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Stepanov-Like Pseudo Almost Periodic Solutions of Class r in α-Norm under the Light of Measure Theory

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Abstract The aim of this work is to present some interesting results on weighted ergodic functions. We also study the existence and uniqueness of (μ, ν)-weighted Stepanov-like pseudo almost periodic solutions class r for some partial differential equations in a Banach space when the delay is distributed using the spectral decomposition of the phase space developed by Adimy and his co-authors.

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1. Introduction

In this work, we study the existence and uniqueness of Stepanov-like pseudo almost periodic solutions of class r for the following partial functional differential equation

$$u'(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators on a Banach space X. The phase space $C_\alpha = C([-r, 0], D(A^\alpha))$, $0 < \alpha < 1$, is the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$, A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be describe later and

$$\|\varphi\|_{C_\alpha} = \|A^\alpha \varphi\|_{C([-r,0],X)}.$$

For $t \geq 0$, and $u \in C([-r, a], D(A^\alpha))$, $a > 0$ and u_t denotes the history function of C_α defined by

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$$u_t(\theta) = u(t+\theta) \text{ for } \; -r \leq \theta \leq 0.$$

and L is a bounded linear operator from C_α into X and $f : \mathbb{R} \to X$ is a continuous function.

Some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. For example in [1], the authors prove the existence and uniqueness theorem of pseudo almost periodic mild solutions to nonautonomous neutral partial evolution equations

$$\begin{aligned}
\frac{d}{dt}[u(t)+f(t,u(t))] &= A(t)[u(t)+f(t,u(t))]+g(t,u(t)), t \in \mathbb{R},\\
\frac{d}{dt}[u(t)+f(t,Bu(t))] &= A(t)[u(t)+f(t,Bu(t))]+g(t,Cu(t)), t \in \mathbb{R},
\end{aligned}$$

where that $A(t)$ satisfy "Acquistapace-Terreni" conditions, the evolution family generated by $A(t)$ has exponential dichotomy, $R(\lambda_0, A(.))$ is almost periodic, B, C are densely defined closed linear operators, f , g are Lipschitz with respect to the second argument uniformly in the first argument, f is pseudo almost periodic in the first argument, g is Stepanov-like pseudo almost periodic in the first argument for $p > 1$ and jointly continuous.

In [2], the author revisits the concept of S^p-pseudo-almost periodicity and he studies the existence of pseudo-almost periodic solutions to some nonautonomous differential equations in the case when the semilinear forcing term is both continuous and S^p-pseudo-almost periodic for $p > 1$.

In [3], the author introduce the concept of weighted pseudo almost periodic, which is more general than the one of the pseudo almost periodicity. He gave some properties of the space of weighted pseudo almost periodic functions such as the completeness and the composition theorem and a new concept of ergodic ϕ functions with respect to some weighted function ρ in the sense that

$$\lim_{\tau \to +\infty} \frac{1}{m(\tau,\rho)} \int_{-\tau}^{\tau} \|\phi(t)\| \rho(t) dt,$$

where $m(\tau,\rho) = \displaystyle\int_{-\tau}^{\tau} \rho(t) dt$ and ρ is assumed to be positive and locally Lebesgue integrable.

However, these results and many others obtained in literature are not correct. For example the decomposition result of weighted pseudo almost periodic functions in classical sense are not unique. The completeness based on the uniqueness decomposition result is not true. It follows that the uniqueness of existence S^p-pseudo-almost periodic solution based on the completeness is also not true.

The aim of this work is to correct many results obtained in the literature and also generalize some results obtained in classical sense on S^p-pseudo-almost periodic functions. Our approach is based on the spectral decomposition of the phase space developed in [4] and a new approach developped in [5].

This work is organised as follow, in sections 2, 3 and 4, we collect some background materials required throughout the paper contained in [6]. In section 5, we recall some prelimary results on pseudo almost periodic and Stepanov like pseudo almost periodic functions that will be used in this work. In section 6, we prove some properties of S^p-pseudo almost periodic function of class r. In section 7, we discuss the main result of this paper. Using the strict contraction principle we study the existence and uniqueness of Stepanov-like pseudo almost periodic solution of class r for equation (1.1). Finally, for illustration, we propose to study the existence and uniqueness of S^p-pseudo almost periodic solution for some model arising in the population dynamics.

2. Analytic Semigroup

The purpose sections 2, 3 and 4 is to collect some background materials required throughout the paper due to I. zabsonre and al [6]. These materials include, on the one hand, the fractional power A^α for $0 < \alpha < 1$ of A.

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α. For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1 ([7]). *Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have*
i) $T(t) : X \to D(A^\alpha)$ for every $t > 0$.
ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$.
iii) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \textit{ for } t > 0.$$

iv) If $0 < \alpha \leq \beta < 1$, $D(A^\beta) \hookrightarrow D(A^\alpha)$.
v) There exists $N_\alpha > 0$ such that

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \textit{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\delta)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$ and Γ is the gamma function
Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

3. Spectral Decomposition

The purpose of this section is to collect some background materials on the spectral decomposition of the phase space and variation of constants formula due to Adimy and al in [4, 8].

To equation (1.1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C_\alpha, \end{cases} \tag{3.1}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.
For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_α by

$$\mathcal{U}(t)\varphi = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in C_\alpha. \end{cases}$$

Proposition 3.1 ([8])**.** *Let $\mathcal{A}_\mathcal{U}$ defined on C_α by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \Big\{\varphi \in C_\alpha,\ \varphi' \in C_\alpha,\ \varphi(0) \in D(A),\ \varphi(0)' \in \overline{D(A)} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } \ \varphi(0)' = -A\varphi(0) + L(\varphi)\Big\} \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \ \text{ for } \ \varphi \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_α.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : \ c \in X\}$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if} \quad \theta \in [-r, 0[, \\ \\ c & \text{if} \quad \theta = 0. \end{cases}$$

Consider the extension $\mathcal{A}_\mathcal{U}$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \Big\{\varphi \in C^1([-r, 0]; X_\alpha) : \ \varphi(0) \in D(A) \ \text{ and } \varphi(0)' \in \overline{D(A)}\Big\} \\ \widetilde{\mathcal{A}_\mathcal{U}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

We make the following assertion:

($\mathbf{H_0}$) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X and satisfies $0 \in \rho(A)$.

Lemma 3.2 ([4])**.** *Assume that* ($\boldsymbol{H_0}$) *holds. Then, $\widetilde{\mathcal{A}_\mathcal{U}}$ satisfies the Hille-Yosida condition on $C_\alpha \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \ \text{ for } \ n \in \mathbb{N} \ \text{ and } \ \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to equation (3.1).

Theorem 3.3 ([8]). *Assume that* ($\boldsymbol{H_0}$) *holds. Then for all* $\varphi \in C_\alpha$, *the solution* u *of equation* (3.1) *is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \ \ \textit{for} \ \ t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-1}$.

Definition 3.4. We say that a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is *hyperbolic* if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset.$$

For the sequel, we make the following assumption:

($\mathbf{H_1}$) $T(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

We get the following result on the spectral decomposition of the phase space C_α.

Proposition 3.5 ([8]). *Assume that* ($\boldsymbol{H_0}$) *and* ($\boldsymbol{H_1}$) *hold. If the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic, then the space* C_α *is decomposed as a direct sum*

$$C_\alpha = S \oplus U$$

of two $\mathcal{U}(t)$ *invariant closed subspaces* S *and* U *such that the restricted semigroup on* U *is a group and there exist positive constants* $\overline{M}$ *and* ω *such that*

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\omega t}|\varphi| \ \ \textit{for} \ \ t \geq 0 \ \ \textit{and} \ \ \varphi \in S \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\omega t}|\varphi| \ \ \textit{for} \ \ t \leq 0 \ \ \textit{and} \ \ \varphi \in U. \end{aligned}$$

The spaces S *and* U *are called respectively the stable and unstable space. By* Π^s *and* Π^u *we denote respectively the projection operator on* S *and* U.

4. Almost Periodic Functions and (μ, ν) Ergodic Functions

In this section, we collect some background materials on the notion of μ-pseudo almost periodicity which generalize the pseudo almost periodicity introduced by Zhang [9–11]; it is also a generalization of weighted pseudo almost periodicity given by Diagana [3]. Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 4.1. A bounded continuous function $\phi : \mathbb{R} \to X$ is called *almost periodic* if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, X)$ such that $|\phi(t + \tau) - \phi(t)| < \varepsilon$ for all $(t, \tau) \in \mathbb{R} \times \mathcal{K}(\varepsilon, \phi, X)$.
We denote by $AP(\mathbb{R}, X)$, the space of all such functions.

