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Generalized Cesàro Vector-Valued Sequence Space Using Modulus Function

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Abstract In this paper, we introduced generalized Cesàro vector-valued sequence space $X(E, f, \Delta^m, p)$ by taking sequence (E_k, q_k) of seminormed spaces, modulus function f, m^{th}-order difference operator Δ^m and bounded sequence (p_k) of strictly positive real numbers. It is proved that the space $X(E, f, \Delta^m, p)$ is complete paranormed space if (E_k, q_k) is a sequence of complete seminormed spaces. Some inclusion relations on the space are obtained. By using composite function f^v, space $X(E, f^v, \Delta^m, p)$ is studied for any $v \in \mathbb{N}$. A result on multiplier space of $X(E, f, \Delta^m, p)$ is also obtained, if $m = 0$.

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1. Introduction

In 1953, Nakano [1] introduced the notion of modulus function. The idea of modulus function is generalized by Ruckle [2] for constructing a class of FK-spaces $L(f)$ as follows:

$$L(f) = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\},$$

where w denotes the space of all sequences $x = (x_k)$ of complex numbers.

He proved that intersection of all such $L(f)$ spaces is empty set, a negative answer to question of Wilansky [3] "Is there a smallest FK-space E in which the set $\{e_1, e_2 \dots\}$ of unit vectors is bounded?"

Kizmaz [4] defined the sequence spaces

$$X(\Delta) = \{x = (x_k \in w : \Delta x \in X\} \text{ for } X = l_\infty,\ c \text{ and } c_0,$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and he introduced that these sequence spaces are Banach spaces with norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty,\ x \in X(\Delta).$$

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Et and Colak [5] introduced sequence spaces using m^{th}-order difference operator defined as follows:

$$X(\Delta^m) = \{x = (x_k) \in w : \Delta^m x \in X\} \text{ for } X = l_\infty,\ c \text{ and } c_0.$$

These sequence spaces are BK-spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty,\ x \in X(\Delta^m),\ \text{where } m \in \mathbb{N}.$$

Furthermore, the notion of difference operator was investigated from different aspects by Ercan and Bektaş [6], Ercan [7], Ercan and Bektaş [8] and many others.
Shuie [9] introduced Cesàro sequence spaces

$$\begin{aligned} Ces_p &= \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=1}^{n} |x_k| \right)^p < \infty, 1 \leqslant p < \infty \right\} \\ \text{and } Ces_\infty &= \left\{ x = (x_k) \in w : \sup_n n^{-1} \sum_{k=1}^{n} |x_k| < \infty \right\}. \end{aligned}$$

He observed that the inclusion $\ell_p \subset Ces_p$ is strict for $1 < p < \infty$.
Orhan [10] generalized spaces Ces_p and Ces_∞ using difference operator Δ by

$$\begin{aligned} Ces_p(\Delta) &= \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=1}^{n} |\Delta x_k| \right)^p < \infty, 1 \leqslant p < \infty \right\} \\ \text{and } Ces_\infty(\Delta) &= \left\{ x = (x_k) \in w : \sup_n n^{-1} \sum_{k=1}^{n} |\Delta x_k| < \infty \right\}. \end{aligned}$$

Et [11] generalized spaces $Ces_p(\Delta)$ and $Ces_\infty(\Delta)$ by using Δ^m operator as follows:

$$\begin{aligned} Ces_p(\Delta^m) &= \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=1}^{n} |\Delta^m x_k| \right)^p < \infty, 1 \leqslant p < \infty \right\} \\ \text{and } Ces_\infty(\Delta^m) &= \left\{ x = (x_k) \in w : \sup_n n^{-1} \sum_{k=1}^{n} |\Delta^m x_k| < \infty \right\}. \end{aligned}$$

Sanhan and Suantai [12] introduced generalized Cesàro sequence space by taking bounded sequence $p = (p_n)$ of strictly positive real numbers as follows:

$$Ces(p) = \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=1}^{n} |x_k| \right)^{p_n} < \infty \right\}.$$

Indu Bala [13] introduced space $Ces(f, p)$ with the help of modulus function f given by

$$Ces(f, p) = \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left[f\left(n^{-1} \sum_{k=1}^{n} |x_k| \right) \right]^{p_n} < \infty \right\}.$$

Sudsukh [14] studied vector-valued Cesàro sequence space, which is defined by

$$Ces(E^*, p) = \left\{ x = (x_k) \in W(E^*) : \sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=1}^{n} ||x_k|| \right)^{p_n} < \infty \right\},$$

where $W(E^*) = \{x = (x_k) : x_k \in E^* \text{ for all } k \in \mathbb{N}\}$ and E^* is a Banach space.

