Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 843–853

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Statistical Convergence in Paranorm Sense on Time Scales

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Abstract In this study, we define statistical convergence and $\lambda-$statistical convergence in paranorm sense on an arbitrary time scale equipped with paranorm. Furthermore, we study on strongly λ_p-summability on time scales in paranorm sense. Eventually, some inclusion theorems are proved.

MSC: 34N05; 40A05; 40A35; 40A30

Keywords: time scales; paranorm; statistical convergence

Submission date: 26.05.2019 / Acceptance date: 16.11.2021

1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. Then, the concept of statistical convergence was introduced by Fast [2] and Steinhaus [3] and later reintroduced by Schoenberg [4] independently for real and complex numbers. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on, it was further investigated from the sequence space point of view and linked with summability theory by various authors (see [5], [6], [7], [8], [9], [10], [11], [12]).

The statistical convergence depends on the density of subsets of $\mathbb{N}$. The natural density of $K \subseteq \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|,$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of $K \subseteq \mathbb{N}$ not exceeding n and if the above limit exists. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(K^c) = 1 - \delta(K)$ (see [9]).

A sequence $x = (x_k)$ is said to be statistically convergent to a real number L if

$$\delta(\{k \leq n : |x_k - L| \geq \varepsilon\}) = 0,$$

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Published by The Mathematical Association of Thailand.

for each $\varepsilon > 0$. In this case, one can write $S - \lim x_k = L$. The set of all statistically convergent sequences is denoted by S (see [6], [9]). It is very well known that every statistical convergent sequence is convergent, but the converse is not true.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. The set of all such sequences is denoted by Λ (see [13], [14], [15], [16]).

$x = (x_k)$ is said to be (V, λ) −summable to a number L if $t_n(x) \to L$ as $n \to \infty$. (V, λ) −summability reduces to $(C, 1)$ summability when $\lambda_n = n$ (see [13], [14]). The sets of sequences $x = (x_k)$ which are strongly Cesàro summable and strongly (V, λ)- summable to L are denoted by

$$[C, 1] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \text{ for some } L \right\},$$

$$[V, \lambda] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \text{ for some } L \right\},$$

respectively. Strong (V, λ) −summability reduces to strong $(C, 1)$ summability when $\lambda_n = n$. Borwein [17] and Maddox [18] introduced and studied strongly summable functions. The notion of λ-statistical convergence was introduced by Mursaleen [13] as follows:

Let $K \subset \mathbb{N}$. λ−density of K is defined by

$$\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|.$$

$\delta_\lambda(K)$ reduces to the natural density $\delta(K)$ in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ (see [13]).

$x = (x_k)$ is said to be λ−statistically convergent to L if

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

for each $\varepsilon > 0$. In this case, one can write $S_\lambda - \lim x_k = L$ (see [13]). If $\lambda_n = n$, S_λ reduces to the set of statistically convergent sequences S. In later years, the concept of almost λ-statistical convergence was studied by Savaş [19]. Nuray [20] studied λ-strong summable and λ-statistically convergent functions.

Before giving the foundations of the subject, let us explain in detail the concept of time scale calculus that we will build on the theory. A time scale is an arbitrary, nonempty, closed subset of real numbers. An arbitrary time scale is denoted by the symbol $\mathbb{T}$. It has the topology that it inherits from the real numbers with the standart topology. The theory of time scale was founded in Hilger's doctoral dissertation in 1988 (see [21], [22]). It allows to unify discrete and continuous analysis. One can replace the range of definition ($\mathbb{R}$) of the functions under consideration by an arbitrary time scale $\mathbb{T}$ [23]. Let us express some notions related to the basics of time scales theory.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ can be defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

for $t \in \mathbb{T}$. And, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. Here, we put $\inf \phi = \sup \mathbb{T}$ where ϕ is an empty set. A closed interval on $\mathbb{T}$ is given by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}{:}a \leq t \leq b\}.$$

Open intervals and half open intervals on time scales are defined similarly (see [24]). Now, it is time to remind Lebesque measure on time scales. It is necessary to generalize the geometric concept of "length" defined for intervals and the generalization is called measure, specifically delta (Δ) measure and nabla (∇) measure on time scales. Measure theory on time scales was first constructed by Guseinov [25]. Then further studies were made by Cabada-Vivero [26].

