi_\psi(\partial) = \xi_\psi(\eta * (\wp * (\wp * \eta))) = \varepsilon, \\ \zeta_\psi((\eta * \wp) * \partial) &= \zeta_\psi(\partial) = \zeta_\psi(\eta * (\wp * (\wp * \eta))) = \delta, \end{aligned}$$

and so

$$\begin{aligned} \Re_A(\eta * (\wp * (\wp * \eta))) &= \operatorname{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\}, \\ \Im_A(\eta * (\wp * (\wp * \eta))) &= \operatorname{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\}, \\ \xi_\psi(\eta * (\wp * (\wp * \eta))) &= \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\}, \\ \zeta_\psi(\eta * (\wp * (\wp * \eta))) &= \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\}. \end{aligned}$$

Case II : If $(\eta * \wp) * \partial \notin P$ and $\partial \notin P$, then

$$\begin{aligned} \Re_A((\eta * \wp) * \partial) &= \Re_A(\partial) = 0, \\ \Im_A((\eta * \wp) * \partial) &= \Im_A(\partial) = 0, \\ \xi_\psi((\eta * \wp) * \partial) &= \xi_\psi(\partial) = 0, \\ \zeta_\psi((\eta * \wp) * \partial) &= \zeta_\psi(\partial) = 0. \end{aligned}$$

Hence

$$\begin{aligned} \Re_A(\eta * (\wp * (\wp * \eta))) &\gg \operatorname{rmin}\{\Re_A((\eta * \wp) * \partial), \Re_A(\partial)\}, \\ \Im_A(\eta * (\wp * (\wp * \eta))) &\ll \operatorname{rmax}\{\Im_A((\eta * \wp) * \partial), \Im_A(\partial)\}, \\ \xi_\psi(\eta * (\wp * (\wp * \eta))) &\leq \max\{\xi_\psi((\eta * \wp) * \partial), \xi_\psi(\partial)\}, \\ \zeta_\psi(\eta * (\wp * (\wp * \eta))) &\geq \min\{\zeta_\psi((\eta * \wp) * \partial), \zeta_\psi(\partial)\}. \end{aligned}$$

Case III : If exactly one of $(\eta * \wp) * \partial$ and ∂ is not in P, then

exactly one of $\Re_A((\eta * \wp) * \partial)$ and $\Re_A(\partial)$ is equal to 0,
exactly one of $\Im_A((\eta * \wp) * \partial)$ and $\Im_A(\partial)$ is equal to 0,
exactly one of $\xi_\psi((\eta * \wp) * \partial)$ and $\xi_\psi(\partial)$ is equal to 0,
exactly one of $\zeta_\psi((\eta * \wp) * \partial)$ and $\zeta_\psi(\partial)$ is equal to 0.

Hence condition (4.1) and (4.2) satisfies.

By combining the preceding three situations, we see that (4.1) and (4.2) hold true for all $\eta, \wp, \partial \in \hbar$. Given that $0 \in I$, it is obvious that $\Re_A(0) = [s_\Re, t_\Re] \gg \Re_A(\eta)$, $\Im_A(0) = [s_\Im, t_\Im] \ll \Im_A(\eta)$, $\xi_\psi(0) = \varepsilon \leq \xi_\psi(\eta)$, $\zeta_\psi(0) = \delta \geq \zeta_\psi(\eta)$, for all $\eta \in \hbar$. Thus condition (3.3) and (3.4) holds. Therefore, $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$ and obviously $\Re_A[s_\Re, t_\Re] = \Im_A[s_\Im, t_\Im] = \xi_\psi[\varepsilon] = \zeta_\psi[\delta] = P$. The proof is complete. ■

5. CI Extension Property

Theorem 5.1 ([13]). *Assume that I and A are ideals of $\hbar$ with $I \subseteq A$. If I is a commutative ideal, then A must be as well.*