Definition 4.2. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called *almost periodic* in $t \in \mathbb{R}$ uniformly in $x \in X_1$ if for each $\varepsilon > 0$ and all compact $K \subset X_1$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, K)$ such that $|\phi(t + \tau, x) - \phi(t, x)| < \varepsilon$ for all $t \in \mathbb{R}, \ x \in K, \ \tau \in \mathcal{K}(\varepsilon, \phi, K)$.
We denote by $AP(\mathbb{R} \times X_1; X_2)$, the space of all such functions.

The next lemma gives a characterization of almost periodic functions.

Lemma 4.3. *A function $\phi \in C(\mathbb{R}, X)$ is almost periodic if and only if the space of functions $\{\phi_\tau : \tau \in \mathbb{R}\}$, where $(\phi_\tau)(t) = \phi(t+\tau)$, is relatively compact in $BC(\mathbb{R}, X)$.*

In the sequel, we recall some preliminary results concerning the (μ,ν)-pseudo almost periodic functions.
The symbol $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu) = \Big\{u \in BC(\mathbb{R}, X_\alpha) : \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} |u(t)|_\alpha d\mu(t) = 0\Big\}.$$

In addition to the above-mentioned spaces, we consider the following spaces

$$\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r) = \Big\{u \in BC(\mathbb{R}, X_\alpha) : \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |u(\theta)|_\alpha\Big) d\mu(t) = 0\Big\}.$$

In addition to above-mentioned space, we consider the following spaces

$$\mathcal{E}(\mathbb{R}\times X_\alpha, X_\alpha, \mu, \nu) = \Big\{u \in BC(\mathbb{R}\times X_\alpha; X_\alpha) : \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} |u(t,x)|_\alpha d\mu(t) = 0\Big\},$$

$$\mathcal{E}(\mathbb{R}\times X_\alpha; X_\alpha, \mu, \nu, r) = \Big\{u \in BC(\mathbb{R}\times X_\alpha; X_\alpha) : \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |u(\theta,x)|_\alpha\Big) d\mu(t) = 0\Big\},$$

where in both cases the limit (as $\tau \to +\infty$) is uniform in compact subset of X_α.
In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ and $\mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$ are continuously embedded in $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$ and $\mathcal{E}(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu)$, respectively.

Example 4.4 ([5])**.** Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt, \quad \text{for } A \in \mathcal{B}, \tag{4.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (4.1) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

On the other hand, one can observe that a ρ-weighted pseudo almost periodic functions is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu([-\tau,\tau]) = 2\tau$ for all $\tau \geq 0$.

5. (μ,ν)-Stepanov-Like Pseudo Almost Periodic Functions

Definition 5.1. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t,s) = f(t+s).$$

Remark 5.2. If $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 5.3. The Bochner transform $F^b(t,s,u),\ t\in\mathbb{R},\ s\in[0,1],\ u\in X$ of a function $F(t,u)$ on $\mathbb{R}\times X$, with values in X, is defined by

$$F^b(t,s,u)=F(t+s,u) \text{ for each } u\in X.$$

Definition 5.4. Let $p\in[1,+\infty[$. The space $BS^p(\mathbb{R},X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in X such that $f^b\in L^\infty(\mathbb{R},L^p([0,1],X))$. This is a Banach space with the norm

$$\|f\|_{S^p}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|f(s)|^p ds\Big)^{\frac{1}{p}}.$$

Definition 5.5. Let $\mu,\nu\in\mathcal{M}$. A bounded continuous function $\phi:\mathbb{R}\to X_\alpha$ is called *(μ,ν)-pseudo almost periodic* if $\phi=\phi_1+\phi_2$, where $\phi_1\in AP(\mathbb{R},X_\alpha)$ and $\phi_2\in\mathcal{E}(\mathbb{R},X_\alpha,\mu,\nu)$.
We denote by $PAP(\mathbb{R},X_\alpha,\mu,\nu)$ the space of all such functions.

Definition 5.6. Let $\mu,\nu\in\mathcal{M}$. A bounded continuous function $\phi:\mathbb{R}\times X_\alpha\to X_\alpha$ is called *uniformly (μ,ν)-pseudo almost periodic* if $\phi=\phi_1+\phi_2$, where
$\phi_1\in AP(\mathbb{R}\times X_\alpha;X_\alpha)$ and $\phi_2\in\mathcal{E}(\mathbb{R}\times X_\alpha,X_\alpha,\mu,\nu)$.
We denote by $PAP(\mathbb{R}\times X_\alpha,X_\alpha,\mu,\nu)$, the space of all such functions.

We now introduce some new spaces used in the sequel.

Definition 5.7. A function $f\in BS^p(\mathbb{R},X_\alpha)$ is called *(μ,ν)-S^p pseudo-almost periodic* (or *Stepanov-like pseudo-almost periodic*) if it can be expressed as $f=h+\varphi$, where $h^b\in AP(\mathbb{R},L^p((0,1),X_\alpha))$ and $\varphi^b\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu)$. The collection of all such functions will be denoted by $PAPS^p(\mathbb{R},X_\alpha,\mu,\nu)$.

In other words, a function $f\in L^p(\mathbb{R},X_\alpha)$ is said to be *S^p-pseudo-almost periodic* if its Bochner transform $f^b:\mathbb{R}\to L^p((0,1),X_\alpha)$ is pseudo-almost periodic in the sense that there exist two functions $h,\varphi:\mathbb{R}\to X_\alpha$ such that $f=h+\varphi$, where $h^b\in AP(\mathbb{R},L^p((0,1),X_\alpha))$ and $\varphi^b\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu)$, i.e., there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{P}(\varepsilon,h)$ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|h(s+\xi)-h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}<\varepsilon \text{ for}(t,\xi)\in\mathbb{R}\times\mathcal{P}(\varepsilon,h).$$

and

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\Big(\int_t^{t+1}|\varphi(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)=0.$$

Definition 5.8. A function $f\in BS^p(\mathbb{R},X_\alpha)$ is called *(μ,ν)-S^p-pseudo-almost periodic of class r* (or *Stepanov-like pseudo-almost periodic of class r*) if it can be expressed as $f=h+\varphi$, where $h^b\in AP(\mathbb{R},L^p((0,1),X_\alpha))$ and $\varphi^b\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ i.e.,

$$\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|\varphi(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)=0.$$

We denote by $PAPS^p(\mathbb{R},X_\alpha,\mu,\nu,r)$ the space of all such functions.

Definition 5.9. A function $f\in BS^p(\mathbb{R}\times X_1,X_2)$ is called *(μ,ν)-S^p-pseudo-almost periodic of class r* (or *Stepanov-like pseudo-almost periodic of class r*) if it can be expressed as $f=h+\varphi$, where $h^b\in AP(\mathbb{R}\times L^p((0,1),X_\alpha),L^p((0,1),X_\alpha))$ and $\varphi^b\in$

$\mathcal{E}[(\mathbb{R} \times L^p((0,1), X_\alpha), L^p((0,1), X_\alpha), \mu, \nu, r]$ i.e.,

$$\lim_{\tau \to +\infty} \frac{1}{\nu[-\tau, \tau]} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \Big(\int_\theta^{\theta+1} |\varphi(s,x)|_\alpha^p ds \Big)^{\frac{1}{p}} d\mu(t) = 0.$$

We denote by $PAPS^p(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu, r)$ the space of all such functions.

Lemma 5.10 ([2]). *Let $f \in AP(\mathbb{R}, X)$, then f is S^p-almost periodic.*

6. Properties of (μ, ν)-Stepanov-Like Pseudo Almost Periodic Functions of Class r

For $\mu, \nu \in \mathcal{M}$, we formulate the following hypothesis.

($\mathbf{H_2}$) Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup_{\tau \to +\infty} \dfrac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \delta < \infty$.