Let A denotes the family of all left closed and right open intervals of $\mathbb{T}$ of the form $[a, b)_{\mathbb{T}}$. Let $m : A \to [0, \infty)$ be a set function on A such that $m([a, b)_{\mathbb{T}}) = b - a$. Then, it is known that m is a countably additive measure on A. Now, the Caratheodory extension of the set function m associated with family A is said to be the Lebesque $\Delta-$measure on $\mathbb{T}$ and is denoted by μ_Δ. In this case, it is known that if $a \in \mathbb{T} - \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(a) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta((a, b)_{\mathbb{T}}) = b - \sigma(a)$. If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$, $a \leq b$; $\mu_\Delta((a, b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a, b])_{\mathbb{T}}) = \sigma(b) - a$ (see [27]).

There are some studies about statistical convergence on time scales in literature. For instance, Seyyidoglu and Tan [28] gave some new notations such as $\Delta-$convergence, $\Delta-$Cauchy by using $\Delta-$density and investigate their relations. Turan and Duman [29] introduced the concept of statistical convergence of delta measurable real-valued functions defined on time scales as follows. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta-$measurable function for a $\Delta-$measurable subset Ω of $\mathbb{T}$, the density of Ω over $\mathbb{T}$ is defined to be number

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t \to \infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0, t]_{\mathbb{T}})},$$

provided that above limit exists. Then, f is statistically convergent to a real number L on $\mathbb{T}$ if for every $\varepsilon > 0$,

$$\delta_{\mathbb{T}}(\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}) = 0.$$

In this case, it can be written as $\boldsymbol{s}_{\mathbb{T}} - \lim_{t \to \infty} f(t) = L$ [29]. Uniform statistical convergence was given by Altin et. al [30]. The notion of λ-statistical convergence on time scales was introduced by Yilmaz and his coworkers [31] as follows:

Let Ω be a $\Delta_\lambda-$ measurable subset of $\mathbb{T}$. Then, the set $\Omega(t, \lambda)$ is defined by

$$\Omega(t, \lambda) = \{s \in [t - \lambda_t + t_0, t]_{\mathbb{T}} : s \in \Omega\},$$

for $t \in \mathbb{T}$. In this case, $\lambda-$ density of Ω on $\mathbb{T}$ is denoted by $\delta_{\mathbb{T}}^{\lambda}(\Omega)$, as follows:

$$\delta_{\mathbb{T}}^{\lambda}(\Omega) = \lim_{t \to \infty} \frac{\mu_{\Delta_\lambda}(\Omega(t, \lambda))}{\mu_{\Delta_\lambda}([t - \lambda_t + t_0, t]_{\mathbb{T}})},$$

provided that the above limit exists. Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta_\lambda-$ measurable function. f is $\lambda-$ statistically convergent on $\mathbb{T}$ to a number L if

$$\lim_{t \to \infty} \frac{\mu_{\Delta_\lambda}(s \in [t - \lambda_t + t_0, t]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon)}{\mu_{\Delta_\lambda}([t - \lambda_t + t_0, t]_{\mathbb{T}})} = 0,$$

for each $\varepsilon > 0$. In this case, one can write $\boldsymbol{s}_{\mathbb{T}}^{\lambda} - \lim_{t\to\infty} (f(t)) = L$. The set of all $\lambda-$statistically convergent functions on $\mathbb{T}$ is denoted by $\boldsymbol{s}_{\mathbb{T}}^{\lambda}$. Except those, Turan and Duman defined Lacunary statistical convergence on time scales [32]. Moreover, Cichon and Yantir studied convergence of sets on time scales [33].

