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$df-$Statistical Convergence of Order α and $df-$Strong Cesàro Summability of Order α in Accordance to a Modulus in Metric Spaces

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Abstract In the current study we present $df-$statistical convergence of order α and $df-$strong Cesàro summability of order α in accordance to a modulus for a sequence in a metric space. Furthermore we introduce the connections between the sets of $df-$statistically convergent sequences of order α and between the sets of $df-$strongly Cesàro summable sequences of order α in accordance to a modulus for various values α and under some conditions on f. Besides this we introduce the relationships between the set of $df-$statistically convergent sequences of order α and the set of $df-$strongly Cesàro summable sequences of order α in accordance to a modulus for various values α and under some conditions on f.

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1. Introduction

The thinking of statistical convergence first was put forward by Zygmund [1] in 1935. Statistical convergence was acquainted for the first time by Steinhaus [2] and Fast [3] and then by Schoenberg [4] detachedly. Some mathematicions studied also the statistical convergence of a sequence along density of subsets of natural numbers that we could mention R. C. Buck [5] for instance. In the last decades and under different names the subject was discussed in many different theories such as in the theory of Fourier analysis, number theory, ergodic theory, measure theory, trigonometric series and Banach spaces. It was additionally studied from the sequence spaces and summability theory point of view and via summability theory by Fridy [6], Connor [7], Savaş [8], Mursaleen [9], Fridy and Orhan [10], Móricz [11], Rath and Tripathy [12], Salat [13], Belen and Mohiuddine [14] and somebody else.

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In general, the statistical convergence has been defined and investigated for the sequences of real or complex numbers. This concept also has been studied in metric spaces by Küçükaslan et al. [15], Bilalov and Nazarova [16] and Kayan and Colak ([17], [18]). In this paper, we study these concepts with order of a real number α between 0 and 1 by using a modulus function in metric spaces.

The order of statistical convergence of a sequence of positive linear operators was introduced by Gadjiev and Orhan [19]. The statistical convergence of order α $(0 < \alpha \leq 1)$ and strong p-Cesàro summability of order α were introduced and investigated by Çolak [20] for number sequences, using the notion α - density of a subset of the set $\mathbb{N}$ of positive integers.

The thinking of a modulus function was introduced by Nakano [21] in 1953. Ruckle [22] and Maddox [23] have acquainted and debated some features of sequence spaces defined by help of a modulus function. Other than them, Connor [24], Ghosh and Srivastava [25], Bhardwaj and Singh ([26], [27], [28]), Çolak [29], Altin and Et [30] and some others have utilized a modulus function to build some sequence spaces. Recently, Aizpuru et. al. [31] defined the statistical convergence with the help of modulus functions and also Bhardwaj and Dhawan [32] studied the statistical convergence of order α and strong Cesàro summability of order α in accordance to a modulus for number sequences.

We now remember some descriptions those will be required afterwards.

Let $\mathbb{N}$ be the set of all positive integers. A number sequence $x = (x_k)$ is called as *statistically convergent* to the number l if for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ has natural density zero, where the natural density of a subset $K \subset \mathbb{N}$ is described by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

and $|\{k \leq n : k \in K\}|$ indicates the number of elements of the set $\{k \leq n : k \in K\}$. It is apparent that $\delta(\mathbb{N}) = 1$ and $\delta(K) = 0$ on condition that $K \subset \mathbb{N}$ is a finite set of positive integers and $\delta(K^c) = \delta(\mathbb{N}) - \delta(K) = 1 - \delta(K)$, where $K^c = \mathbb{N} - K$. When a sequence is statistically convergent to l we use the notation $S - \lim x_k = l$ to show it. Furthermore we indicate by S the set of all statistically convergent sequences.

Definition 1.1. [20] Let $\alpha \in (0, 1]$ be any real number. The $\alpha -$ *density* of a subset $H \subset \mathbb{N}$ is described by

$$\delta_\alpha (H) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : k \in H\}|$$

where the limit exists (finite or infinite).

Obviously we have $\delta_\alpha (H) = 0$ for every $\alpha \in (0, 1]$ provided that $H \subset \mathbb{N}$ is a finite subset. Although $\delta_\alpha (H^c) = 1 - \delta_\alpha (H)$ for $\alpha = 1$, the equality $\delta_\alpha (H^c) = 1 - \delta_\alpha (H)$ is not true for $0 < \alpha < 1$ in general.

Also $\alpha-$density $\delta_\alpha (H)$ degrades to the natural density $\delta (H)$ of a subset $H \subset \mathbb{N}$ in case $\alpha = 1$. Note that

$$\delta_\alpha (\mathbb{N}) = \begin{cases} 1, & \alpha = 1 \\ \infty, & \alpha < 1 \end{cases}$$

and for every subset $H \subseteq \mathbb{N}$, $\delta_\alpha (H) = 0$ if $\alpha > 1$.

Let $0 < \alpha \leq 1$ be given. A number sequence (x_k) is called as *statistically convergent of order* α if there exists a number l that provides

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right| = 0$$

for every $\varepsilon > 0$ [20]. Then we say that (x_k) is statistically convergent of order α, to l. We use the notation $S^{\alpha} - \lim x_k = l$ in case the sequence (x_k) is statistically convergent of order α, to l. We will use the notation S^{α} to indicate the set of all sequences which are statistically convergent of order α.

Lemma 1.2. [20] *Let* $E \subseteq \mathbb{N}$. *Then* $\delta_{\beta}(E) \leq \delta_{\alpha}(E)$ *if* $0 < \alpha \leq \beta \leq 1$.