Lemma 6.1. *Assume ($\boldsymbol{H_2}$) holds. $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ is a closed subspace of $BS^p(\mathbb{R}, X)$.*

Proof. Let $(x_n)_n$ be a sequence in $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ such that $\lim_{n \to \infty} x_n = x$ in $BS^p(\mathbb{R}, X_\alpha)$. For each n, let $x_n = y_n + z_n$ with $y_n^b \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$ and $z_n^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. Then $(y_n)_n$ converges to someone $y \in BS^p(\mathbb{R}, X_\alpha)$ and $(z_n)_n$ also converges to some $z \in BS^p(\mathbb{R}, X_\alpha)$. Since $y_n^b \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$, then for each n there exists a relatively dense subset $\mathcal{P}(\varepsilon, y_n)$ of $\mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} |y_n(s+\xi) - y_n(s)|_\alpha^p ds \Big)^{\frac{1}{p}} < \varepsilon \ \text{ for} (s, \xi) \in \mathbb{R} \times \mathcal{P}(\varepsilon, y_n).$$

Let $\xi \in \mathcal{P}(\varepsilon, y_n)$ and $\|f\|_{S^p,\alpha} = \sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} |f(s)|_\alpha^p ds \Big)^{\frac{1}{p}}$, by Minkowski inequality we have

$$\begin{aligned} \Big(\int_t^{t+1} |y(s+\xi) - y(s)|_\alpha^p ds \Big)^{\frac{1}{p}} &= \Big(\int_t^{t+1} |y(s+\xi) - y_n(s+\xi) + y_n(s+\xi) - y_n(s) \\ &\quad + y_n(s) - y(s)|_\alpha^p ds \Big)^{\frac{1}{p}} \\ &\le \Big(\int_t^{t+1} |y(s+\xi) - y_n(s+\xi)|_\alpha^p ds \Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_t^{t+1} |y_n(s+\xi) - y_n(s)|_\alpha^p ds \Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_t^{t+1} |y_n(s) - y(s)|_\alpha^p ds \Big)^{\frac{1}{p}}. \end{aligned}$$

It follows that

$$\sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} |y(s+\xi) - y(s)|_\alpha^p ds \Big)^{\frac{1}{p}} dt \le 2\|y_n - y\|_{S^p,\alpha} + \sup_{t \in \mathbb{R}} \Big(\int_t^{t+1} |y_n(s+\xi) - y_n(s)|_\alpha^p ds \Big)^{\frac{1}{p}},$$

which shows that $y \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$.
Since

$$\int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |z(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |z(s) - z_n(s) + z_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t),$$

then by Minkowski inequality, we also have

$$\begin{aligned}
&\frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |z(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \\
&\leq \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big[\Big(\int_\theta^{\theta+1} |z(s) - z_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} + \Big(\int_\theta^{\theta+1} |z_n(s)|^p) ds\Big)^{\frac{1}{p}}\Big] d\mu(t) \\
&\leq \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |z(s) - z_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \\
&\quad + \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |z_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \\
&\leq \|z - z_n\|_{S^p,\alpha} \times \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |z_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t).
\end{aligned}$$

So $z \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$, and hence $x \in PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. ■

Consequently, we have the following lemma:

Lemma 6.2. *The space $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ endowed with the $\| \,.\, \|_{S^p}$ norm is a Banach space.*

The next result is a characterization of (μ, ν)-ergodic functions of class r.

Theorem 6.3. *Assume that $(\boldsymbol{H_2})$ holds and let I be a bounded interval (we do not exclude the case $I = \emptyset$). Assume that $f \in BS^p(\mathbb{R}, X)$. Then the following assertions are equivalent:*

i) $f \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$.
ii) $\displaystyle\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0.$
iii) for any $\varepsilon > 0$ we have

$$\lim_{\tau\to+\infty} \frac{\mu\Big(\Big\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau]\setminus I)} = 0.$$

Proof. The proof runs along the same lines as the proof of Theorem 2.13 in [5].
i) $\Leftrightarrow$ *ii)* Let us pose $A = \nu(I)$, $B = \displaystyle\int_I \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t)$. Since the interval I is bounded and the function f is bounded and continuous then A and $B \in \mathbb{R}$.

For $\tau > 0$ such that $I \subset [-\tau, \tau]$ and $\nu([-\tau, \tau] \setminus I) > 0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
&= \frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)-B\Big]\\
&= \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].
\end{aligned}$$

From above and the fact that $\nu(\mathbb{R}) = +\infty$, we conclude that *ii*) is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)=0,$$

that is *i*).

iii) $\Rightarrow$ *ii*) Let us pose A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}>\varepsilon\Big\}$$

and

$$B_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I):\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}\le\varepsilon\Big\}.$$

Assume that *iii*) holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{6.1}$$

From the following equality

$$\begin{aligned}
\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t) &= \int_{A_\tau^\varepsilon}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\quad+\int_{B_\tau^\varepsilon}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t),
\end{aligned}$$

it follows that

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}\mu(t)\le\|f\|_{S^p}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}$$

for τ sufficiently large. By $(\mathbf{H_2})$, it follows that for for all $\varepsilon > 0$ we have

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|f(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\le\alpha\varepsilon.$$

Consequently (ii) holds.

$ii) \Rightarrow iii)$ Assume that $ii)$ holds. From the following inequalities

$$\begin{aligned}
\int_{[-\tau,\tau]\setminus I} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) &\geq \int_{A_\tau^\varepsilon} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \\
\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) &\geq \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} \\
\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) &\geq \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},
\end{aligned}$$

which hold for τ sufficiently large, we obtain equation (6.1). So, $iii)$ holds. ■

For $\mu \in \mathcal{M}$, we formulate the following hypotheses.

($\mathbf{H_3}$) For all a, b and $c \in \mathbb{R}$, such that $0 \leq a < b \leq c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \mu(a+\delta, b+\delta) \geq \alpha_0 \mu(\delta, c+\delta).$$

($\mathbf{H_4}$) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a+\tau : a \in A\}) \leq \beta\mu(A) \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

We have the following result due to [5].

Lemma 6.4 ([5]). *Hypothesis ($\boldsymbol{H_4}$) implies ($\boldsymbol{H_3}$).*

Lemma 6.5 ([5, 12]). *$\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$) and let $f \in PAP(\mathbb{R}, X, \mu, \nu)$ be such that*

$$f = g + h$$

where $g \in AP(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then

$$\{g(t),\ t \in \mathbb{R}\} \subset \overline{\{f(t),\ t \in \mathbb{R}\}} \text{ (the closure of the range of } f).$$

Lemma 6.6 ([13]). *Let $\mu, \nu \in \mathcal{M}$. Assume ($\boldsymbol{H_4}$) holds. Then the decomposition of a (μ,ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu, r)$, is unique.*

Definition 6.7. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 *is equivalent to* μ_2, denoting this as $\mu_1 \sim \mu_2$ if there exist constants $\alpha > 0$ and $\beta > 0$ and a bounded interval I (we allow also the situation when $I = \emptyset$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

From [5] we know that $\sim$ is a binary equivalence relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \{\varpi \in \mathcal{M} : \mu \sim \varpi\}.$$

Theorem 6.8. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then the spaces $PAPS^p(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r)$ and $PAPS^p(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$ coincide, that is, $PAPS^p(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = PAPS^p(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (we allow also the situation when $I = \emptyset$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfying $A \cap I = \emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for τ sufficiently large, we obtain

$$\frac{\alpha_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau] \setminus I)}$$

$$\leq \frac{\mu_2\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau] \setminus I)}$$

$$\leq \frac{\beta_1\mu_1\Big(\Big\{t \in [-\tau,\tau] \setminus I : \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau] \setminus I)}$$

By Theorem 6.3 we deduce that $\mathcal{E}(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = \mathcal{E}(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$. According to Definition 5.8, we deduce that $PAPS^p(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = PAPS^p(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$. ■

For $\mu, \nu \in \mathcal{M}$ set

$$cl(\mu,\nu) = \{\varpi_1, \varpi_2 \in \mathcal{M} : \ \mu \sim \varpi_2 \text{ and } \ \nu \sim \varpi_2\}.$$

Lemma 6.9 ([12])**.** *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$). Then $PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ is translation invariant, that is $f \in PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ implies $f_\gamma \in PAP(\mathbb{R}, X_\alpha, \mu, \nu)$ for all $\gamma \in \mathbb{R}$.*

Corollary 6.10. *Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_5}$). Then $PAPS^p(\mathbb{R}, X_\alpha, \mu, r)$ is translation invariant, that is $f \in PAAS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ implies $f_\gamma \in PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ for all $\gamma \in \mathbb{R}$.*

Proof. It suffices to prove that $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ is translation invariant. Let $f \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ and

$$F^t(\theta) = \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s)|_\alpha^p ds\Big)^{\frac{1}{p}}.$$

Then $F^t \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$. But since $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ is translation invariant, it follows that

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} F^t(\theta+\gamma)d\mu(t) = \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |f(s+\gamma)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0.$$

This implies that $f(.+\gamma) \in PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$ and ends the proof. ■

In what follows, we prove some preliminary results concerning the composition of (μ,ν)-Stepanov-pseudo almost periodic functions of class r.