Definition 5.2. Assume that $\mathcal{A} = \langle A, \psi \rangle$ and $\mathcal{B} = \langle B, \vartheta \rangle$ are two CISs of $\hbar$. Then $\mathcal{B} = \langle B, \vartheta \rangle$ is referred to as the CI extension of $\mathcal{A} = \langle A, \psi \rangle$, indicated as $\mathcal{A} \lesssim \mathcal{B}$, if $\Re_A(\eta) \ll \Re_B(\eta)$, $\Im_A(\eta) \gg \Im_B(\eta)$, $\xi_\psi(\eta) \geq \xi_\vartheta(\eta)$ and $\zeta_\psi(\eta) \leq \zeta_\vartheta(\eta)$, for all $\eta \in \hbar$.

Following that, we demonstrate the CI extension theorem for CI commutative ideals.

Theorem 5.3. *Let $\mathcal{A} = \langle A, \psi \rangle$ and $\mathcal{B} = \langle B, \vartheta \rangle$ be two CI ideals of $\hbar$ such that $\mathcal{A} \lesssim \mathcal{B}$ and $\Re_A(0) = \Re_B(0)$, $\Im_A(0) = \Im_B(0)$, $\xi_\psi(0) = \xi_\vartheta(0)$ and $\zeta_\psi(0) = \zeta_\vartheta(0)$. If $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$, then so is $\mathcal{B} = \langle B, \vartheta \rangle$.*

Proof. To establish that $\mathcal{B} = \langle B, \vartheta \rangle$ is a CI commutative ideal of $\hbar$ it suffices to demonstrate that for every $([s_\Re, t_\Re], [s_\Im, t_\Im]) \in D[0,1] \times D[0,1]$ and $(\varepsilon, \delta) \in [0,1] \times [0,1]$, $\Re_B[s_\Re, t_\Re]$, $\Im_B[s_\Im, t_\Im]$, $\xi_\vartheta[\varepsilon]$ and $\zeta_\vartheta[\delta]$ are either empty or commutative ideals of $\hbar$. Assume that $\Re_B[s_\Re, t_\Re]$, $\Im_B[s_\Im, t_\Im]$, $\xi_\vartheta[\varepsilon]$ and $\zeta_\vartheta[\delta]$ are not empty and $\mathcal{A} \lesssim \mathcal{B}$. Obviously, if $\eta \in \Re_A[s_\Re, t_\Re] \cap \Im_A[s_\Im, t_\Im] \cap \xi_\psi[\varepsilon] \cap \zeta_\psi[\delta]$ then $\Re_A(\eta) \gg [s_\Re, t_\Re]$, $\Im_A(\eta) \ll [s_\Im, t_\Im]$, $\xi_\psi(\eta) \leq \varepsilon$ and $\zeta_\psi(\eta) \geq \delta$. Hence $\Re_B(\eta) \gg [s_\Re, t_\Re]$, $\Im_B(\eta) \ll [s_\Im, t_\Im]$, $\xi_\vartheta(\eta) \leq \varepsilon$ and $\zeta_\vartheta(\eta) \geq \delta$, that is $\eta \in \Re_B[s_\Re, t_\Re] \cap \Im_B[s_\Im, t_\Im] \cap \xi_\vartheta[\varepsilon] \cap \zeta_\vartheta[\delta]$. So $\Re_A[s_\Re, t_\Re] \subseteq \Re_B[s_\Re, t_\Re]$, $\Im_A[s_\Im, t_\Im] \subseteq \Im_B[s_\Im, t_\Im]$, $\xi_\psi[\varepsilon] \subseteq \xi_\vartheta[\varepsilon]$ and $\zeta_\psi[\delta] \subseteq \zeta_\vartheta[\delta]$. By the hypothesis, $\mathcal{A} = \langle A, \psi \rangle$ is a CI commutative ideal of $\hbar$, it follows from Theorem 4.7 that the sets $\Re_A[s_\Re, t_\Re]$, $\Im_A[s_\Im, t_\Im]$, $\xi_\psi[\varepsilon]$ and $\zeta_\psi[\delta]$ are commutative ideals of $\hbar$. By Theorem 5.1, $\Re_B[s_\Re, t_\Re]$, $\Im_B[s_\Im, t_\Im]$, $\xi_\vartheta[\varepsilon]$ and $\zeta_\vartheta[\delta]$ are also commutative ideals of $\hbar$. As a result of Theorem 4.8, we get $\mathcal{B} = \langle B, \vartheta \rangle$ is a CI commutative ideal of $\hbar$. The evidence is conclusive. ∎