Statistical convergence in a paranormed space was introduced by Alotaibi and Alroqi in 2012 [34]. They defined the concept of statistical convergence and strongly $p-$Cesàro summability in a paranormed space. $\lambda-$statistical convergence and strongly $\lambda-$summability in a paranormed space were defined by Alghamdi and Mursaleen in 2013 [35]. In this study, our main goal is to define firstly a paranorm on an arbitrary time scale and construct the structure of classical statistical convergence, $\lambda-$statistical convergence and $\lambda p-$summability on that time scale equipped with a paranorm.

2. Main Results

In this section, we try to give some basic notions related to paranorm, statistical convergence, $\lambda-$statistical convergence and $\lambda-$summability on a time scale equipped with a paranorm.

Definition 2.1. Let $(V,+,.)$ be a linear space of the functions $f,g:\mathbb{T}\to\mathbb{R}$ on a time scale $\mathbb{T}$ and $q:V\to\mathbb{R}$. If q satisfies following conditions, it is called a paranorm.

P1) $q(f(s))=0$, then $f(s)=0$.
P2) $q(-f(s))=q(f(s))$.
P3) $q(f(s)+g(s))\le q(f(s))+q(g(s))$.
P4) If (α_n) be a sequence of scalars with $\alpha_n\to\alpha_0$ as $n\to\infty$ and $f(s),L\in V$, with $f(s)\to L$ in the sense that $q(f(s)-L)\to 0$ as $s\to\infty$, then $q(\alpha_n f(s)-\alpha_0 L)\to 0$ as $s,n\to\infty$.

Definition 2.2. Let $f:\mathbb{T}\to V$ be a $\Delta-$ measurable function. f is statistically convergent in paranorm sense on $\mathbb{T}$ to a number L if

$$\lim_{t\to\infty}\frac{\mu_{\Delta}\left(s\in[t_0,t]_{\mathbb{T}}:q\left(f\left(s\right)-L\right)\ge\varepsilon\right)}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)}=0,$$

for each $\varepsilon>0$.

In this case, we write $\boldsymbol{p}_{\mathbb{T}}(st)-\lim_{t\to\infty} f(t)=L$. The set of all statistically convergent functions in paranorm sense on $\mathbb{T}$ will be denoted by $\boldsymbol{p}_{\mathbb{T}}(st)$.

Definition 2.3. Let $f:\mathbb{T}\to V$ be a $\Delta_{\lambda}-$ measurable function. f is $\lambda-$statistically convergent in paranorm sense on $\mathbb{T}$ to a number L if

$$\lim_{t\to\infty}\frac{\mu_{\Delta_\lambda}\left(s\in I:q\left(f\left(s\right)-L\right)\ge\varepsilon\right)}{\mu_{\Delta_\lambda}\left(I\right)}=0,$$

for each $\varepsilon>0$ where $I=[t-\lambda_t+t_0,t]_{\mathbb{T}}$.

In this case, we write $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)-\lim_{t\to\infty} f(t)=L$. The set of all $\lambda-$statistically convergent functions in paranorm sense on $\mathbb{T}$ will be denoted by $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)$.

Now we have the following important theorems.

Theorem 2.4. *Let $f:\mathbb{T}\to V$ be a $\Delta_\lambda-$measurable function. Then, $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)-\lim_{t\to\infty} f(t)=L$ and the limit is unique.*

Proof. Assume that we have two limits $L_1 \neq L_2$. Hence, by definition,

$$\Omega_1(t) = \left\{ s \in I : q(f(s) - L_1) \geq \frac{\varepsilon}{2} \right\},$$
$$\Omega_2(t) = \left\{ s \in I : q(f(s) - L_2) \geq \frac{\varepsilon}{2} \right\}.$$

On the other hand, $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L_1$ then, $\delta_{\mathbb{T}}^{\lambda}(\Omega_1(t)) = 0$ and $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L_2$ then, $\delta_{\mathbb{T}}^{\lambda}(\Omega_2(t)) = 0$. Let $\Omega(t) = \Omega_1(t) \cap \Omega_2(t)$. Therefore, $\Omega^c(t) = \phi$ and $\delta_{\mathbb{T}}^{\lambda}(\Omega^c(t)) = 1$. If $s \in I \setminus \Omega(t)$, $q(L_1 - L_2) \leq q(f(s) - L_1) + q(f(s) - L_2) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since ε is arbitrary, $q(L_1 - L_2) = 0$ or $L_1 = L_2$. This completes the proof. ■