Authors studied various properties including geometry of these spaces. Inclusion relations on these spaces are also obtained. These spaces lead us to introduce a new vector-valued Cesàro difference sequence space as discussed in present paper.

2. A New Sequence Space $X(E, f, \Delta^m, p)$

Let X be a normal AK-sequence algebra with absolutely monotone norm g_X and $E = (E_k, q_k)$ be a sequence of seminormed spaces such that $E_{k+1} \subseteq E_k$ for each $k \in \mathbb{N}$. We define

$$W(E) = \{x = (x_k) : x_k \in E_k \text{ for each } k \in \mathbb{N}\}.$$

Clearly, $W(E)$ is a linear space under usual co-ordinate wise operations of vector addition and scalar multiplication.
We introduced a new vector-valued Cesàro sequence space as follows:

$$X(E, f, \Delta^m, p) = \left\{ x \in W(E) : \left(\left\{ f\left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \in X \right\},$$

where f is a modulus function, $p = (p_k)$ is a bounded sequence of positive real numbers and

$$\Delta^m x_k = \sum_{l=0}^{m} (-1)^l \binom{m}{l} x_{k+l}, \ m \in \mathbb{N} \cup \{0\}.$$

Particular Cases: Some well-known sequence spaces are obtained by taking particular values of X, E_k, f, p_n and m as follows:
(i) If $X = \ell_1$ or $\ell_\infty, f(x) = x, E_k = \mathbb{C}$ for all k, $p_n = p$ $(1 \leqslant p < \infty)$ and $m = 0$, then $X(E, f, \Delta^m, p) = Ces_p$ or Ces_∞, respectively (Shuie [9]).
(ii) If $X = \ell_1$ or $\ell_\infty, f(x) = x, E_k = \mathbb{C}$ for all k, $p_n = p$ $(1 \leqslant p < \infty)$ and $m = 1$, then $X(E, f, \Delta^m, p) = Ces_p(\Delta)$ or $Ces_\infty(\Delta)$, respectively (Orhan [10]).
(iii) If $X = \ell_1$ or $\ell_\infty, f(x) = x, E_k = \mathbb{C}$ for all k and $p_n = p$ $(1 \leqslant p < \infty)$, then $X(E, f, \Delta^m, p) = Ces_p(\Delta^m)$ or $Ces_\infty(\Delta^m)$, respectively (Et [11]).
(iv) If $X = \ell_1, f(x) = x, E_k = \mathbb{C}$ for all k and $m = 0$, then $X(E, f, \Delta^m, p) = Ces(p)$ (Sanhan and Suantai [12]).
(v) If $X = \ell_1, E_k = \mathbb{C}$ for all k and $m = 0$, then $X(E, f, \Delta^m, p) = Ces(f, p)$ (Bala [13]).
(vi) If $X = \ell_1, f(x) = x, E_k = E^*$ for all k, where E^* is a Banach space and $m = 0$, then $X(E, f, \Delta^m, p) = Ces(E^*, p)$ (Sudsukh [14]).

3. Definitions and Some Known Results

Definition 3.1 ([2]). A function $f : [0, \infty) \to [0, \infty)$ is called *modulus function* if it satisfies following conditions:
(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leqslant f(x) + f(y)$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Definition 3.2 ([15]). A sequence space X is called a *K-space* if the co-ordinate function $p_k : X \to \mathbb{K}$ given by $p_k(x) = x_k$ is continuous for each $k \in \mathbb{N}$.

Definition 3.3 ([15]). A complete metric linear space is called a *Frechet space*. An FK-space is a Frechet sequence space with continuous co-ordinates.

Definition 3.4 ([16]). An FK-space X is said to be an AK*-space* if $X \supset \phi$, where ϕ denotes space of finite sequences and (e_n) is a basis for X, i.e., $x^{[n]} = \sum_{k=1}^{n} x_k e_k \to x$ for all $x \in X$. For example, $\ell(p)$ $(1 \leqslant p < \infty)$ and c_0 are AK-spaces. Also, a normed FK-space is called BK-space.

