Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 855–859

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Some Fixed Point Theorems in $K-$Metric Type Spaces

A. Taheri[1], **Ali Farajzadeh**[2,*] **and Suthep Suantai**[3]

[1] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*
e-mail : taheri@yahoo.com

[2] *Department of Mathematics, Faculty of Science, Razi University, Kermanshah, Iran*
e-mail : farajzadehali@gmail.com

[3] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
e-mail : suthep.s@cmu.ac.th

Abstract In this paper, $K-$ metric type spaces, generalized KKM mappings, KKM property and fixed point property are introduced. Moreover a relationship between KKM property and fixed point property is established and finally some fixed point theorems in $K-$ metric type spaces are presented.

MSC: 49K35; 47H10; 20M12

Keywords: $K-$ type metric type space; fixed point

Submission date: 24.04.2019 / Acceptance date: 25.07.2019

1. Introduction

By a $K-$ metric space we mean an ordered pair (X, d) where, X is any nonempty set and $d : X \times X \to E$ is a mapping, called $K-$ metric, satisfying the following conditions:

(i) $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark that in 2007 Huang and Zhang [1] introduced cone metric spaces. Although this definition already under the name $K-$ metric, K indicates to the cone of a vector space, and $K-$ normed spaces were introduced in [2–4]. In this article, we first introduce $K-$ metric type mapping which is an extension of $K-$ metric mapping in the setting of cone metric type spaces and then we present some fixed point theorems. We need the following concepts in the sequel, for more details refer to [5–18].

Definition 1.1. Let E be a real Banach space and K is a subset of E. Then K is called a cone if and only if:

(i) K is closed, nonempty and $K \neq 0$,
(ii) if $a, b \geq 0$ and $x, y \in K$, then $ax + by \in K$,
(iii) if $x \in K$ and $-x \in K$, then $x = 0$, the zero element of E.

*Corresponding author.

Published by The Mathematical Association of Thailand.

For given a cone K in a Banach space E, we define a partial ordering $\leq$ with respect to K by $x \leq y \Longleftrightarrow y - x \in P$, we also write $x << y$ whenever $y - x \in intK$ (where $intK$ denotes the interior of K).

Definition 1.2. Let X be a nonempty set and E be a real Banach space with cone K. A vector-valued function $d : X \times X \to E$ is said to $K-$ metric type function on X with constant $m \geq 1$,if the the following conditions are satisfied:

(i) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq m(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called $K-$ metric type space (for short $KMTS$).

Remark 1.3. The Definition 1.2 coincides to the definition $K-$ metric by taking $m = 1$. There are many examples of $K-$ metric type spaces which are not $K-$ metric spaces.

Definition 1.4. Let (X, d) be a $KMTS$ and x_n be a sequence in X:

(i) $\{x_n\}$ converges to $x \in X$ if and only if for every $c \in E$ with $\theta << c$ there exists $n_0 \in N$ such that $d(x, x_n) << c$ for all $n > n_0$,and we write $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$,
(ii) If for every $c \in intP$ there exists $n_0 \in N$ such that $d(x_n, x_m) << c$ for all $n > n_0$, then $\{x_n\}$ is called a Cauchy sequence in X,
(iii) X is complete if and only if any Cauchy sequence in X is convergent.

2. KKM Maps in KMTS

In this section, let X and Y be two topological spaces and $F : X \to 2^Y$ be a multifunction with nonempty values, where 2^Y denotes the set of all subsets of Y.

Definition 2.1. A multifunction $F : X \to 2^Y$ is said to be:

(i) closed if its graph $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed.
(ii) compact if the closure of $F(X) = \cup_{x\in X} F(x)$ is a compact subset of Y.

For a set X, we denote the set of all nonempty finite subsets of X by $< X >$.