Theorem 2.5. *Let $f, g : \mathbb{T} \to V$ be $\Delta_\lambda-$ measurable functions, $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L_1$ and $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} g(t) = L_2$. Following relations are satisfied;*

(1) $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} (f(t) + g(t)) = L_1 + L_2$.

(2) $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} (cf(t)) = cL_1,\ (c \in \mathbb{R})$.

Proof. (1)

$$\frac{\mu_{\Delta_\lambda}(s \in I : q(f(s) + g(s) - (L_1 + L_2)) \geq \varepsilon)}{\mu_{\Delta_\lambda}(I)} \leq \frac{\mu_{\Delta_\lambda}\left(s \in I : q(f(s) - L_1) \geq \frac{\varepsilon}{2}\right)}{\mu_{\Delta_\lambda}(I)} + \frac{\mu_{\Delta_\lambda}\left(s \in I : q(g(s) - L_2) \geq \frac{\varepsilon}{2}\right)}{\mu_{\Delta_\lambda}(I)}.$$

For $t \to \infty$, $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L_1$ and $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} g(t) = L_2$ implies

$$\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} (f(t) + g(t)) = L_1 + L_2.$$

(2) It is evident when $c = 0$. For $c \neq 0$,

$$\frac{\mu_{\Delta_\lambda}(s \in I : q(cf(s) - cL_1) \geq \varepsilon)}{\mu_{\Delta_\lambda}(I)} \leq \frac{\mu_{\Delta_\lambda}\left(s \in I : q(cf(s) - cL_1) \geq \frac{\varepsilon}{c}\right)}{\mu_{\Delta_\lambda}(I)}.$$

Then, we get $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} cf(t) = cL_1$ as $t \to \infty$. ■

Theorem 2.6. *If $\boldsymbol{p}_{\mathbb{T}}(st) \leq \boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)$, then*

$$\lim_{t\to\infty} \inf \frac{\mu_{\Delta_\lambda}(I)}{\mu_\Delta([t_0, t])_{\mathbb{T}}} > 0. \tag{2.1}$$

Proof. For $\varepsilon > 0$,

$$\mu_\Delta(s \in [t_0, t]_{\mathbb{T}} : q(f(s) - L) \geq \varepsilon) \geq \mu_{\Delta_\lambda}(s \in I : q(f(s) - L) \geq \varepsilon).$$

Then,

$$\frac{\mu_\Delta(s \in [t_0, t]_{\mathbb{T}} : q(f(s) - L) \geq \varepsilon)}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \geq \frac{\mu_{\Delta_\lambda}(s \in I : q(f(s) - L) \geq \varepsilon)}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \frac{\mu_{\Delta_\lambda}(I)}{\mu_{\Delta_\lambda}(I)}$$
$$= \frac{\mu_{\Delta_\lambda}(I)}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \frac{1}{\mu_{\Delta_\lambda}(I)} \mu_{\Delta_\lambda}(s \in I : q(f(s) - L) \geq \varepsilon).$$

Hence by using (2.1) and taking limit as $t \to \infty$, $\boldsymbol{p}_{\mathbb{T}}(st) - \lim_{t\to\infty} f(t) = L$ implies

$$\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L.$$

This completes proof. ∎

Theorem 2.7. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_\lambda} \le \mu_{\Delta_\beta}$ for all $t \in \mathbb{T}$.*

(1) If

$$\liminf_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} > 0, \tag{2.2}$$

then $\boldsymbol{p}_{\mathbb{T}}^{\beta}(st) \subseteq \boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)$.

(2) If

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1, \tag{2.3}$$

then $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) \subseteq \boldsymbol{p}_{\mathbb{T}}^{\beta}(st)$.