Definition 3.5 ([15]). A sequence space X is called *normal* (or *solid*) space if

$$x = (x_k) \in X \text{ and } |\lambda_k| \leqslant 1 \text{ for each } k \in \mathbb{N} \Rightarrow \lambda x = (\lambda_k x_k) \in X.$$

Example: $\ell(p)$ and $c_0(p)$ are normal spaces.

Definition 3.6 ([15]). A normed algebra X is an algebra with normed linear space satisfying the condition $||xy|| \leqslant ||x||||y||$, for all $x, y \in X$.

Definition 3.7 ([15]). A norm g on a normal sequence space X is said to be *absolutely monotone* if

$$x = (x_k), y = (y_k) \in X \text{ and } |x_k| \leqslant |y_k| \text{ for each } k \in \mathbb{N} \Rightarrow g(x) \leqslant g(y).$$

Result 3.8 ([17]). For $a_k, b_k \in \mathbb{C}$, the following inequalities hold:

$$|a_k + b_k|^{p_k} \leqslant T\{|a_k|^{p_k} + |b_k|^{p_k}\} \qquad (3.1)$$

$$\text{and } |\lambda|^{p_k} \leqslant \max(1, |\lambda|^H), \qquad (3.2)$$

where (p_k) is a bounded sequence of real numbers with $0 < p_k \leqslant \sup_k p_k = H$, $T = \max(1, 2^{H-1})$ and $\lambda \in \mathbb{C}$.

Result 3.9 ([17]). For $a_k, b_k \in \mathbb{C}$ and $0 < p \leqslant 1$,

$$\sum_{k=1}^{n}(|a_k| + |b_k|)^p \leqslant \sum_{k=1}^{n}|a_k|^p + \sum_{k=1}^{n}|b_k|^p. \qquad (3.3)$$

Result 3.10 ([18]). If F is normal sequence algebra and $||\ ||_F$ be absolutely monotone seminorm on F, then for every $u = (u_n), v = (v_n) \in F$ and $p \geqslant 1$,

$$||(u+v)^p||_F^{\frac{1}{p}} \leqslant ||u^p||_F^{\frac{1}{p}} + ||v^p||_F^{\frac{1}{p}}, \qquad (3.4)$$

where $(u+v)^p = ((u_n + v_n)^p)$.

Lemma 3.11 ([19]). If f is a modulus function, then f^r is also modulus function for each $r \in \mathbb{N}$, where $f^r = f \circ f \circ f \cdots \circ f$ (r-times composition of f with itself).

Lemma 3.12 ([20]). Let f_1 and f_2 be modulus functions and $0 < \delta < 1$. If $f_1(t) > \delta$ for $t \in [0, \infty)$. Then

$$(f_2 \circ f_1)(t) < \left(\frac{2f_2(1)}{\delta}\right) f_1(t).$$

Lemma 3.13 ([21]). For any modulus function f,

$$\lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}.$$

4. Main Results on the Sequence Space $X(E, f, \Delta^m, p)$

Theorem 4.1. $X(E, f, \Delta^m, p)$ is a linear space over $\mathbb{C}$.

Proof. Let $x, y \in X(E, f, \Delta^m, p)$ and $\lambda, \mu \in \mathbb{C}$. For every $n \in \mathbb{N}$, by properties of seminorm and modulus function, we get

$$\left[f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(\lambda x_k + \mu y_k))\right)\right]^{p_n}$$
$$\leqslant \left[f\left(|\lambda| n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right) + f\left(|\mu| n^{-1}\sum_{k=1}^{n} q_k(\Delta^m y_k)\right)\right]^{p_n}$$
$$\leqslant \max(1, M_1^H) T\left[f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right]^{p_n}$$
$$+ \max(1, M_2^H) T\left[f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m y_k)\right)\right]^{p_n}, \text{ using inequalities (3.1) and (3.2),}$$

where $H = \sup_k p_k$, $T = \max(1, 2^{H-1})$, $M_1 = \{1 + [|\lambda|]\}$, $M_2 = \{1 + [|\mu|]\}$ and $[|\lambda|]$ denotes integral part of $|\lambda|$. Since X is a normal space, so $\lambda x + \mu y \in X(E, f, \Delta^m, p)$. Hence $X(E, f, \Delta^m, p)$ is a linear space. ∎

Lemma 4.2. Let (E_k, q_k) be a sequence of seminormed spaces and X be normal AK-space with absolutely monotone seminorm g_X. Then the map $\tilde{h}_r : [0, \infty) \to [0, \infty)$ defined by

$$\tilde{h}_r(u) = g_X\left[\sum_{n=1}^{r}\left\{f\left(n^{-1}\sum_{k=1}^{n} u q_k(\Delta^m x_k)\right)\right\}^{p_n} e_n\right]$$

is a continuous function of u, for every positive integer r, where $x = (x_k) \in X$ and (e_k) is a unit vector basis in space X.