Theorem 6.11. *Let $\mu, \nu \in \mathcal{M}$, Let $\phi \in PAPS^p(\mathbb{R} \times X_\alpha, X_\alpha, \mu, \nu, r)$ and $\phi = \phi_1 + \phi_2 \in PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Assume:*

(i) $\phi_1(t,x)$ is uniformly continuous on any bounded subset uniformly for $t \in \mathbb{R}$,
(ii) that there exists a function $L_\phi : \mathbb{R} \to [0, +\infty[$ satisfying

$$|\phi(t,x_1) - \phi(t,x_2)| \leq L_\phi(t)|x_1(t) - x_2(t)|_\alpha \quad \text{for} \quad t \in \mathbb{R} \quad \text{and for} \quad x_1, x_2 \in L^p((0,1), X_\alpha). \tag{6.2}$$

If $\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)d\mu(t) < \infty$ *and*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)\Big(\int_\theta^{\theta+1} |\xi(s)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0 \quad (6.3)$$

for $\xi \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu)$*, then the function* $t \to \phi(t, h(t))$ *belongs to* $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$*.*

Proof. Assume that $\phi = \phi_1 + \phi_2, h = h_1 + h_2$
where $\phi_1^b \in AP(\mathbb{R}\times L^p((0,1), X_\alpha), L^p((0,1), X_\alpha))$, $\phi_2^b \in \mathcal{E}(\mathbb{R}\times L^p((0,1), X_\alpha), \mu, \nu, r)$ and $h_1^b \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$, $h_2^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

From [2], $\phi_1^b(., h_1^b(.)) \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$. Now, we need to prove that both $\phi^b(., h(.)) - \phi^b(., h_1^b(.))$ and $\phi_2^b(., h_1^b(.))$ belong to $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$.
By equation (6.2), it follows that

$$\begin{aligned}
&\frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\\
&\le \frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1} (L_\phi(\theta)|h_2(\theta)|_\alpha)^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}\\
&\le \frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1} |h_2(\theta)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.
\end{aligned}$$

Since h_2 is (μ,ν)-ergodic of class r, Theorem 6.3 and equation (6.3) yield that for the above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \Big(\sup_{\theta\in[t-r,t+1]} L_\phi(\theta)\Big)\Big(\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1} |h_2(\theta)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big) > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0,$$

and then we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])} = 0, \quad (6.4)$$

By Theorem 6.3, equation (6.4) shows that $t \mapsto \phi(t, h(t)) - \phi(t, h_1(t))$ belongs to $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$.

Since ϕ_2^b is uniformly continuous on the compact set $K = \overline{\{h_1^b(t) : t \in \mathbb{R}\}}$ with respect to the second variable x, we deduce that for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\xi_1 - \xi_2\|_{L^p} \le \delta \Rightarrow |\phi_2^b(t, \xi_1(t)) - \phi_2^b(t, \xi_2(t))| \le \varepsilon$$

for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$. Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i, \delta).$$

Then by Minkowski inequality we have

$$\begin{aligned}
&\Big(\int_t^{t+1} |\phi_2(t, h_1(t))|_\alpha^p ds\Big)^{\frac{1}{p}} \\
&\quad\leq \quad \Big(\int_t^{t+1} |\phi_2(t, h_1(t)) - \phi_2(t, z_i)|_\alpha^p ds\Big)^{\frac{1}{p}} + \Big(\int_t^{t+1} |\phi_2(t, z_i))|_\alpha^p ds\Big)^{\frac{1}{p}} \\
&\quad\leq \quad \varepsilon + \sum_{i=1}^{n(\varepsilon)} \Big(\int_t^{t+1} |\phi_2(t, z_i)|_\alpha^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

Since

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |\phi_2(\theta, z_i)|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0$$

for every $i \in \{1, ..., n(\varepsilon)\}$, we deduce that for all $\varepsilon > 0$ we have

$$\limsup_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |\phi_2(\theta, h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) \leq \varepsilon.$$

This implies that

$$\forall\varepsilon > 0, \quad \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} |\phi_2(\theta, h_1(\theta))|_\alpha^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0.$$

Consequently $t \mapsto \phi_2(t, h(t))$ belongs to $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. ∎

For $\mu \in \mathcal{M}$ and $\delta \in \mathbb{R}$, we denote μ_δ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\delta(A) = \mu([a + \delta : \ a \in A]) \tag{6.5}$$

Lemma 6.12 ([5])**.** *Let $\mu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$). Then the measures μ and μ_δ are equivalent for all $\delta \in \mathbb{R}$.*

Lemma 6.13 ([5])**.** ($\boldsymbol{H_4}$) *implies*

$$\text{for all } \sigma > 0 \ \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma, \tau+\sigma])}{\mu([-\tau,\tau])} < +\infty.$$

We have the following result.

Theorem 6.14. *Let $u \in PAPS^p(\mathbb{R}, L^p((0,1), X_\alpha, \mu, \nu, r)$. Then the function $t \to u_t$ belongs to $PAPS^p(C_\alpha, \mu, \nu, r)$.*

Proof. Assume that $u = g + h$ where $g^b \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$ and $h^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. From [2], $u_t^b = g_t^b + h_t^b$. Since $g \in AP(\mathbb{R}, L^p((0,1), X_\alpha))$, then there exists a relatively dense subset $\mathcal{P}(\varepsilon, g)$ of $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} |g(s+\xi) - g(s)|_\alpha^p ds\Big)^{\frac{1}{p}} < \varepsilon \ \ \text{for}(t,\tau) \in \mathbb{R} \times \mathcal{P}(\varepsilon, h).$$

Let $\xi \in \mathcal{P}(\varepsilon, g)$ and $\theta \in [-r, 0]$ we have

$$\begin{aligned}\Big(\int_t^{t+1}\Big(\sup_{\theta\in[-r,0]}(|g_s(\theta+\xi)-g_s(\theta)|_\alpha)\Big)^p ds\Big)^{\frac{1}{p}} &= \Big(\int_t^{t+1}\Big(\sup_{\theta\in[-r,0]}(|g(s+\theta+\xi)-g(s+\theta)|_\alpha)\Big)^p ds\Big)^{\frac{1}{p}}\\ &\le \Big(\sup_{\theta\in[-r,0]}\int_{t+\theta}^{t+1+\theta}|g(s+\xi)-g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}.\end{aligned}$$

Consequently

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\Big(\sup_{\theta\in[-r,0]}(|g_s(\theta+\xi)-g_s(\theta)|_\alpha)\Big)^p ds\Big)^{\frac{1}{p}} \le \sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|g(s+\xi)-g(s)|_\alpha^p ds\Big)^{\frac{1}{p}},$$

which shows that g_t is S^p-almost periodic.

Let us denote by

$$M_\delta(\tau) = \frac{1}{\nu_\alpha([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h(\theta)|_\alpha\Big)^{\frac{1}{p}}d\mu_\delta(t),$$

where μ_δ and ν_δ are the positive measure defined by equation (6.5). By Lemma 6.12, it follows that μ_α and μ are equivalent and ν_δ and ν are also equivalent. Then by Theorem 6.8 we have $\mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu_\alpha, \nu_\alpha, r) = \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. Consequently $h \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu_\alpha, \nu_\alpha, r)$, that is

$$\lim_{\tau\to+\infty} M_\delta(\tau) = 0, \quad \text{for all } \alpha \in \mathbb{R}.$$

On the other hand, for $r > 0$ we have

$$\begin{aligned}&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}\Big(\sup_{\xi\in[-r,0]}|h(s+\xi)|_\alpha\Big)^p ds\Big)^{\frac{1}{p}}d\mu(t)\\ &\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\sup_{\xi\in[-r,0]}\int_{\theta+\xi}^{\theta+1+\xi}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\ &\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-2r,t]}\Big(\int_\theta^{\theta+1}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\ &\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-2r,t-r]}\Big(\int_\theta^{\theta+1}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\ &\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau-r}^{+\tau+r}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t+r)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_\theta^{\theta+1}|h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t).\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}\Big(\sup_{\xi\in[-r,0]}|h(s+\xi)|\Big)^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|^{p}ds\Big)^{p}d\mu(t)\\
&+\frac{1}{\nu([-\tau-r,\tau+r]}\int_{-\tau-r}^{+\tau+r}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|h(s)|^{p}ds\Big)^{p}d\mu(t+r)\times\Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big].
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}|h(\theta+\xi)|\Big]\Big)d\mu(t) \quad &\leq \quad \Big[\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big]\times M_r(\tau+r)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|h(\theta)|\Big)d\mu(t).
\end{aligned}$$

According to Lemma 6.12 and Lemma 6.13, the function $t\to u_t$ belongs to $PAPS^p(C_\alpha,\mu,\nu,r)$. Thus, we obtain the desired result. ∎

7. Weighted Stepanov Like Pseudo Almost Periodic Solutions of Class r

In what follows, we will be looking at the existence of bounded integral solution of equation (1.1).