Definition 2.2. Let (M, d) be a KMTS. A subset $A \subset M$ is called bounded if $\delta(A) = sup\{d(x, y), x, y \in A\}$ exists in E. Also A is called totally bounded if for any $c >> 0$, that is $c \in intP$, :

$$A \subset \bigcup N_i, \quad \delta(N_i) << c$$

Definition 2.3. Let (M, d) be a KMTS, for every $x \in M$ and $t \in intP$, the closed and open balls of M are defined as

$$B(x, r) = \{y \in M, d(x, y) \leq t\} \quad , B_o(x, r) = \{y \in M, d(x, y) \ll t\}.$$

Definition 2.4. Let A be a nonempty bounded subset of a $K-$ metric type space (M, d). Then:

(i)

$$Co(A) = \bigcap\{B \subset M : B \text{ is a closed ball in} M, \ A \subseteq B\}$$

.

(ii)

$$A(M) = \{A \subset M : A = Co(A)\}, \text{ i.e.} A \in A(M)$$

if and only if A is an intersection of all closed balls containing A. In this case, we say that A is an admissible set in M.

(iii) A is called subadmissible, if for each $D \in < A >, Co(D) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

Definition 2.5. Let M be a $K-$ metric type space and X a subadmissible subset of M. A multifunction $G : X \to 2^X$ is called a KKM mapping, if for each $A \in < X >$, we have $Co(A) \subset G(A)$, where $G(A) = \bigcup\{G(a), a \in A\}$. More generally, if Y is a topological space and $G : X \to 2^Y$,$F : X \to 2^Y$ are two multifunctions such that for any $A \in < X >$, we gave $F(Co(A) \subset G(A)$ then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \to 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \to 2^Y$ with respect to F the family $\{\overline{G(X)}, x \in X\}$ has the finite intersection property, then F is said to have KKM property. We define
$KKM(X, Y) = \{F : X \to 2^Y, F \textit{ has the KKM property}\}$

Definition 2.6. Let X be a nonempty subset of a $K-$ metric type space M, a multifunction $F : X \to 2^X$ is said to have the approximate fixed point property if for any $\epsilon \in M$, $\epsilon \in intP$, there exists an $x_\epsilon \in X$ such that $F(x_\epsilon) \bigcap B_0(x_\epsilon, \epsilon) \neq \emptyset$, i.e. there exists $y \in F(x_\epsilon)$ such that $d(x_\epsilon, y) << \epsilon$

Theorem 2.7. *Let (M, d) be a KMTS and X a nonempty subadmissible subset of M.Let $F \in KKM(X, X)$ be such that $\overline{F(X)}$ is totally bounded. Then F has approximate fixed point property.*

Proof. Let $Y = \overline{F(X)} \subset \overline{X}$. Since Y is totally bounded, then for any $\epsilon \in M$, $\epsilon \in intP$, there exists a finite subset $A \subset X$ such that $Y \subset \bigcup_{x\in A} B_o(x, \epsilon)$. We define multifunction $G : X \to 2^Y$ as $G(x) = Y \bigcap \overline{B_o(x, \epsilon)^c}$, where W^c is the complement of W in M. Clearly $G(x)$ is closed. On the other hand, since $Y \subset \bigcup_{x\in A} B_o(x, \epsilon)$, then we have $\bigcap_{x\in A} G(x) = \emptyset$. So G is not a generalized KKM mapping with respect to F. Since $F \in KKM(X, X)$, there exists a finite nonempty subset $B \subset X$ such that $F(Co(B)) \nsubseteq \bigcup_{x\in B} G(x)$. So there exists $x_0 \in F(Co(B))$ such that $x_0 \notin G(x)$ for any $x \in B$. In other words, we have $x_0 \in \overline{B_o(x, \epsilon)^c}^c$, for any $x \in B$. Hence $x_0 \in B_o(x, \epsilon)$ for any $x \in B$, then $B \subset B_o(x_0, \epsilon)$. By the definition of $Co(B)$, we have $Co(B) \subset B_o(x_0, \epsilon)$. Since $x_0 \in F(Co(B))$ there exists $x_\epsilon \in Co(B)$ such that $x_0 \in F(x_\epsilon)$. But $x_\epsilon \in Co(B) \subset B_0(x_0, \epsilon)$, gives $d(x_0, x_\epsilon) \ll \epsilon$. Then we have proved

$$x_0 \in F(x_\epsilon) \bigcap B_0(x_\epsilon, \epsilon) \neq \emptyset.$$

Since ϵ was arbitrary, the proof of the theorem is completed. ∎

Theorem 2.8. *Let (M, d) be a $K-$ metric type space and X be a nonempty subadmissible subset of M. Let $F \in KKM(X, X)$ be closed and compact. Then F has a fixed point, i.e. there exists $x \in X$ such that $x \in F(x)$.*