Proof. As norm function is continuous, it is sufficient to show that mapping defined by

$$g_n : [0, \infty) \to X,\ \ g_n(u) = \left\{f\left(n^{-1}\sum_{k=1}^{n} u q_k(\Delta^m x_k)\right)\right\}^{p_n} e_n$$

is continuous function for each $n = 1, 2, \cdots r$. For this, let $u_i \to 0$ as $i \to \infty$. Then

$$g_n(u_i) = \left\{f\left(n^{-1}\sum_{k=1}^{n} u_i q_k(\Delta^m x_k)\right)\right\}^{p_n} e_n \to (0, 0, \cdots) \text{ as } i \to \infty.$$

It is true for each $n = 1, 2, \cdots r$. Thus, each g_n is a sequentially continuous function. Hence the function $\tilde{h}_r$ is a continuous function for each $r \in \mathbb{N}$. ∎

Theorem 4.3. Sequence space $X(E, f, \Delta^m, p)$ is paranormed space under paranorm g given by

$$g(x) = \sum_{i=1}^{m} f(q_i(x_i)) + \left(g_X\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right)^{\frac{1}{M}},$$

where $H=\sup_k p_k$, $M=\max(1,H)$.

Proof. (i) Clearly, $g(\theta)=0$ as $q_k(\theta_k)=0$ and $f(0)=0$, where θ_k is a zero vector in $E_k, k\in\mathbb{N}$ and $\theta=(\theta_1,\theta_2,\cdots,\theta_n,\cdots)$.
(ii) $g(-x)=g(x)$ as $q_k(-x)=q_k(x)$, for any $k\in\mathbb{N}$.
(iii) For showing $g(x+y)\leqslant g(x)+g(y)$, let x and y be any arbitrary elements of $X(E,f,\Delta^m,p)$. Then

$$\begin{aligned}
g(x+y) \leqslant\ & \sum_{i=1}^{m} f(q_i(x_i))+\sum_{i=1}^{m} f(q_i(y_i)) \\
&+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)+f\left(n^{-1}\sum_{k=1}^{n}(q_k(\Delta^m y_k))\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}} \\
\leqslant\ & \sum_{i=1}^{m} f(q_i(x_i))+\sum_{i=1}^{m} f(q_i(y_i)) \\
&+\left(g_X\left[\left(\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{\frac{p_n}{M}}+\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m y_k)\right)\right\}^{\frac{p_n}{M}}\right)^{M}\right)\right]\right)^{\frac{1}{M}} \\
&\qquad\qquad \text{using inequality (3.3)} \\
\leqslant\ & \sum_{i=1}^{m} f(q_i(x_i))+\sum_{i=1}^{m} f(q_i(y_i))+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}} \\
&+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m y_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}},\ \text{using inequality (3.4)} \\
=\ & g(x)+g(y).
\end{aligned}$$

(iv) Suppose $\lambda_r\to\lambda$ as $r\to\infty$ and $x^r\to x$ as $r\to\infty$ in $X(E,f,\Delta^m,p)$. Then it is required to prove that $g(\lambda_r x^r-\lambda x)\to 0$ as $r\to\infty$. For this, consider

$$g(\lambda_r x^r-\lambda x)=\sum_{i=1}^{m} f(q_i(\lambda_r x_i^r-\lambda x_i))+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(\lambda_r x_k^r-\lambda x_k))\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}}$$

$$\begin{aligned}
\Rightarrow g(\lambda_r x^r-\lambda x) \leqslant\ & \sum_{i=1}^{m} f(|\lambda_r|q_i(x_i^r-x_i)+|\lambda_r-\lambda|q_i(x_i)) \\
&+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n}|\lambda_r|q_k(\Delta^m(x_k^r-x_k))+n^{-1}\sum_{k=1}^{n}|\lambda_r-\lambda|q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}} \\
\Rightarrow g(\lambda_r x^r-\lambda x) \leqslant\ & \sum_{i=1}^{m} f\left(|\lambda_r|q_i(x_i^r-x_i)+|\lambda_r-\lambda|q_i(x_i)\right) \\
&+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n}|\lambda_r|q_k(\Delta^m(x_k^r-x_k))\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}} \\
&+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n}|\lambda_r-\lambda|q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}},\ \text{by inequalities (3.3) and (3.4)}
\end{aligned}$$