A sequence $x = (x_k)$ is called as *strongly Cesàro summable* to L, if there exists a complex number L that provides

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$$

and the set of all strongly Cesàro summable sequences is indicated by $[C, 1]$, that is

$$[C,1] = \left\{ x = (x_k) : \lim_{n} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ for some } L \in \mathbb{C} \right\}.$$

The idea of a modulus function was established by Nakano [21]. We remember that a function $f : [0, \infty) \to [0, \infty)$ is called a modulus (see [22], [23]) if

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

In accordance with these features it is obvious that a modulus function must be continuous all over $[0, \infty)$. It is easy to check that a modulus may be bounded or unbounded. For instance, $f(x) = x^p$, where $0 < p \leq 1$, is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded.

Furthermore we have $f(mu) \leq \ mf(u)$ and so that $f(m) \leq \ m\, f(1)$ for every $m \in \mathbb{N}$ from (ii) and $\lim_{t\to\infty} \frac{f(t)}{t}$ exists by Proposition 1 of Maddox [33].

2. *df*−Statistical Convergence of Order α in Accordance to a Modulus in a Metric Space

In this section using an unbounded modulus function f we define and study the *df* -statistical convergence of order α for sequences in a metric space.

Definition 2.1. [32] Let f be an unbounded modulus function and α be any real number that provides $0 < \alpha \leq 1$. f_{α}*-density* of a subset A of $\mathbb{N}$ is described by

$$\delta_f^{\alpha}(A) = \lim_{n\to\infty} \frac{1}{f(n^{\alpha})} f\left(\left|\{k \leq n : \ k \in A\}\right|\right)$$

if this limit is existing.

As will be noted, when $\alpha = 1$, f_{α}-density returns to f-density. In case $f(x) = x$, f_{α}-density becomes α-density. If $\alpha = 1$ and $f(x) = x$, then f_{α}-density reduces to the natural density.

The equality $\delta_f^{\alpha}(A)+\delta_f^{\alpha}(\mathbb{N}-A)=1$ does not hold for $\alpha\in(0,1]$ and an unbounded modulus f, in general. For instance, if we take $f(x)=x^p$, $0<p\leq 1$, $\alpha\in(0,1)$ and $A=\{2n:\ n\in\mathbb{N}\}$, then $\delta_f^{\alpha}(A)=\delta_f^{\alpha}(\mathbb{N}-A)=\infty$. Also, finite sets have zero f_α-density for any unbounded modulus f and $\alpha\in(0,1]$ (see [32]).

Lemma 2.2. *Let α be any real number such that $0<\alpha\leq 1$, $E\subset\mathbb{N}$ and f be an unbounded modulus function. If $\delta_f^{\alpha}(E)=0$, then $\mathbb{N}-E\neq\emptyset$.*

For any unbounded modulus f and $\alpha\in(0,1]$, if $\delta_f^{\alpha}(A)=0$ then $\delta^{\alpha}(A)=0$, but the inverse of this need not be true (see [32]). Namely, a set having zero α-density for some $\alpha\in(0,1]$ might have non-zero f_α-density for some unbounded modulus f, with the same α. Similarly a set having zero natural density might have non-zero f_α-density for some unbounded modulus f and $\alpha\in(0,1]$. For example, let $f(x)=\log(x+1)$ and $A=\{1,4,9,\ldots\}$. Then $\delta(A)=0$ and $\delta^{\alpha}(A)=0$ for $\alpha\in(\frac{1}{2},1]$ but $\delta_f^{\alpha}(A)\geq\delta_f(A)=\frac{1}{2}$ and therefore $\delta_f^{\alpha}(A)\neq 0$.

If $A\subseteq\mathbb{N}$ has zero f_α-density for some unbounded modulus f and for some $\alpha\in(0,1]$, then it has zero α-density and hence zero natural density (see [32].)

Lemma 2.3. [32] *Let f be an unbounded modulus and $A\subseteq\mathbb{N}$. If $0<\alpha\leq\beta\leq 1$, then $\delta_f^{\beta}(A)\leq\delta_f^{\alpha}(A)$.*

Thus, for any unbounded modulus f and $0<\alpha\leq\beta\leq 1$, if A has zero f_α-density in that case, it has zero f_β-density. Specially, a set having zero f_α-density for some $\alpha\in(0,1]$ has zero f-density. But, the inverse is not correct. For instance, let $f(x)=x^p$ for $0<p\leq 1$ and $A=\{1,4,9,\ldots\}$. Then

$$\delta_f(A)=\lim_{n\to\infty}\frac{f(|\{k\leq n:\ k\in A\}|)}{f(n)}\leq\lim_{n\to\infty}\frac{f([\sqrt{n}])}{f(n)}=\lim_{n\to\infty}\frac{[\sqrt{n}]^p}{n^p}=0$$

but, since

$$\frac{f(|\{k\leq n:\ k\in A\}|)}{f(n^{\alpha})}=\frac{[\sqrt{n}]^p}{(n^{\alpha})^p}=\frac{[\sqrt{n}]^p}{(\sqrt{n})^p}\cdot\frac{(\sqrt{n})^p}{n^{\alpha p}}=\frac{[\sqrt{n}]^p}{(\sqrt{n})^p}\cdot\frac{1}{n^{p(\alpha-\frac{1}{2})}}$$

taking limit as $n\to\infty$ on both sides for any $\alpha\in(0,\frac{1}{2})$ we get $\delta_f^{\alpha}(A)=\infty$ as $\lim_{n\to\infty}\frac{[\sqrt{n}]^p}{(\sqrt{n})^p}$ is finite. Here $[r]$ denotes the integer part of real number r.