Proof. Since $intP$ is nonempty there exists $c \in intP$. Then $c_n = \frac{c}{n} \in intP$, for each $n \in \mathbb{N}$, the positive integer numbers, note P is a convex cone. Hence it follows from the previous theorem that for each $n \in \mathbb{N}$ there exists $(x_n, y_n) \in X \times X$ such that

$y_n \in F(x_n) \cap B_0(c_n)$. The compactness condition of F, that is $\overline{F(X)}$ is compact, implies that there exists a subsequence (y_{n_k}) of y_n which converges to a point, we say x, of $\overline{F(X)}$. This and $y_{n_k} \in B_0(x_{n_k}, c_{n_k})$ imply that x_{n_k} converges to x. On the other hand from $y_{n_k} \in F(x_{n_k}) \cap B_0(c_n)$ we obtain $(x_{n_k}, y_{n_k}) \in G_r F$ and since $(x_{n_k}, y_{n_k}) \to (x, x)$ and F is closed, that is its graph is closed, we get $(x, x) \in G_r$. This means that $x \in F(x)$ and the proof completes. ■

References

[1] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. 32 (2007) 1468–1476.

[2] L.V. Kantorovich, The majorant principle and Newton's method, Dokl. Akad. Nauk SSSR (NS) 76 (1951) 17–20.

[3] J.S. Vandergraft, Newton's method for convex operators in partially ordered spaces, SIAM J. Numer. Anal. 4 (1967) 406–432.

[4] P.P. Zabreiko, $K-$ Metric and $K-$ normed spaces: survey, Collect. Math. 48 (4–6) (1997) 825–859.

[5] M. Tavakoli, A.P. Farajzadeh, T. Abdeljawad, S. Suantai, Some notes on cone metric spaces, Thai. J. Math. 16 (1) (2018) 229–242.

[6] A.P. Farajzadeh, On the scalarization method in cone metric spaces, Positivity 18 (4) (2014) 703–708.

[7] A.P. Farajzadeh, A. Amini-Harandi, D. Baleanu, Fixed point theory for generalized contractions in cone metric spaces, Comm. Nonl. Sci. Num. Sim. 17 (2) (2012) 708–712.

[8] P. Zangenehmehr, A.P. Farajzadeh, S.M. Vaezpour, On fixed point theorems for monotone increasing vector valued mappings via scalarizing, Positivity 19 (2) (2015) 333–340.

[9] P. Zangenehmehr, A. Farajzadeh, R. Lashkaripour, A. Karamian, On fixed point theory for generalized contractions in cone rectangular metric spaces via scalarizing, Thai. J. Math. 15 (1) (2016) 33–45.

[10] C. Gert (Tammer), P. Weinder, Nonconvex separation theorems and some applications in vector optimization, J. Optim. Theory Appl. 67 (2) (1990) 297–320.

[11] W. Rudin, Functional Analysis, McGraw-Hill Company, United States of America, 1972.

[12] A. Amini, M. Fakhar, J. Zafarani, KKM mappings in metric spaces, Nonl. Anal. TMA 60 (2005) 1045–1052.

[13] M.A. Khamsia, N. Hussain, KKM mappings in metric type spaces, Nonl. Anal. 73 (2010) 3123–3129.

[14] H. Long-Guang, Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.

[15] D. Turkoglu, M. Abuloha, T. Abdeljawad, KKM mappings in cone metric spaces and some fixed point theorems, Nonl. Anal. TMA 72 (2010) 348–355

[16] G.Y. Chen, X.Q. Yang, H. Yu, A nonlinear scalarization function and generralized quasi-vector equilibrium, J. Glob. Optim. 32 (2005) 451–466.

[17] K. Deimling, Nonlinear Functinal Analysis, Springer-Verlag, Berlin, 1988.

[18] W.-S. Du, A note on cone metric fixed point theory and its equivalence, Nonl. Anal. 72 (2010) 2259–2261.