Theorem 7.1. *Assume that* ($\boldsymbol{H_0}$) *and* ($\boldsymbol{H_1}$) *hold and that the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic. If* $f\in BS^p(\mathbb{R},X)$*, then there exists a unique bounded solution* u *of equation* (1.1) *on* $\mathbb{R}$*, given by*

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))ds, \quad (7.1)$$

for $t\in\mathbb{R}$*, where* $\widetilde{B}_\lambda=\lambda(\lambda I-\mathcal{A}_{\mathcal{U}})^{-1}$ *for* $\lambda>\widetilde{\omega}$*,* Π^s *and* Π^u *are the projections of* C_α *onto the stable and unstable subspaces, respectively.*

Proof. Let us first prove that the limits in equation (7.1) exist. For $t\in\mathbb{R}$, we have

$$\begin{aligned}
\int_{-\infty}^{t}|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))|ds \quad &\leq \quad \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}|f(s)|ds\\
&\leq \quad \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\Big(\int_{t-n}^{t-n+1}e^{-\omega(t-s)}|f(s)|ds\Big).
\end{aligned}$$

Let q be such that $\dfrac{1}{q}+\dfrac{1}{p}=1$. Using Hölder inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{t} |\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|ds \\
&\qquad \leq \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\Big[\Big(\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\qquad \leq \frac{\overline{M}\widetilde{M}|\Pi^s|}{\sqrt[q]{q\omega}}\sum_{n=1}^{\infty}\Big[(e^{-q\omega(n-1)}-e^{-q\omega n})^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\qquad \leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}-1)^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{-\omega n} \\
&\qquad \leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\sum_{n=1}^{\infty}e^{-\omega n}.
\end{aligned}$$

Since the series $\displaystyle\sum_{n=1}^{+\infty}e^{-q\omega n}$ is convergent, it follows that

$$\int_{-\infty}^{t} |\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))|ds \quad < \quad \gamma \tag{7.2}$$

with $\quad \gamma=\dfrac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\displaystyle\sum_{n=1}^{\infty}e^{-\omega n}.$

Set $F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))$ for $n\in\mathbb{N}$ and $s\leq t$. For n is sufficiently large and $\sigma\leq t$, we have

$$\begin{aligned}
&\Big|\int_{-\infty}^{\sigma} F(n,s,t)ds\Big| \\
&\qquad \leq \overline{M}\widetilde{M}|\Pi^s|\sum_{n=1}^{\infty}\Big[\Big(\int_{\sigma-n}^{\sigma-n+1} e^{-q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{\sigma-n}^{\sigma-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\qquad \leq \frac{\overline{M}\widetilde{M}|\Pi^s|}{\sqrt[q]{q\omega}}\sum_{n=1}^{\infty}\Big[(e^{-q\omega(t-\sigma+n-1)}-e^{-q\omega(t-\sigma+n)})^{\frac{1}{q}}\Big(\int_{\sigma-n}^{\sigma-n+1}|f(s)|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\qquad \leq \frac{\overline{M}\widetilde{M}|\Pi^s|\|f\|_{S^p}}{\sqrt[q]{q\omega}}(e^{q\omega}+1)^{\frac{1}{q}}\times e^{-\omega(t-\sigma)}\sum_{n=1}^{\infty}e^{-\omega n} \\
&\qquad \leq \gamma e^{-\omega(t-\sigma)}.
\end{aligned}$$

It follows that for n and m sufficiently large and $\sigma \leq t$, we have

$$\begin{aligned}\Big|\int_{-\infty}^{t} F(n,s,t)ds - \int_{-\infty}^{t} F(m,s,t)ds\Big| &\leq \Big|\int_{-\infty}^{\sigma} F(n,s,t)ds\Big| + \Big|\int_{-\infty}^{\sigma} F(m,s,t)ds\Big| \\ &\quad + \Big|\int_{\sigma}^{t} F(n,s,t)ds - \int_{\sigma}^{t} F(m,s,t)ds\Big| \\ &\leq 2\gamma e^{-\omega(t-\sigma)} + \Big|\int_{\sigma}^{t} F(n,s,t)ds - \int_{\sigma}^{t} F(m,s,t)ds\Big|.\end{aligned}$$

Since $\lim_{n\to+\infty}\int_{\sigma}^{t} F(n,s,t)ds$ exists, we infer that

$$\limsup_{n,m\to+\infty}\Big|\int_{-\infty}^{t} F(n,s,t)ds - \int_{-\infty}^{t} F(m,s,t)ds\Big| \leq 2\gamma e^{-\omega(t-\sigma)}.$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty}\Big|\int_{-\infty}^{t} F(n,s,t)ds - \int_{-\infty}^{t} F(m,s,t)ds\Big| = 0.$$

Thus, we deduce that the limit

$$\lim_{n\to+\infty}\int_{-\infty}^{t} F(n,s,t)ds = \lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))ds$$

exists. In addition, one can see from equation (7.2) that the function

$$\eta_1 : t \to \lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))ds$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$\eta_2 : t \to \lim_{n\to+\infty}\int_{t}^{\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 f(s))ds$$

is well defined and bounded on $\mathbb{R}$. Using the same argument as in the proof of [8, Theorem 5.9], it can be shown that the integral solution u given by the formula

$$u_t = \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$$

for $t \in \mathbb{R}$ is the only bounded integral solution of equation (1.1) on $\mathbb{R}$. ■

Theorem 7.2. *Let $g \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ and let Γ be the mapping defined by*

$$\Gamma g(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))ds.$$

for $t \in \mathbb{R}$. If $p > 1$, then $\Gamma g \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$.

Proof. For each $n = 1, 2, 3, ...$ and $t \in \mathbb{R}$, set

$$X_n(t) = \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))ds - \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))ds$$

for $t \in \mathbb{R}$. We have

$$
\begin{aligned}
|X_n(t)|_\alpha \ &\leq \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1} |\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))|_\alpha ds \\
&\quad + \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n} |\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))|_\alpha ds\Big] \\
&\leq \lim_{\lambda\to+\infty}\int_{t-n}^{t-n+1} \|\mathcal{A}_{\mathcal{U}}^\alpha \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\| ds \\
&\quad + \lim_{\lambda\to+\infty}\int_{t+n-1}^{t+n} \|\mathcal{A}_{\mathcal{U}}^\alpha \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))\| ds\Big] \\
&\leq \overline{M}\widetilde{M}\int_{t-n}^{t-n+1} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|\Pi^s|\ \|g(s)\|ds + \overline{M}\widetilde{M}\int_{t+n-1}^{t+n} \frac{e^{\omega(t-s)}}{(s-t)^\alpha}|\Pi^u|\ \|g(s)\|ds.
\end{aligned}
$$

Set

$$
K = \max(\overline{M}\widetilde{M}|\Pi^s|, \overline{M}\widetilde{M}|\Pi^u|).
$$

Let q be such that $\dfrac{1}{q}+\dfrac{1}{p}=1$. Using Hölder inequality, we obtain

$$
\begin{aligned}
|\Gamma g(t)|_\alpha \quad &\leq \quad K\Big(\int_{t-n}^{t-n+1} \frac{e^{-q\omega(t-s)}}{(t-s)^{q\alpha}} ds\Big)^{\frac{1}{q}}\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} \\
&\quad +K\Big(\int_{t+n-1}^{t+n} \frac{e^{q\omega(t-s)}}{(s-t)^{q\alpha}} ds\Big)^{\frac{1}{q}}\Big(\int_{t+n-1}^{t+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \quad K\Big(\int_{q\omega(n-1)}^{q\omega n} \frac{e^{-s}}{s^{q\alpha}} \times (\omega q)^{\alpha q-1} ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t+n-1}^{t+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\leq \quad \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha} ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t+n-1}^{t+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\leq \quad \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_0^{q\omega n} e^{-s}s^{-q\alpha} ds - \int_0^{q\omega(n-1)} e^{-s}s^{-q\alpha} ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} \\
&\quad +\Big(\int_{t+n-1}^{t+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big].
\end{aligned}
$$