Definition 2.4. Let f be an unbounded modulus function and let $0<\alpha\leq 1$ be given. In that case, a sequence $x=(x_k)$ in metric space (X,d) is called as *df -statistically convergent of order* α to $x_\circ$ or $S_{df}^{\alpha}-$*convergent* to $x_\circ$ if there exists a point $x_\circ\in X$ that provides

$$\delta_f^{\alpha}(\{k\in\mathbb{N}:x_k\notin B_\varepsilon(x_\circ)\})=0$$

that is

$$\lim_{n\to\infty}\frac{1}{f(n^{\alpha})}f(|\{k\leq n:x_k\notin B_\varepsilon(x_\circ)\}|)=0$$

for every $\varepsilon>0$, where $B_\varepsilon(x_\circ)=\{x\in X:d(x,x_\circ)<\varepsilon\}$ is the open ball of radius r and center $x_\circ$.

If a sequence (x_k) is *df* -statistically convergent of order α to $x_\circ$, we show it with $S_{df}^{\alpha}-\lim x_k=x_\circ$. The set of all *df* -statistically convergent sequences of order α in the metric space (X,d) will be indicated by $S_{df}^{\alpha}(X)$. As will be noted;

In case $\alpha = 1$, S_{df}^{α}−convergence returns to S_{df}− convergence [34].
In case $f(x) = x$, S_{df}^{α}−convergence becomes S_{d}^{α}−convergence [18].
In case $\alpha = 1$ and $f(x) = x$, S_{df}^{α}−convergence becomes S_{d}−convergence. [15].

Lemma 2.5. *Let f be an unbounded modulus function and let $0 < \alpha \leq 1$ be given. If a sequence $x = (x_k)$ is df -statistically convergent of order α, in that case, its limit is unique.*

Proof. Suppose that $S_{df}^{\alpha} - \lim x_k = x_\circ$ and $S_{df}^{\alpha} - \lim x_k = x'_\circ$. Let $\varepsilon > 0$ be given and define the sets $K_1(\varepsilon) = \left\{k \leq n : x_k \notin B_{\frac{\varepsilon}{2}}(x_\circ)\right\}$ and $K_2(\varepsilon) = \left\{k \leq n : x_k \notin B_{\frac{\varepsilon}{2}}(x'_\circ)\right\}$. Since $S_{df}^{\alpha} - \lim x_k = x_\circ$, we have $\delta_f^{\alpha}(K_1(\varepsilon)) = 0$ and similarly, since $S_{df}^{\alpha} - \lim x_k = x'_\circ$, we have $\delta_f^{\alpha}(K_2(\varepsilon)) = 0$. Let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. In that case, $\delta_f^{\alpha}(K(\varepsilon)) = 0$ and from Lemma 2.2 it follows that $\mathbb{N} - K(\varepsilon) \neq \emptyset$. Thus for any $k \in \mathbb{N} - K(\varepsilon)$, we may write

$$d(x_\circ, x'_\circ) \leq d(x_\circ, x_k) + d(x_k, x'_\circ) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we get $d(x_\circ, x'_\circ) = 0$, i.e. $x_\circ = x'_\circ$. ■

Remark 2.6. *df* -statistical convergence of order α is not well defined for $\alpha > 1$. To show this let f be an unbounded modulus such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\alpha > 1$. Then it can easily be shown that for the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} a, & k = 2m \\ b, & k \neq 2m \end{cases} \quad m = 1, 2, 3, \ldots$$

we have $S_{df}^{\alpha} - \lim x_k = a$ and $S_{df}^{\alpha} - \lim x_k = b$. But this contradicts to Lemma 2.5.

Remark 2.7. It is wide-open that any convergent sequence is *df* -statistically convergent of order α for any unbounded modulus f and $\alpha \in (0, 1]$ in a metric space (X, d). However the inverse is not correct in general. For instance, take into account the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & k = m^2 \\ 0, & k \neq m^2 \end{cases} \quad m = 1, 2, 3, \ldots$$

in the space $X = \mathbb{R}$ with the usual (absolute value) metric and the unbounded modulus $f(x) = x^p$, $0 < p \leq 1$. Since

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{f(n^\alpha)} f(|\{k \leq n : \ d(x_k, 0) \geq \varepsilon\}|) &\leq \lim_{n\to\infty} \frac{f(\sqrt{n})}{f(n^\alpha)} \\ &= \lim_{n\to\infty} \frac{(\sqrt{n})^p}{(n^\alpha)^p} = 0, \end{aligned}$$

we have $x = (x_k) \in S_{df}^{\alpha}(X)$ for $\alpha \in (\frac{1}{2}, 1]$, but it is not convergent.

Theorem 2.8. *Let (X, d) be a metric space, f be an unbounded modulus function and $0 < \alpha \leq \beta \leq 1$. Then $S_{df}^{\alpha}(X) \subseteq S_{df}^{\beta}(X)$ and the inclusion is exact for some $\alpha < \beta$.*

Proof. It is easy to see that $S_{df}^{\alpha}(X) \subseteq S_{df}^{\beta}(X)$, since $0 < \alpha \leq \beta \leq 1$ and f is increasing. To show that the containment is exact, take into account the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} a, & k = m^3 \\ & \\ b, & k \neq m^3 \end{cases} \quad m = 1, 2, 3, \ldots$$

where $a, b \in X$ are fixed points with $a \neq b$ and take the modulus $f(x) = x^p$, $0 < p \leq 1$. Then $x \in S_{df}^{\beta}(X)$ for $\beta \in (\frac{1}{3}, 1]$, but $x \notin S_{df}^{\alpha}(X)$ for $\alpha \in \left(0, \frac{1}{3}\right)$. ■

Corollary 2.9. *Let (X, d) be a metric space, $0 < \alpha \leq 1$ and f be an unbounded modulus. Then if a sequence (x_k) is df -statistically convergent of order α to a point $x_\circ \in X$, in that case, it is df -statistically convergent to $x_\circ$, i.e., $S_{df}^{\alpha}(X) \subseteq S_{df}(X)$ and the containment is exact for some $\alpha < 1$.*