Proof. *(1)* Suppose that $\mu_{\Delta_\lambda} \le \mu_{\Delta_\beta}$ for all $t \in \mathbb{T}$ and (2.2) is satisfied. Then $I_t \subset J_t$ and for $\varepsilon > 0$, we have

$$\begin{aligned}&\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)\\ &\qquad \ge \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right).\end{aligned}$$

where $J_t = [t-\beta_t+1,t]$. Therefore,

$$\begin{aligned}\frac{\mu_{\Delta_\beta}\left(s \in [t-j_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \ge& \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \frac{1}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}\\ &\times \mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right),\end{aligned}$$

for all $t \in \mathbb{T}$. Hence by using (2.2) and taking the limit as $t \to \infty$, we get $\boldsymbol{p}_{\mathbb{T}}^{\beta}(st) \subseteq \boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)$.

(2) Let f be a Δ_λ measurable function and, $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim f(t) = L$. Since $I_t \subset J_t$, we can write

$$\begin{aligned}\frac{\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} =& \frac{\mu_{\Delta_\beta}\left(t-\beta_t+t_0 \le s \le t : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\\ &+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\\ \le& \frac{\mu_{\Delta_\beta}\left(s \in [t-\beta_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\\ &- \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\\ &+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}\\ \le& \left(1 - \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\right)\\ &+ \frac{\mu_{\Delta_\lambda}\left(s \in [t-\lambda_t+t_0,t]_{\mathbb{T}} : q\left(f(s)-L\right) \ge \varepsilon\right)}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)},\end{aligned}$$

for all $t \in \mathbb{T}$. Since $\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1$ by *(2)*, the term in above inequality tends to 0 as $t \to \infty$. Furthermore, since $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim f(t) = L$, the second term of the right hand side of the above inequality goes to 0

$$\frac{\mu_{\Delta_\beta}\left(t - \beta_t + t_0 \le s \le t : q\left(f\left(s\right) - L\right) \ge \varepsilon\right)}{\mu_{\Delta_\beta}\left([t - \beta_t + t_0, t]_{\mathbb{T}}\right)} \to 0,$$

as $t \to \infty$. Therefore $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) \subseteq \boldsymbol{p}_{\mathbb{T}}^{\beta}(st)$. From Theorem 2.7, we have the following result. ∎

Corollary 2.8. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_\lambda} \le \mu_{\Delta_\beta}$ for all $t \in \mathbb{T}$. If the second property of Theorem 2.7 holds, then $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) = \boldsymbol{p}_{\mathbb{T}}^{\beta}(st)$. If we take $\mu_{\Delta_{\lambda(t)}} = \lambda(t)$ for $t \in \mathbb{T}$ in above corollary, we get the following result.*

Corollary 2.9. *Let $\mu_{\Delta_{\lambda(t)}} \in \Lambda$. If $\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\lambda(t)} = 1$, then $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) = \boldsymbol{p}_{\mathbb{T}}(st)$.*

Definition 2.10. Let $f : \mathbb{T} \to V$ be a $\Delta_\lambda-$measurable function and $0 < p < \infty$. If there exists a $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{\mu_{\Delta_\lambda}\left([t - \lambda_t + t_0, t]_{\mathbb{T}}\right)} \int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} q\left(f\left(s\right) - L\right)^p \Delta s = 0,$$

f is strongly $\lambda p-$summable function in paranorm sense on $\mathbb{T}$. The set of all strongly $\lambda p-$summable functions in paranorm sense on $\mathbb{T}$ will be denoted by $[W, \lambda_p, q]_{\mathbb{T}}$.