6. Conclusions

In this paper, taking into account CIS, we have presented CI commutative ideals in BCK-algebras, and discussed their different properties. We have discussed the relationship between a CI subalgebra, a CI ideal, and a CI commutative ideal. There is a criterion for a CI ideal to be a CI commutative ideal. The extension theorem of CI commutative ideals has been given. In our future study of the CI structure of BCK-algebra, we will look into the following:
(i) to obtain the product of CI subalgebras, ideals and commutative ideals of a BCK-algebra,
(ii) to obtain images and preimages of CI commutative ideals under homomorphism,
(iii) to assess relationship between CI implicative ideals, CI positive implicative ideals and CI commutative ideals of a BCK-algebra.

Acknowledgements

This work was supported by the revenue budget in 2022, School of Science, University of Phayao.

References

[1] Y.B. Jun, C.S. Kim, K.O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (2012) 83–98.

[2] Y.B. Jun, G. Muhiuddin, M.A. Ozturk, E.H. Roh, Cubic soft ideals in BCK/BCI-algebras, J. Comput. Anal. Appl. 22 (2017) 929–940.

[3] G. Muhiuddin, A.M. Al-roqi, Cubic soft sets with applications in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. 8 (2014) 291–304.

[4] G. Muhiuddin, F. Feng, Y.B. Jun, Subalgebras of BCK/BCI-algebras based on cubic soft sets, The Scientific World Journal 2014 (2014) Article ID 458638.

[5] T. Senapati, C.S. Kim, M. Bhowmik, M. Pal, Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy Inf. Eng. 7 (2) (2015) 129–149.

[6] T. Senapati, Y.B. Jun, K.P. Shum, Cubic set structure applied in UP-algebras, Discrete Math. Algorithms Appl. 10 (4) (2018) Article ID 1850049.

[7] Y.B. Jun, A novel extension of cubic sets and its applications in BCK/BCI-algebras, Ann. Fuzzy. Math. Inform. 14 (5) (2017) 475–486.

[8] T. Senapati, C. Jana, M. Pal, Y.B. Jun, Cubic intuitionistic q-ideals of BCI-algebras, Symmetry 10 (12) (2018) Article no. 752.

[9] T. Senapati, Y.B. Jun, G. Muhiuddin, K.P. Shum, Cubic intuitionistic structures applied to ideals of BCI-algebras, An. St. Univ. Ovidius Constanta 25 (2) (2019) 213–232.

[10] T. Senapati, Y.B. Jun, K.P. Shum, Cubic intuitionistic subalgebras and closed cubic intuitionistic ideals of B-algebras, J. Intell. Fuzzy Systems 36 (2) (2019) 1563–1571.

[11] T. Senapati, Y.B. Jun, K.P. Shum, Cubic intuitionistic structure of KU-algebras, Afr. Mat. 31 (2020) 237–248.

[12] T. Senapati, Y.B. Jun, K.P. Shum, Cubic intuitionistic implicative ideals of BCK-algebras, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. 91 (2021) 273–282.

[13] J. Meng, Commutative ideals in BCK-algebras, Pure Appl. Math. (in China) 9 (1991) 49–53.

[14] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986) 87–96.

[15] K.T. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31 (1989) 343–349.