Corollary 2.10. *Let (X, d) be a metric space and f be an unbounded modulus function and $\alpha, \beta \in (0, 1]$. If $S_{df}^{\alpha} - \lim x_k = x_\circ$ and $S_{df}^{\beta} - \lim x_k = x'_\circ$ then $x_\circ = x'_\circ$.*

Proof. Let $S_{df}^{\alpha} - \lim x_k = x_\circ$ and $S_{df}^{\beta} - \lim x_k = x'_\circ$. In view of Corollary 2.9, we know that $S_{df}^{\alpha}(X) \subset S_{df}(X)$ for every $\alpha \in (0, 1]$. Hence the sequence $x = (x_k)$ is *df* -statistically convergent both to $x_\circ$ and to $x'_\circ$, independently of α and β. Since the limit of a *df*-statistically convergent sequence is unique, we get $x_\circ = x'_\circ$. ■

Now we can give the next theorem with an example for strictness of inclusion, without proof.

Theorem 2.11. *Let (X, d) be a metric space and f be an unbounded modulus function and $0 < \alpha \leq 1$. Then*
i) $S_{df}^{\alpha}(X) \subseteq S_{d}^{\alpha}(X)$ and the containment is exact for some α,
ii) $S_{df}^{\alpha}(X) \subseteq S_{d}(X)$ and the containment is exact for some α.

Proof. One may give the proof easily. We only show that the inclusions are strict. For this consider the metric space $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and the sequence (x_k) defined by

$$x_k = \begin{cases} k, & k = m^2 \\ & \\ 0, & k \neq m^2 \end{cases} \quad m = 1, 2, 3, \ldots . \tag{1}$$

Let $f(x) = \log(x + 1)$. Then $x \in S^{\alpha}$ for $\alpha \in (\frac{1}{2}, 1]$ and hence $x \in S$. But $x \notin S_f^{\alpha}$ for any α, since

$$\delta_f^{\alpha}(\{k \in \mathbb{N} : \ |x_k - 0| \geq \varepsilon\}) \geq \delta_f(\{k \in \mathbb{N} : \ |x_k - 0| \geq \varepsilon\}) = \frac{1}{2}(\neq 0).$$

■

In summary, the inclusion relations among the sets $c(X)$, $S_d(X)$, $S_d^{\alpha}(X)$, $S_{df}(X)$ and $S_{df}^{\alpha}(X)$ are as in the schema:

$$\begin{array}{ccccc} & & S_{df}(X) & \subseteq & S_d(X) \\ & & \cup & & \\ c(X) & \subseteq & S_{df}^{\alpha}(X) & \subseteq & S_d^{\alpha}(X) \end{array}$$

Corollary 2.12. *Let (X, d) be a metric space, f and g be two unbounded modulus functions and $0 < \alpha \leq 1$. If $S_{df}^{\alpha} - \lim x_k = x_\circ$ and $S_{dg}^{\alpha} - \lim x_k = x'_\circ$, then $x_\circ = x'_\circ$.*

Proof. In the light of Theorem 2.11, we know that $S_{df}^{\alpha}(X) \subseteq S_{d}^{\alpha}(X)$ for every unbounded modulus f. Therefore any sequence $(x_k) \in S_{df}^{\alpha}(X) \cap S_{dg}^{\alpha}(X)$ is d-statistically convergent of order α both to $x_{\circ}$ and to $x_{\circ}'$, independently of f and g. Since the limit of a d-statistically convergent sequence of order α is unique [18] we get $x_{\circ} = x_{\circ}'$. ∎

Theorem 2.13. *Let (X, d) be a metric space and $0 < \alpha \leq \beta \leq 1$ and f be an unbounded modulus that provides $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence (x_k) is d-statistically convergent of order α to a point $x_{\circ} \in X$, in that case, it is df -statistically convergent of order β to $x_{\circ}$, namely, $S_{d}^{\alpha}(X) \subseteq S_{df}^{\beta}(X)$.*

Proof. Let $\varepsilon > 0$ be given. Since $|\{k \leq n : \ x_k \notin B_{\varepsilon}(x_{\circ})\}|$ is a natural number, we have

$$f(|\{k \leq n : \ x_k \notin B_{\varepsilon}(x_{\circ})\}|) \leq |\{k \leq n : \ x_k \notin B_{\varepsilon}(x_{\circ})\}| \, f(1)$$

and hence, using that $0 < \alpha \leq \beta \leq 1$ we may write

$$\begin{aligned} \frac{f(|\{k \leq n : \ x_k \notin B_{\varepsilon}(x_{\circ})\}|)}{f(n^{\beta})} &\leq \frac{f(|\{k \leq n : \ x_k \notin B_{\varepsilon}(x_{\circ})\}|)}{f(n^{\alpha})} \\ &\leq \frac{|\{k \leq n : \ x_k \notin B(x_{\circ}, \varepsilon)\}|}{n^{\alpha}} \frac{n^{\alpha}}{f(n^{\alpha})} f(1). \end{aligned}$$

Since $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$ and $x \in S_{d}^{\alpha}(X)$, we get $x \in S_{df}^{\beta}(X)$. ∎

If we get $\beta = \alpha$ in Theorem 2.13 we get the next conclusion.

Corollary 2.14. *Let (X, d) be a metric space, $0 < \alpha \leq 1$ be given and f be an unbounded modulus such that $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence (x_k) is d-statistically convergent of order α to an element $x_{\circ} \in X$, in that case, it is df -statistically convergent of order α to $x_{\circ}$, i.e., $S_{d}^{\alpha}(X) \subseteq S_{df}^{\alpha}(X)$.*

If we get $\alpha = 1$ in Corollary 2.14 we get the next conclusion.