Lemma 2.11. *Let $f : \mathbb{T} \to V$ be a $\Delta_\lambda-$measurable function and $\Omega(t, \lambda) = \{s \in I : q(f(s) - L) \ge \varepsilon\}$. Then, the following inequality holds for $\varepsilon > 0$,*

$$\mu_{\Delta_\lambda}\left(\Omega(t, \lambda)\right) \frac{1}{\varepsilon} \int_{\Omega(t,\lambda)} q\left(f\left(s\right) - L\right) \Delta s \le \frac{1}{\varepsilon} \int_I q\left(f\left(s\right) - L\right) \Delta s.$$

Theorem 2.12. *Let $f : \mathbb{T} \to V$ be a $\Delta_\lambda-$measurable function, $L \in V$ and $0 < p < \infty$. Then, the followings are satisfied.*

(1) $[W, \lambda_p, q]_{\mathbb{T}} \subset \boldsymbol{p}_{\mathbb{T}}^{\lambda}(st)$.

(2) $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L$ *and if f is bounded, f is strongly λ_p-summable to L in paranorm sense.*

Proof. *(1)* Let us take as $f(s) \to [W, \lambda_p, q]_{\mathbb{T}}$. Then,

$$\int_{[t-\lambda_t+t_0,t]_{\mathbb{T}}} q\left(f\left(s\right) - L\right)^p \Delta s \ge \int_{\Omega(t,\lambda)} q\left(f\left(s\right) - L\right)^p \Delta s \ge \varepsilon^p \mu_{\Delta_\lambda}\left(\Omega(t, \lambda)\right).$$

Therefore $[W, \lambda_p, q]_{\mathbb{T}} - \lim_{t\to\infty} f(t) = L$ implies $\boldsymbol{p}_{\mathbb{T}}^{\lambda}(st) - \lim_{t\to\infty} f(t) = L$.

(2) Let f be bounded and $\lambda-$statistically convergent to L in paranorm sense on $\mathbb{T}$. Then, there exists a positive number $M > 0$ such that $q\left(f\left(s\right) - L\right) \le M$, $s \in \mathbb{T}$ and

$$\lim_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left(\Omega(t, \lambda)\right)}{\mu_{\Delta_\lambda}(I)} = 0$$

$$\begin{aligned}\int_I q\left(f\left(s\right) - L\right)^p \Delta s &= \int_{\Omega(t,\lambda)} q\left(f\left(s\right) - L\right)^p \Delta s + \int_{I/\Omega(t,\lambda)} q\left(f\left(s\right) - L\right)^p \Delta s \\ &\le M^p \mu_{\Delta_\lambda}\left(\Omega(t, \lambda)\right) + \varepsilon^p \mu_{\Delta_\lambda}(I).\end{aligned}$$

Then,

$$\lim_{t\to\infty}\frac{1}{\mu_{\Delta_\lambda}(I)}\int_I q\left(f\left(s\right)-L\right)^p\Delta s\leq M^p\lim_{t\to\infty}\frac{\mu_{\Delta_\lambda}\left(\Omega(t,\lambda)\right)}{\mu_{\Delta_\lambda}(I)}+\varepsilon^p.$$

Since ε is arbitrary, we obtain $f(s)\to[W,\lambda_p,q]_{\mathbb{T}}$. ∎

Theorem 2.13. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_\lambda}\leq\mu_{\Delta_\beta}$ for all $t\in\mathbb{T}$. Then,*

(1) If (2.1) holds, then $[W,\lambda_p,q]_{\mathbb{T}}\subseteq[W,\beta_p,q]_{\mathbb{T}}$.
(2) If (2.2) holds, then

$$\ell_\infty\left(\mathbb{T},q\right)\cap[W,\lambda_p,q]_{\mathbb{T}}\subseteq[W,\beta_p,q]_{\mathbb{T}}$$

$$\text{where } \ell_\infty\left(\mathbb{T},q\right)=\left\{f/f:\mathbb{T}\to\mathbb{R},\sup_{s\in\mathbb{T}}q(f(s))<\infty\right\}.$$