Since the series

$$
\frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\times\sum_{n=1}^{+\infty}\Big(\int_0^{q\omega n} e^{-s}s^{-q\alpha} ds\Big) \ \text{ and } \ \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\times\sum_{n=1}^{+\infty}\Big(\int_0^{q\omega(n-1)} e^{-s}s^{-q\alpha} ds\Big)^{\frac{1}{q}}
$$

are both convergent, then the series

$$
\frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}} \times \sum_{n=1}^{+\infty}\Big(\int_0^{q\omega n} e^{-s}s^{-q\alpha} ds - \int_0^{q\omega(n-1)} e^{-s}s^{-q\alpha} ds\Big)^{\frac{1}{q}}
$$

is convergent. It follows from the Weierstrass M-test that the sequence of functions $\sum_{n=1}^{N} X_n(t)$ is uniformly convergent on $\mathbb{R}$. Since $g \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$ and

$$|X_n(t)|_\alpha \leq C_q(K,\omega)\Big[\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t+n-1}^{t+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big],$$

where $C_q(K,\omega) = \dfrac{K}{(\omega q)^{\frac{1-\alpha q}{q}}} \times \sum_{n=1}^{+\infty} \Big(\int_0^{q\omega n} e^{-s} s^{-q\alpha} ds - \int_0^{q\omega(n-1)} e^{-s} s^{-q\alpha} ds\Big)^{\frac{1}{q}}$, we conclude that $X_n \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Thus $\sum_{n=1}^{N} X_n(t) \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ by Lemma 6.1. Observing that $\Gamma g(t) = \sum_{n=1}^{+\infty} X_n(t)$, we deduce that $\Gamma g(t) \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$. ■

Theorem 7.3. *Let $g \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$. If $p > 1$, then $\Gamma g \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_\alpha), \mu, \nu, r)$.*

Proof. For each $n = 1, 2, 3, ...,$ let be X_n defined as in the proof of Theorem 7.2. We have

$$|X_n(t)|_\alpha^p \leq C_q^p(K,\omega)\Big[\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t+n-1}^{t+n} |g(s)|_\alpha^p ds\Big)^{\frac{1}{p}}\Big]^p.$$

Using Minkowski inequality, we obtain

$$\begin{aligned}
\Big(\int_\theta^{\theta+1} |X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} &\leq C_q(K,\omega)\Big[\Big(\int_\theta^{\theta+1}\Big[\Big(\int_{s-n}^{s-n+1} \|g(\xi)\|^p d\xi\Big)^{\frac{1}{p}}\Big]^p ds\Big)^{\frac{1}{p}} \\
&\quad + \Big(\int_\theta^{\theta+1}\Big[\Big(\int_{s+n-1}^{s+n} \|g(\xi)\|^p d\xi\Big)^{\frac{1}{p}}\Big]^p ds\Big)^{\frac{1}{p}}\Big] \\
&\leq C_q(K,\omega)\Big[\Big(\sup_{s\in[\theta,\theta+1]} \int_{s-n}^{s-n+1} \|g(\xi)\|^p d\xi\Big)^{\frac{1}{p}} \\
&\quad + \Big(\sup_{s\in[\theta,\theta+1]} \int_{s+n-1}^{s+n} \|g(\xi)\|^p d\xi\Big)^{\frac{1}{p}}\Big] \\
&\leq C_q(K,\omega)\Big[\Big(\int_{\theta-n}^{\theta-n+2} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{\theta+n-1}^{\theta+n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big] \\
&\leq C_q(K,\omega)\Big[\Big(\int_{\theta-n}^{\theta-n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{\theta-n+1}^{\theta-n+2} \|g(s)\|^p ds\Big)^{\frac{1}{p}} \\
&\quad + \Big(\int_{\theta+n-1}^{\theta+n} \|g(s)\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{\theta+n}^{\theta+n+1} \|g(s)\|^p ds\Big)^{\frac{1}{p}}\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\quad\leq\ C_q(K,\omega)\Big[\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n}^{\theta-n+1}\|g(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n+1}^{\theta-n+2}\|g(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\qquad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n-1}^{\theta+n}\|g(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)+\\
&\qquad+\frac{1}{\mu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta+n}^{\theta+n+1}\|g(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\Big].
\end{aligned}$$

We conclude that $X_n\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$.

Thus $\sum_{n=1}^{N}X_n(t)\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ and its uniform limit belongs $\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$ by Lemma 6.1. Observing that

$$\Gamma g(t)=\sum_{n=1}^{+\infty}X_n(t),$$

we deduce that $\Gamma g(t)\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$. ∎

Theorem 7.4. *Let $h\in AP(\mathbb{R},L^p((0,1),X))$. If $p>1$, then $\Gamma h\in AP(\mathbb{R},L^p((0,1),X))$.*

Proof. Since $h\in AP(\mathbb{R},L^p((0,1),X_\alpha))$, then there exists a relatively dense subset $\mathcal{P}(\varepsilon,h)$ of $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|h(s+\xi)-h(s)|_\alpha^p ds\Big)^{\frac{1}{p}}<\varepsilon\ \text{ for}(t,\xi)\in\mathbb{R}\times\mathcal{P}(\varepsilon,h).$$

In [14], the authors proved that for $\tau\in\mathbb{R}$ $(\Gamma h)_\tau=(\Gamma h_\tau)$, thus for $\xi\in\mathcal{P}(\varepsilon,h)$ we have

$$\begin{aligned}
(\Gamma h)(s+\xi)-(\Gamma h)(s)\ &=\ \lim_{\lambda\to+\infty}\int_{-\infty}^{s}\mathcal{U}^s(s-\theta)\Pi^s(\widetilde{B}_\lambda X_0[h(\theta+\xi)-h(\theta)]d\theta\\
&\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{s}\mathcal{U}^u(s-\theta)\Pi^u(\widetilde{B}_\lambda X_0[h(\theta+\xi)-h(\theta)]d\theta\\
|(\Gamma h)(s+\xi)-(\Gamma h)(s)|_\alpha\ &\leq\ K\int_{-\infty}^{s}\frac{e^{-\omega(s-\theta)}}{(s-\theta)^\alpha}\|h(\theta+\xi)-h(\theta)\|d\theta\\
&\quad+K\int_{+\infty}^{s}\frac{e^{\omega(\theta-s)}}{(s-\theta)^\alpha}\|h(\theta+\xi)-h(\theta)\|d\theta,
\end{aligned}$$

where

$$K=\max(\overline{M}\widetilde{M}|\Pi^s|,\overline{M}\widetilde{M}|\Pi^u|).$$

For each $n = 1, 2, 3, ...,$ set

$$X_n(s) = K\int_{s-n}^{s-n+1} \frac{e^{-\omega(s-\theta)}}{(s-\theta)^{\alpha}}\|h(\theta+\xi)-h(\theta)\|d\theta + K\int_{s+n-1}^{s+n} \frac{e^{\omega(\theta-s)}}{(s-\theta)^{\alpha}}\|h(\theta+\xi)-h(\theta)\|d\theta.$$

Using Hölder inequality as in the proof of Theorem 7.2, we obtain

$$\begin{aligned} X_n(s) &\leq \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{s-n}^{s-n+1} \|h(\theta+\xi)-h(\theta)\|^p d\theta\Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_{s+n-1}^{+n} |h(\theta+\xi)-h(\theta)|^p d\theta\Big)^{\frac{1}{p}}\Big]. \end{aligned}$$

By Minkowski inequality, we have

$$\begin{aligned} &\Big(\int_t^{t+1} |X_n(s)|_{\alpha}^p ds\Big)^{\frac{1}{p}} \\ &\leq \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_t^{t+1}\Big[\Big(\int_{s-n}^{s-n+1} \|h(\theta+\xi)-h(\theta)\|^p d\theta\Big)^{\frac{1}{p}}\Big]^p ds\Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_t^{t+1}\Big[\Big(\int_{s+n-1}^{s+n} \|h(\theta+\xi)-h(\theta)\|^p d\theta\Big)^{\frac{1}{p}}\Big]^p ds\Big)^{\frac{1}{p}}\Big] \\ &\leq \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\sup_{s\in[t,t+1]}\int_{s-n}^{s-n+1} \|h(\theta+\xi)-h(\theta)\|^p d\theta\Big)^{\frac{1}{p}} \\ &\quad + \Big(\sup_{s\in[t,t+1]}\int_{s+n-1}^{s+n} \|h(\theta+\xi)-h(\theta)\|^p d\theta\Big)^{\frac{1}{p}}\Big] \\ &\leq \frac{K}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1} \|h(s+\xi)-h(s)\|^p ds\Big)^{\frac{1}{p}}. \end{aligned}$$