Corollary 2.15. *Let (X, d) be a metric space and f be an unbounded modulus such that $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence (x_k) is d-statistically convergent to a point $x_{\circ} \in X$, in that case, it is df -statistically convergent to $x_{\circ}$, i.e., $S_{d}(X) \subseteq S_{df}(X)$ [34].*

3. *df*–Strong Cesàro Summability of Order α in Accordance to a Modulus in a Metric Space

In this section using a modulus function f we define and study the *df*- strong Cesàro summability of order α $(\alpha > 0)$ for sequences given in a metric space. We also establish the relationships between *df*- strong Cesàro summability of order α and *df*- statistical convergence of order α $(0 < \alpha \leq 1)$ in a metric space.

Definition 3.1. Let (X, d) be a metric space and f be a modulus function, and let a real number $\alpha > 0$ be given. A sequence (x_k) in space X is called as *df -strongly Cesàro summable* of order α, to $x_{\circ}$ if

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} f[d(x_k, x_{\circ})] = 0$$

for a point $x_{\circ} \in X$.

Note that the modulus f need not be unbounded in this Definition. The set of all df -strongly Cesàro summable sequences of order α in the metric space (X,d) will be indicated by $w_{df}^{\alpha}(X)$ that is

$$w_{df}^{\alpha}(X)=\left\{x=(x_k):\ \lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(x_k,x_\circ\right)\right]=0,\ \exists x_\circ\in X\right\}.$$

Note that we have obtained some sets previously defined for special cases of α and f. For example, we get the set

$$w_{df}(X)=\left\{x=(x_k):\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left[d\left(x_k,x_\circ\right)\right]=0,\ \exists x_\circ\in X\right\},$$

for $\alpha=1$ [16], the set

$$w_{d}^{\alpha}(X)=\left\{x=(x_k):\ \lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}d\left(x_k,x_\circ\right)=0,\ \exists x_\circ\in X\right\}$$

for $f(x)=x$ [18] and the set

$$w_{d}(X)=\left\{x=(x_k):\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}d\left(x_k,x_\circ\right)=0,\ \exists x_\circ\in X\right\}$$

for $\alpha=1$ and $f(x)=x$ [16].

As stated in Remark 3.2 in [32], the authors of that paper use α as any positive real number neglecting the condition "$\alpha\leq 1$" in their spaces $w_{\alpha,o}^{f}$ and w_{α}^{f}, although α was taken as a positive real number less than or equal to 1 in the spaces w_{p}^{α} and w_{op}^{α} of Çolak (see [20]).

First of all we have to point out that to take real α as $0<\alpha\leq 1$ is essential in the studies of strong Cesàro summability of order α, which is first given by Çolak [20]. Indeed it is not difficult to illustrate that the df -strong Cesàro summability of order α is not well defined for $\alpha>1$. To prove this consider the constant sequence $(x_k)=(a,a,a,...)$ in space X, which is convergent and df -strongly Cesàro summable of order α to a, and a modulus f. Then for any arbitrary x_o in X we have

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(x_k,x_\circ\right)\right]=\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(a,x_\circ\right)\right]=\lim_{n\to\infty}\frac{n}{n^{\alpha}}f\left[d\left(a,x_\circ\right)\right]=0$$

and since $\lim\limits_{n\to\infty}\frac{n}{n^{\alpha}}=0$ for $\alpha>1$ this means that (x_k) is df - strongly Cesaro summable of order α to any x_o . But this is not possible, because $w_{df}^{\alpha}-\lim x_k$ is unique by Lemma 3.4 given below.

Theorem 3.2. *Let (X,d) be a metric space and f be a modulus. Then*

$$w_{d}^{\alpha}(X)\subseteq w_{df}^{\alpha}(X)$$

in case $\alpha=1$.

Proof. Since the proof is alike to that of Theorem 1.18 in [34] we do not repeat it here. ∎

Remark 3.3. Theorem 3.2 does not have to be true for $\alpha < 1$. For example, let $X = \mathbb{R}$, $d(x,y) = |x-y|$ and consider the sequence $(x_k) = \left(\frac{1}{k^2}\right)$ and the modulus $f(x) = x^{\frac{1}{4}}$. Since the inequality

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} d(x_k, 0) = \frac{1}{n^\alpha}\sum_{k=1}^{n}\frac{1}{k^2} \le \frac{1}{n^\alpha}\frac{\pi}{6}$$

is satisfied, then we get that the sequence $(x_k) = \left(\frac{1}{k^2}\right)$ is d-strongly Cesàro summable of order α, to 0 for every $\alpha > 0$ and therefore for $0 < \alpha < \frac{1}{2}$, that is $x \in w_d^\alpha(\mathbb{R})$. At the same time, using the inequality

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}} > \sqrt{n}$$

we may write

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, 0)] = \frac{1}{n^\alpha}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} > \frac{1}{n^\alpha}\sqrt{n}.$$

This means that the sequence $(x_k) = \left(\frac{1}{k^2}\right)$ is not *df* -strongly Cesàro summable of order α, to 0 for $\alpha \le \frac{1}{2}$, i.e., $x \notin w_{df}^\alpha(\mathbb{R})$.