Proof. *(1)* Suppose that $\mu_{\Delta_\lambda}\leq\mu_{\Delta_\beta}$ for all $t\in\mathbb{T}$. Then $I_t\subset J_t$ so that we may write

$$\begin{aligned}\frac{1}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}&\int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}}q\left(f\left(s\right)-L\right)^p\Delta s\\&\geq\frac{1}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}}q\left(f\left(s\right)-L\right)^p\Delta s,\end{aligned}$$

for all $t\in\mathbb{T}$. This implies that

$$\begin{aligned}\frac{1}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}&\int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}}q\left(f\left(s\right)-L\right)^p\Delta s\\&\geq\frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\frac{1}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}\int\limits_{[t-\lambda_t+t_0,t]_{\mathbb{T}}}q\left(f\left(s\right)-L\right)^p\Delta s.\end{aligned}$$

Then taking limit of the last inequality as $t\to\infty$ and using (2.3), we obtain $[W,\lambda_p,q]_{\mathbb{T}}\subseteq[W,\beta_p,q]_{\mathbb{T}}$.

(2) Let $f\in\ell_\infty\left(\mathbb{T},q\right)\cap[W,\lambda_p,q]_{\mathbb{T}}$. Suppose that (2.2) holds. Since $f\in\ell_\infty\left(\mathbb{T},q\right)$, then there exists a positive number M such that $q(f(s))\leq M$ for all $s\in\mathbb{T}$ and also now,

since $\mu_{\Delta_\lambda} \leq \mu_{\Delta_\beta}$ and so that $\frac{1}{\mu_{\Delta_\beta}} \leq \frac{1}{\mu_{\Delta_\lambda}}$ and $I_t \subset J_t$, we may write

$$\begin{aligned}
&\frac{1}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t-\beta_t+t_0,t]_{\mathbb{T}}} q\left(f\left(s\right)-L\right)^p \Delta s \\
&\qquad \geq \frac{1}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{J_t/I_t} q\left(f\left(s\right)-L\right)^p \Delta s \\
&\qquad\quad + \frac{1}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{I_t} q\left(f\left(s\right)-L\right)^p \Delta s \\
&\qquad \leq \left(\frac{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)-\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)}\right) M \\
&\qquad\quad + \frac{1}{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)} \int\limits_{I_t} q\left(f\left(s\right)-L\right)^p \Delta s
\end{aligned}$$

for all $t \in \mathbb{T}$. Since $\lim\limits_{t\to\infty} \frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta_\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} = 1$ by (2.2) the first term and since $f \in [W,\lambda_p,q]_{\mathbb{T}}$ the second term of right hand side of above inequality tend to 0 as $t \to \infty$.

$$\left(\text{Note that: } 1-\frac{\mu_{\Delta_\lambda}\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_{\Delta\beta}\left([t-\beta_t+t_0,t]_{\mathbb{T}}\right)} \geq 0 \text{ for all } t \in \mathbb{T}\right).$$

This implies that $\ell_\infty\left(\mathbb{T},q\right) \cap [W,\lambda_p,q]_{\mathbb{T}} \subseteq [W,\beta_p,q]_{\mathbb{T}}$ and so $\ell_\infty\left(\mathbb{T},q\right) \cap [W,\lambda_p,q]_{\mathbb{T}} \subseteq \ell_\infty\left(\mathbb{T}\right) \cap [W,\beta_p,q]_{\mathbb{T}}$. From Theorem 2.5 we have the following result. ■

Corollary 2.14. *Let $\mu_{\Delta_{\lambda(t)}}$ and $\mu_{\Delta_{\beta(t)}}$ be two sequences in Λ such that $\mu_{\Delta_\lambda} \leq \mu_{\Delta_\beta}$ for all $t \in \mathbb{T}$. If (2.2) holds, then $\ell_\infty\left(\mathbb{T},q\right) \cap [W,\lambda_p,q]_{\mathbb{T}} = \ell_\infty\left(\mathbb{T}\right) \cap [W,\beta_p,q]_{\mathbb{T}}$.*

Conclusions

Important concepts such as statistical convergence and $\lambda-$statistical convergence, which have a very important effect for the summability theory, are discussed again in the paranorm case on an arbitrary time scale. Likewise, λ_p-summability and its properties are studied in paranorm sense on an arbitrary time scale. We think that these results will bring a new breath to the summability theory.

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