Since

$$\sum_{n=1}^{+\infty} X_n(s) = K\int_{-\infty}^{s} \frac{e^{-\omega(s-\theta)}}{(s-\theta)^{\alpha}}\|h(\theta+\xi)-h(\theta)\|d\theta + K\int_{+\infty}^{s} \frac{e^{\omega(s-\theta)}}{(s-\theta)^{\alpha}}\|h(\theta+\xi)-h(\theta)\|d\theta,$$

it follows that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1} |(\Gamma h)(s+\xi)-(\Gamma h)(s)|_{\alpha}^p ds\Big)^{\frac{1}{p}} \leq 2C_q(K,\omega)\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1} \|h(s+\xi)-h(s)\|^p ds\Big)^{\frac{1}{p}}.$$

This implies that $\Gamma h \in AP(\mathbb{R}, L^p((0,1), X_{\alpha}))$. ∎

For the existence of pseudo almost periodic solution, we make the following assertion.

($\mathbf{H_5}$) $f : \mathbb{R} \to X$ is $cl(\mu,\nu)$-S^p-pseudo almost periodic of class r.

Theorem 7.5. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_3}$) and ($\boldsymbol{H_5}$) hold. If $p > 1$, then equation (1.1) has a unique $cl(\mu,\nu)$-S^p-pseudo almost periodic solution of class r.*

Proof. Since f is S^p-pseudo almost periodic function, f has a decomposition $f = f_1 + f_2$ where $f_1^b \in AP(\mathbb{R}, L^p((0,1), X_{\alpha}))$ and $f_2^b \in \mathcal{E}(\mathbb{R}, L^p((0,1), X_{\alpha}), \mu, \nu, r)$. Using Theorem 7.1, Theorem 7.3 and Theorem 7.4, we get the desired result. ∎

Our next objective is to show the existence of pseudo almost periodic solutions of class r for the following problem

$$u'(t) = -Au(t) + L(u_t) + f(t,u_t) \text{ for } t \in \mathbb{R} \tag{7.3}$$

where $f : \mathbb{R} \times C_\alpha \to X$ is a continuous.

For the sequel, we make the following assertion.

$(\mathbf{H_6})$ $f : \mathbb{R} \times C_\alpha \to X$ is uniformly $cl(\mu,\nu)$-S^p-pseudo almost periodic of class r such that there exists a positive constant L_f such that

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| \leq L_f \|\varphi_1 - \varphi_2\|_{C_\alpha} \quad \text{for all } \ t \in \mathbb{R} \ \text{ and } \ \varphi_1, \varphi_2 \in C_\alpha.$$

Theorem 7.6. *Assume $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H_3})$, $(\boldsymbol{H_4})$, $(\boldsymbol{H_6})$ and $(\boldsymbol{H_8})$ hold. If*

$$2C_q(K,\omega)L_f < 1,$$

where $C_q(K,\omega) = \dfrac{K}{(\omega q)^{\frac{1-\alpha q}{q}}} \times \displaystyle\sum_{n=1}^{+\infty} \Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s} s^{-q\alpha} ds \Big)^{\frac{1}{q}}$ *and $p > 1$, then equation (7.3) has a unique $cl(\mu,\nu)$-S^p-pseudo almost periodic solution of class r.*

Proof. Let x be a function in $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. From Theorem 6.14 it follows that the function $t \to x_t$ belongs to $PAPS^p(C_\alpha, \mu, \nu, r)$. Hence Theorem 6.11 implies that the function $g(.) := f(., x_.)$ is in $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Consider the mapping

$$\mathcal{H} : PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r) \to PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$$

defined for $t \in \mathbb{R}$ by

$$\begin{aligned}(\mathcal{H}x)(t) \quad = \quad & \Big[\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds \\ & + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s,x_s))ds \Big].\end{aligned}$$

From Theorem 7.1, Theorem 7.3 and taking into account Theorem 7.4, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Let $x_1, x_2 \in PAPS^p(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Then we have

$$\begin{aligned}|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)|_\alpha \ \leq \ & \Big| \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f((s,x_{1s})) - f((s,x_{1s}))]ds \Big|_\alpha \\ & + \Big| \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda X_0[f((s,x_{2s})) - f((s,x_{2s}))]ds \Big|_\alpha \\ \leq \ & KL_f \Big(\int_{-\infty}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} \|x_{1s} - x_{2s}\| ds + \int_{t}^{+\infty} \frac{e^{\omega(t-s)}}{(s-t)^\alpha} \|x_{1s} - x_{2s}\| ds \Big).\end{aligned}$$

For each $n = 1, 2, 3, \ldots$, set

$$X_n(t) = KL_f \Big(\int_{t-n}^{t-n+1} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} \|x_{1s} - x_{2s}\| ds + \int_{t+n-1}^{t+n} \frac{e^{\omega(t-s)}}{(s-t)^\alpha} \|x_{1s} - x_{2s}\| ds \Big).$$

Then for each $n = 1, 2, 3, ...,$ using a same reasoning as in the proof of Theorem 7.4, we have

$$X_n(t) \leq \frac{KL_f}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}}\Big[\Big(\int_{t-n}^{t-n+1}\|x_{1s}-x_{2s}\|^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t+n-1}^{t+n}\|x_{1s}-x_{2s}\|^p ds\Big)^{\frac{1}{p}}\Big]$$

and

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|X_n(s)|_\alpha^p ds\Big)^{\frac{1}{p}} \leq \frac{2KL_f}{(\omega q)^{\frac{1-\alpha q}{q}}}\Big(\int_{q\omega(n-1)}^{q\omega n} e^{-s}s^{-q\alpha}ds\Big)^{\frac{1}{q}} \sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|x_1(s)-x_2(s)\|^p ds\Big)^{\frac{1}{p}}\Big].$$

Since

$$\sum_{n=1}^{+\infty} X_n(t) = KL_f\Big(\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}\|x_{1s}-x_{2s}\|ds + \int_t^{+\infty}\frac{e^{\omega(t-s)}}{(s-t)^\alpha}\|x_{1s}-x_{2s}\|ds\Big),$$

it follows that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}|\mathcal{H}x_1(s)-\mathcal{H}x_2(s)|_\alpha^p ds\Big)^{\frac{1}{p}} \leq 2C_q(K,\omega)L_f\|x_1-x_2\|_{S^p}.$$

Thus, $\mathcal{H}$ is a contractive mapping. We conclude that equation (7.3), has one and only one $cl(\mu,\nu)$-S^p-pseudo almost periodic solution of class r. This ends the proof. ∎

8. Example

For illustration, we will study the existence of solutions for the following model

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x))d\theta - \cos t - \frac{1}{\sqrt{2}}\cos(\sqrt{2}t) + g(t) \\ \qquad\qquad + h\Big(t, \dfrac{\partial}{\partial x}z(t+\theta,x)\Big) \text{ for } t\in\mathbb{R} \text{ and } x\in[0,\pi] \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R}, \end{cases} \tag{8.1}$$

where $G : [-r, 0]$ into $\mathbb{R}$ is a continuous function, $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies Lipschitz condition with respect to the second argument and $g : \mathbb{R}\times[0,\pi] \to \mathbb{R}$ is a bounded continuous function defined by

$$g(t) = \begin{cases} 0 \ \text{ for } \ t \leq 0 \\ -(t+1)e^{-t} \ \text{ for } \ t \geq 0. \end{cases}$$

To rewrite equation (8.1) in the abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $A : X \to X$ be defined by

$$\begin{cases} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \\ Ay = y''. \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 :\ n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns),\ s \in [0,\pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2(y,e_n)e_n,\ y \in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X :\ \sum_{n=1}^{+\infty} n(y,e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,e_n)e_n,\ y \in D(A).$$

Lemma 8.1 ([15]). *If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and $|y'| = |A^{\frac{1}{2}}y|$.*

It is well known that $-A$ is the generator of a compact analytic semigroup semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x,e_n)e_n,\ x \in X.$$

Then $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satified. Here we choose $\alpha = \dfrac{1}{2}$.