Lemma 3.4. *Let (X,d) be a metric space and f be an unbounded modulus.Then the df-strong Cesàro sum of a df-strongly Cesàro summable sequence of order α is unique for $\alpha \le 1$.*

Proof. Let $0 < \alpha \le 1$ and suppose the sequence (x_k) be *df*-strongly Cesàro summable of order α to $x_\circ$ and $x'_\circ$. In that case, we may write

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, x_\circ)] = \lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, x'_\circ)] = 0.$$

Using the fact that f is subadditive, from the inequality

$$d(x_\circ, x'_\circ) \le d(x_\circ, x_k) + d(x_k, x'_\circ)$$

we may write

$$f[d(x_\circ, x'_\circ)] \le f[d(x_k, x_\circ)] + f[d(x_k, x'_\circ)]$$

and hence

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_\circ, x'_\circ)] \le \frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, x_\circ)] + \frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, x'_\circ)].$$

Since both terms on the right side tend to 0 as $n \to \infty$, then the term on the left side also tends to 0 as $n \to \infty$, hence we have

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_\circ, x'_\circ)] = \lim_{n\to\infty}\frac{n}{n^\alpha} f[d(x_\circ, x'_\circ)] = 0.$$

From the fact

$$\lim_{n\to\infty}\frac{n}{n^\alpha} = \begin{cases} \infty & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

the equality $\lim_{n\to\infty}\frac{n}{n^{\alpha}}f\left[d\left(x_{\circ},x_{\circ}'\right)\right]=0$ is satisfied only for $f\left[d\left(x_{\circ},x_{\circ}'\right)\right]=0$. Thus we get $f\left[d\left(x_{\circ},x_{\circ}'\right)\right]=0 \Longleftrightarrow d\left(x_{\circ},x_{\circ}'\right)=0 \Longleftrightarrow x_{\circ}=x_{\circ}'$. ∎

Theorem 3.5. *Let (X,d) be a metric space, f be a modulus and $0<\alpha\leq 1$. If $0<\lim_{t\to\infty}\frac{f(t)}{t}$ then*

$$w_{df}^{\alpha}(X)\subseteq w_{d}^{\alpha}(X).$$

Proof. Let us define $m=\lim_{t\to\infty}\frac{f(t)}{t}=\inf\left\{\frac{f(t)}{t}:\ t>0\right\}$ and suppose $(x_k)\in w_{df}^{\alpha}(X)$. By definition of m, we have $f(t)\geq mt$ for every $t\geq 0$. Since $m>0$, we have $t\leq\frac{1}{m}f(t)$, for every $t\geq 0$ and hence we have

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}d\left(x_k,x_{\circ}\right)\leq\frac{1}{m}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(x_k,x_{\circ}\right)\right].$$

From this inequality we get $x\in w_{d}^{\alpha}(X)$ whenever $x\in w_{df}^{\alpha}(X)$. ∎

Remark 3.6. The condition $\lim_{t\to\infty}\frac{f(t)}{t}>0$ in Theorem 3.5 can not be removed, i.e., the inclusion $w_{df}^{\alpha}(X)\subseteq w_{d}^{\alpha}(X)$ does not have to be provided for any modulus function f yielding the condition $\lim_{t\to\infty}\frac{f(t)}{t}=0$. For example, take into account the space $X=\mathbb{R}$ with the metric $d(x,y)=|x-y|$, the sequence $x=(x_k)$ defined by (1) and the modulus $f(x)=\frac{x}{1+x}$. Now $\lim_{t\to\infty}\frac{f(t)}{t}=0$ is provided and, since

$$\frac{1}{n^{\alpha}}\sum_{k=1}^{n}d\left(x_k,0\right)=\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k=m^2}}^{n}k+\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k\neq m^2}}^{n}0\geq\frac{1}{n^{\alpha}}\frac{\left(\sqrt{n}-1\right)\sqrt{n}\sqrt{n}}{6}$$

we have $x\notin w_{d}^{\alpha}(\mathbb{R})$ for $\alpha\leq\frac{3}{2}$. However, since

$$\begin{aligned}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(x_k,0\right)\right]&=\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k=m^2}}^{n}f(k)+\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k\neq m^2}}^{n}f(0)\\&=\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k=m^2}}^{n}\frac{k}{1+k}\leq\frac{1}{n^{\alpha}}\sqrt{n}\end{aligned}$$

we get $x\in w_{df}^{\alpha}(\mathbb{R})$ for $\alpha>\frac{1}{2}$ and therefore for $\frac{1}{2}<\alpha\leq\frac{3}{2}$.

By combining Theorem 3.2 and Theorem 3.5 we get the next result.

Corollary 3.7. *Let (X,d) be a metric space, f be a modulus providing $\lim_{t\to\infty}\frac{f(t)}{t}>0$. Then*

$$w_{d}^{\alpha}(X)=w_{df}^{\alpha}(X).$$

in case $\alpha=1$.

Theorem 3.8. *Let (X,d) be a metric space and f be any modulus. Then*

$$w_{df}^{\alpha}(X)\subseteq w_{df}^{\beta}(X)$$

in case $1\geq\beta\geq\alpha>0$ and the inclusion may be strict for some $\alpha<\beta$.

Proof. It is easy to see that $w_{df}^{\alpha}(X) \subseteq w_{df}^{\beta}(X)$ for $\beta \geq \alpha$. To show that the inclusion is strict, let f be a modulus and let us consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} a, & k = m^2 \\ & \qquad m = 1, 2, 3, \ldots \\ b, & k \neq m^2 \end{cases} \tag{2}$$

where $a, b \in X$ and $a \neq b$. Using the fact that $f(0) = 0$ we may write

$$\begin{aligned} \frac{1}{n^{\beta}} \sum_{k=1}^{n} f\left[d\left(x_k, b\right)\right] &= \frac{1}{n^{\beta}} \sum_{\substack{k=1 \\ k=m^2}}^{n} f\left[d\left(a, b\right)\right] + \frac{1}{n^{\beta}} \sum_{\substack{k=1 \\ k\neq m^2}}^{n} f\left[d\left(b, b\right)\right] \\ &\leq \frac{\sqrt{n}}{n^{\beta}} f\left[d\left(a, b\right)\right] = \frac{1}{n^{\beta-\frac{1}{2}}} f\left[d\left(a, b\right)\right]. \end{aligned}$$

Taking limit as $n \to \infty$ we get $\frac{1}{n^{\beta-\frac{1}{2}}} f\left[d\left(a, b\right)\right] \to 0$ for $\beta > \frac{1}{2}$ so that $x \in w_{df}^{\beta}(X)$. Also, since

$$\frac{1}{n^{\alpha}} \sum_{k=1}^{n} f\left[d\left(x_k, b\right)\right] \geq \frac{\sqrt{n}-1}{n^{\alpha}} f\left[d\left(a, b\right)\right]$$

and so that $\frac{\sqrt{n}-1}{n^{\alpha}} f\left[d\left(a, b\right)\right] \to \infty$ as $n \to \infty$ for $0 < \alpha < \frac{1}{2}$ we get $x \notin w_{df}^{\alpha}(X)$. ■

4. Relationship between *df*–Statistical Convergence of Order α and *df*–Strong Cesàro Summability of Order α in Accordance to a Modulus in Metric Spaces

In this part we establish the relationships between *df*- strong Cesàro summability of order α and *df*- statistical convergence of order α $(0 < \alpha \leq 1)$ in a metric space.

Theorem 4.1. *Let (X, d) be a metric space, f be an unbounded modulus providing the condition $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $0 < \alpha \leq \beta \leq 1$. If a sequence (x_k) is df -strongly Cesàro summable of order α to a point $x_\circ \in X$, in that case, it is df -statistically convergent of order β to $x_\circ$, that is, $w_{df}^{\alpha}(X) \subseteq S_{df}^{\beta}(X)$.*

Proof. Let $\varepsilon > 0$ be given and let us choose $K(\varepsilon) = \{k \leq n : d(x_k, x_\circ) \geq \varepsilon\}$. Since $|K(\varepsilon)|$ is a positive integer we may write $f(|K(\varepsilon)|) \leq |K(\varepsilon)| f(1)$ and since f is increasing the following inequalities hold.

$$\begin{aligned} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} f\left[d\left(x_k, x_\circ\right)\right] &\geq \frac{1}{n^{\alpha}} \sum_{\substack{k=1 \\ k\in K(\varepsilon)}}^{n} f\left[d\left(x_k, x_\circ\right)\right] \geq \frac{1}{n^{\alpha}} \left|K(\varepsilon)\right| f(\varepsilon) \\ &\geq \frac{1}{n^{\beta}} \left|K(\varepsilon)\right| f(\varepsilon) \geq \frac{f\left(n^{\beta}\right)}{n^{\beta}} \frac{f\left(\left|K(\varepsilon)\right|\right)}{f\left(n^{\beta}\right)} \frac{f(\varepsilon)}{f(1)}. \end{aligned}$$

Since $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $x \in w_{df}^{\alpha}(X)$, from this inequality we get $x \in S_{df}^{\beta}(X)$. ■

If we take $\beta = \alpha$ in Theorem 4.1, we get the next conclusion.

Corollary 4.2. *Let f be an unbounded modulus that provides the condition $\lim_{t\to\infty}\frac{f(t)}{t}>0$ and a real number $0<\alpha\leq 1$ be given. If a sequence (x_k) in a metric space (X,d) is df -strongly Cesàro summable of order α to a point $x_\circ\in X$, in that case, the sequence is df -statistically convergent of order α to $x_\circ$, i.e., $w_{df}^{\alpha}(X)\subseteq S_{df}^{\alpha}(X)$.*

The special case $\alpha=1$ in Corollary 4.2, gives the next conclusion which is Theorem 3.6 in [34].

Corollary 4.3. *Let f be an unbounded modulus that provides the condition $\lim_{t\to\infty}\frac{f(t)}{t}>0$. If a sequence (x_k) in a metric space (X,d) is df -strongly Cesàro summable to a point $x_\circ\in X$, in that case, the sequence is df -statistically convergent to $x_\circ$, i.e., $w_{df}(X)\subseteq S_{df}(X)$.*

Remark 4.4. The condition $\lim_{t\to\infty}\frac{f(t)}{t}>0$ in Theorem 4.1 can not be removed. That is, for an unbounded modulus f which provides $\lim_{t\to\infty}\frac{f(t)}{t}=0$, a *df* -strongly Cesàro summable sequence of order α does not have to be *df* -statistically convergent of order β, where $0<\alpha\leq\beta\leq 1$. For example, consider a metric space (X,d) and the sequence $x=(x_k)$ defined by (2) where $a,b\in X$ $(a\neq b)$ and the unbounded modulus $f(x)=\log(x+1)$ that provides $\lim_{t\to\infty}\frac{f(t)}{t}=\lim_{t\to\infty}\frac{\log(t+1)}{t}=0$. Since

$$\begin{aligned}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left[d\left(x_k,b\right)\right]&=\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k=m^2}}^{n}f\left[d\left(a,b\right)\right]+\frac{1}{n^{\alpha}}\sum_{\substack{k=1\\k\neq m^2}}^{n}f\left[d\left(b,b\right)\right]\\&\leq\frac{\sqrt{n}}{n^{\alpha}}\log\left[d\left(a,b\right)+1\right],\end{aligned}$$

taking limit in this last inequality as $n\to\infty$, we obtain $x\in w_{df}^{\alpha}(X)$ for $\alpha>\frac{1}{2}$. However, since

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{f\left(n^{\beta}\right)}f\left(\left|\left\{k\leq n:\ d\left(x_k,b\right)\geq\varepsilon\right\}\right|\right)&=\lim_{n\to\infty}\frac{1}{f\left(n^{\beta}\right)}f\left(\left|\left\{k\leq n:\ k=n^2\right\}\right|\right)\\&=\lim_{n\to\infty}\frac{f\left(\sqrt{n}\right)}{f\left(n^{\beta}\right)}=\lim_{n\to\infty}\frac{\log\left(\sqrt{n}+1\right)}{\log\left(n^{\beta}+1\right)}\\&=\frac{1}{2\beta}\neq 0\end{aligned}$$

we get $x\notin S_{df}^{\beta}(X)$ for any β such that $0<\beta\leq 1$.