We define $f : \mathbb{R} \times C_{\frac{1}{2}} \to X$ and $L : C_{\frac{1}{2}} \to X$ as follows

$$f(t,\varphi)(x) \quad = \quad -\cos t - \frac{1}{\sqrt{2}}\cos(\sqrt{2}t) + g(t) + h\Big(t, \frac{\partial}{\partial x}\varphi(\theta,x)\Big) \text{ for } x \in [0,\pi] \text{ and } t \in \mathbb{R}$$

and

$$L(\varphi)(x) \;=\; \int_{-r}^{0} G(\theta)\varphi(\theta,x))d\theta \;\text{ for }\; -r\leq\theta\leq 0 \;\text{ and }\; x\in[0,\pi].$$

Let us set $v(t)=z(t,x)$. Then equation (8.1) takes the following abstract form

$$v'(t)=Av(t)+L(v_t)+f(t,v_t) \text{ for } t\in\mathbb{R}. \tag{8.2}$$

Consider the measures μ and ν whose Radon-Nikodym derivatives are respectively are respectively $\rho_1,\rho_2:\mathbb{R}\to\mathbb{R}$ defined by

$$\rho_1(t)=\begin{cases}1 \;\text{ for }\; t>0\\ e^t \;\text{ for }\; t\leq 0.\end{cases}$$

and

$$\rho_2(t)=|t| \;\text{ for }\; t\in\mathbb{R}$$

i.e $d\mu(t)=\rho_1(t)dt$ and $d\nu(t)=\rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A)=\int_A \rho_1(t)dt \;\text{ for }\; \nu(A)=\int_A\rho_2(t)dt \;\text{ for }\; A\in\mathcal{B}.$$

From [5] we know that $\mu,\nu\in\mathcal{M}$ and that μ,ν satisfy hypothesis $(\mathbf{H_4})$.
Since $A^{\frac{1}{2}}\Big(-\cos t-\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}t)\Big)=\sin t+\sin(\sqrt{2}t)$ and the function $t\to\sin t+\sin(\sqrt{2}t)$ belongs to $AP(\mathbb{R},X)$ from [16], then by Proposition 5.10 $t\to(\sin t+\sin(\sqrt{2}t))$ is S^p-almost periodic. It follows that he function $t\to\Big(-\cos t-\dfrac{1}{\sqrt{2}}\cos(\sqrt{2}t)\Big)$ belongs to $AP(\mathbb{R},L^p((0,1),X_{\frac{1}{2}}))$.

We have

$$\limsup_{\tau\to+\infty}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\limsup_{\tau\to+\infty}\frac{\displaystyle\int_{-\tau}^{0}e^t dt+\int_0^{\tau}dt}{\displaystyle 2\int_0^{\tau}tdt}=\limsup_{\tau\to+\infty}\frac{1-e^{-\tau}+\tau}{\tau^2}=0<\infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.

Let $p\geq 1$. Since r is given then we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|g(s)|_{\frac{1}{2}}^{p}ds\Big)^{\frac{1}{p}}d\mu(t)\\
&\quad= \frac{1}{\mu([-\tau,\tau])}\int_{0}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}\|A^{\frac{1}{2}}g(s)\|^{p}ds\Big)^{\frac{1}{p}}dt\\
&\quad= \frac{1}{\mu([-\tau,\tau])}\int_{0}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}\|g'(s)\|^{p}ds\Big)^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\nu([-\tau,\tau])}\int_{0}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{t+1}s^{p}e^{-ps}ds\Big)^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\nu([-\tau,\tau])}\int_{0}^{+\tau}\Big(\int_{t-r}^{t+1}s^{p}e^{-s}ds\Big)^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\nu([-\tau,\tau])}\int_{0}^{+\tau}\Big[\Big(\int_{t-r}^{t+1}s^{p^2}\Big)^{\frac{1}{p}}\Big(\int_{t-r}^{t+1}e^{-qs}ds\Big)^{\frac{1}{q}}\Big]^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\mu([-\tau,\tau])}\int_{0}^{+\tau}\Big[[(t+1)^{p^2+1}]^{\frac{1}{p}}e^{-(t-r)}\Big]^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\nu([-\tau,\tau])}\int_{0}^{+\infty}\Big[(t+1)^{p+1}e^{-(t-r)}\Big]^{\frac{1}{p}}dt\\
&\quad\le \frac{1}{\nu([-\tau,\tau])}\int_{0}^{+\infty}(t+1)^{2}e^{-\frac{(t-r)}{p}}dt\\
&\quad\le \frac{5e^{\frac{r}{p}}}{p\tau^2}.
\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}|g(s)|_{\alpha}^{p}ds\Big)^{\frac{1}{p}}dt=0.$$

It follows that $g\in\mathcal{E}(\mathbb{R},L^p((0,1),X_\alpha),\mu,\nu,r)$. Consequently, f is uniformly μ-S^p-pseudo almost periodic of class r. Moreover, L is a bounded linear operator from $C_{\frac{1}{2}}$ to X. Let k be the lipschiz constant of h, then for every $\varphi_1,\varphi_2\in C_{\frac{1}{2}}$ and $t\ge 0$, we have

$$\begin{aligned}
\|f(t,\varphi_1)(x)-f(t,\varphi_2)(x)\| &= \Big(\int_0^{\pi}\Big[h\Big(\theta,\frac{\partial}{\partial x}\varphi_1(\theta,x)\Big)-h\Big(t,\frac{\partial}{\partial x}\varphi_1(t,x)\Big)\Big]^2dx\Big)^{\frac{1}{2}}\\
&\le L_h\Big[\int_0^{\pi}\Big(\frac{\partial}{\partial x}\varphi_1(\theta,x)-\frac{\partial}{\partial x}\varphi_2(\theta,x)\Big)^2dx\Big]^{\frac{1}{2}}\\
&\le L_h\sup_{-r\le\theta\le 0}\Big[\int_0^{\pi}\Big(\frac{\partial}{\partial x}\varphi_1(\theta,x)-\frac{\partial}{\partial x}\varphi_2(\theta,x)\Big)^2dx\Big]^{\frac{1}{2}}\\
&\le L_h\|\varphi_1-\varphi_2\|_{C_\alpha}
\end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of class r.

Lemma 8.2 ([17]). *If* $\int_{-r}^{0}|G(\theta)|d\theta < 1$, *then the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic.*

For example, let us set $G(\theta) = \dfrac{\theta^2-1}{(\theta^2+1)^2}$ for $\theta \in [-r, 0]$. We can see that in our case we have

$$\int_{-r}^{0}|G(\theta)|d\theta = \int_{-r}^{0}\left|\frac{\theta^2-1}{(\theta^2+1)^2}\right|d\theta = \left[\frac{\theta}{\theta^2+1}\right]_{-r}^{0} = \frac{r}{r^2+1} < 1 \text{ if } r < 1$$

and

$$\begin{aligned}\int_{-r}^{0}|G(\theta)|d\theta = \int_{-r}^{0}\left|\frac{\theta^2-1}{(\theta^2+1)^2}\right|d\theta &= \int_{-r}^{-1}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta + \int_{-1}^{0}\frac{-\theta^2+1}{(\theta^2+1)^2}d\theta \\ &= 1 - \frac{r}{r^2+1} \\ &< 1 \text{ if } r \geq 1,\end{aligned}$$

Lemma 8.3. *Under the above assumptions, if Lip*(h) *is small enough, then equation* (8.1) *has a unique* $cl(\mu,\nu)$-S^p*-pseudo almost periodic solution* v *of class* r.

9. Conclusion

In this paper we give a new approach to study Stepanov-like pseudo almost periodic functions in α-norm using the measure theory. We also study the existence and uniqueness of (μ,ν)- Stepanov-like pseudo almost periodic solutions of class r for some partial functional differential equations in a Banach space.

Since C^n-(μ,ν)- Stepanov-like pseudo almost periodic functions is more general than (μ,ν)- Stepanov-like pseudo almost periodic functions, we desire to find sufficient conditions to extend this work in the case of C^n-(μ,ν)- Stepanov-like pseudo almost periodic functions. On the other hand, the existence of (μ,ν)- Stepanov-like pseudo almost periodic solution of class r studied in this work, gives a unique $cl(\mu,\nu)$-(μ,ν)- Stepanov-like pseudo almost periodic which can contain many solutions. Next works should be to find some efficient hypotheses to get a unique pseudo almost periodic solution.

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