If we take $f(x)=x$ in Theorem 4.1, we get the next conclusion which is Theorem 4.1 for $p=1$ in [18].

Corollary 4.5. *Let $0<\alpha\leq\beta\leq 1$ be given. If a sequence (x_k) in a metric space (X,d) is d-strongly Cesàro summable of order α to an element $x_\circ\in X$, in that case, that sequence is d-statistically convergent of order β to $x_\circ$, i.e., $w_{d}^{\alpha}(X)\subseteq S_{d}^{\beta}(X)$.*

Remark 4.6. The inverse of Theorem 4.1 is usually not correct. That is, for $0<\alpha\leq\beta\leq 1$ and an unbounded modulus f such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$, a *df* -statistically convergent sequence of order β does not have to be *df* -strongly Cesàro summable of order α. For example, consider a metric space (X,d) with different points $a,b\in X$ and the sequence $x = (x_k)$ defined by (2) and the unbounded modulus $f(x) = x$ that provides $\lim\limits_{t\to\infty} \frac{f(t)}{t} = \lim\limits_{t\to\infty} \frac{t}{t} = 1 > 0$. Since

$$\frac{1}{f(n^\beta)} f(|\{k \le n : \ d(x_k, b) \ge \varepsilon\}|) \le \frac{f(\sqrt{n})}{f(n^\beta)} = \frac{\sqrt{n}}{n^\beta}$$

holds true, then taking limit as $n \to \infty$,we obtain $x \in S^\beta_{df}(X)$ for $\beta \in (\frac{1}{2}, 1]$. However, since the inequality

$$\begin{aligned}\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, b)] &= \frac{1}{n^\alpha}\sum_{\substack{k=1\\k=m^2}}^{n} f[d(a, b)] + \frac{1}{n^\alpha}\sum_{\substack{k=1\\k\ne m^2}}^{n} f[d(b, b)]\\ &= \frac{1}{n^\alpha}\sum_{\substack{k=1\\k=m^2}}^{n} d(a, b) \ge \frac{\sqrt{n}-1}{n^\alpha} d(a, b)\end{aligned}$$

holds true, then taking limit as $n \to \infty$, we get $x \notin w^\alpha_{df}(X)$ for $\alpha < \frac{1}{2}$.

Remark 4.7. A bounded and *df* -statistically convergent sequence of order α does not have to be *df* -strongly Cesàro summable of order α. To show this fact, consider the space $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$, the sequence $(x_k) = \left(\frac{1}{k^2}\right)$ and the unbounded modulus $f(x) = x^{\frac{1}{4}}$. The sequence $\left(\frac{1}{k^2}\right)$ is convergent and therefore, it is *df* -statistically convergent of order α to 0, for any $\alpha \in (0, 1]$ and every f. But, the sequence (x_k) is not *df* -strongly Cesàro summable of order α to 0, for $\alpha \le \frac{1}{2}$.

Theorem 4.8. *Let f be a modulus and the real numbers $0 < \alpha \le \beta \le 1$ be given. If a sequence (x_k) in a metric space (X, d) is df -strongly Cesàro summable of order α to a point $x_\circ \in X$, in that case, that sequence is d -statistically convergent of order β to $x_\circ$, i.e., $w^\alpha_{df}(X) \subseteq S^\beta_d(X)$.*

Proof. Let $\varepsilon > 0$ be given and let us define $K(\varepsilon) = \{k \le n : \ d(x_k, x_\circ) \ge \varepsilon\}$. Using the fact that f is increasing, we can write

$$\frac{1}{n^\alpha}\sum_{k=1}^{n} f[d(x_k, x_\circ)] \ge \frac{1}{n^\alpha}\sum_{\substack{k=1\\k\in K(\varepsilon)}}^{n} f[d(x_k, x_\circ)] \ge \frac{1}{n^\alpha}|K(\varepsilon)| f(\varepsilon) \ge \frac{1}{n^\beta}|K(\varepsilon)| f(\varepsilon).$$

From here, it follows that $x \in S^\beta_d(X)$ whenever $x \in w^\alpha_{df}(X)$. ∎

If we take $\beta = \alpha$ in Theorem 4.8, we get the next conclusion.

Corollary 4.9. *Let f be a modulus and a real number $0 < \alpha \le 1$ be given. If a sequence (x_k) in a metric space (X, d) is df -strongly Cesàro summable of order α to a point $x_\circ \in X$, in that case, that sequence is d -statistically convergent of order α to $x_\circ$, i.e., $w^\alpha_{df}(X) \subseteq S^\alpha_d(X)$.*

The special case $\alpha = 1$ in Corollary 4.9, gives the next conclusion which is Theorem 3 in [16].

Corollary 4.10. *Let (X, d) be a metric space and f be a modulus. If a sequence (x_k) is df -strongly Cesàro summable to an element $x_\circ \in X$, in that case, it is d -statistically convergent to $x_\circ$, i.e., $w_{df}(X) \subseteq S_d(X)$.*

The special case $f(x) = x$ and $\alpha = 1$ in Theorem 4.8, gives the next conclusion which is Theorem 2 (i) for $p = 1$ in [16].

Corollary 4.11. *Let (X, d) be a metric space. If a sequence (x_k) is d -strongly Cesàro summable to an element $x_\circ \in X$, in that case, it is d -statistically convergent to $x_\circ$, i.e., $w_d(X) \subseteq S_d(X)$.*

Note: It should be noted that in Theorem 4.1, Corollary 4.2 and Corollary 4.3, f is an unbounded modulus such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ while in Theorem 4.8, Corollary 4.9 and Corollary 4.10, there is no any restriction on the modulus f.

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