$$\Rightarrow g(\lambda_r x^r - \lambda x) \leqslant M_3 \sum_{i=1}^{m} f(q_i(x_i^r - x_i))$$
$$+ M_3^{H/M} \left(g_X \left[\left(\left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m (x_k^r - x_k)) \right) \right\}^{p_n} \right) \right] \right)^{\frac{1}{M}}$$
$$+ \sum_{i=1}^{m} f(|\lambda_r - \lambda| q_i(x_i)) + \left(g_X \left[\left(\left\{ f \left(n^{-1} \sum_{k=1}^{n} |\lambda_r - \lambda| q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \right] \right)^{\frac{1}{M}},$$

where $M_3 = \sup_r (1 + [|\lambda_r|])$.

$$\Rightarrow g(\lambda_r x^r - \lambda x) \leqslant M_3 g(x^r - x) + \sum_{i=1}^{m} f(|\lambda_r - \lambda| q_i(x_i))$$
$$+ \left(g_X \left[\left(\left\{ f \left(|\lambda_r - \lambda| n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \right] \right)^{\frac{1}{M}} \tag{4.1}$$

The first term of R.H.S. tends to 0 as $r \to \infty$, the second term of R.H.S. also tends to 0 as $r \to \infty$ due to the continuity of f. Now, proof is complete if we will show

$$\left(g_X \left[\left(\left\{ f \left(|\lambda_r - \lambda| n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \right] \right)^{\frac{1}{M}} \to 0 \text{ as } r \to \infty.$$

Since X is an AK-sequence space and $\left(\left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \in X$, we get

$$g_X \left[\left(\left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) - \sum_{n=1}^{m'} \left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} e_n \right] \to 0$$
$$\Rightarrow g_X \left[\sum_{n=m'+1}^{\infty} \left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} e_n \right] \to 0, \text{ as } m' \to \infty.$$

Thus, for every $\varepsilon > 0$, there exists a positive integer m_0 such that

$$g_X \left[\sum_{n=m_0+1}^{\infty} \left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} e_n \right] < \left(\frac{\varepsilon}{2} \right)^M \tag{4.2}$$

Now, $\lambda_r \to \lambda$ as $r \to \infty$ implies that $|\lambda_r - \lambda| < 1$ for all $r \geqslant r_1$. As f is an increasing function implies for all $r \geqslant r_1$,

$$\sum_{n=m_0+1}^{\infty} \left\{ f \left(|\lambda_r - \lambda| n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \leqslant \sum_{n=m_0+1}^{\infty} \left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n}.$$

From monotonicity of g_X, we have

$$g_X \left[\sum_{n=m_0+1}^{\infty} \left\{ f \left(|\lambda_r - \lambda| n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} e_n \right] \leqslant g_X \left[\sum_{n=m_0+1}^{\infty} \left\{ f \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} e_n \right].$$

By inequality (4.2), for all $r \geqslant r_1$

$$g_X\left[\sum_{n=m_0+1}^{\infty}\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]<\left(\frac{\varepsilon}{2}\right)^M \tag{4.3}$$

Now, from Lemma 4.2, the function

$$\tilde{h}_{m_0}(u)=g_X\left[\sum_{n=1}^{m_0}\left\{f\left(n^{-1}\sum_{k=1}^{n}uq_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]$$

is continuous function in u. Hence, there exist a δ $(0<\delta<1)$ such that

$$\tilde{h}_{m_0}(u)<\left(\frac{\varepsilon}{2}\right)^M, \text{ for } 0<u<\delta.$$

For given $\delta>0$, there exist a $r_2\in\mathbb{N}$ such that $|\lambda_r-\lambda|<\delta$, for all $r\geqslant r_2$. This implies that for all $r\geqslant r_2$,

$$\tilde{h}_{m_0}(|\lambda_r-\lambda|)=g_X\left[\sum_{n=1}^{m_0}\left\{f\left(n^{-1}\sum_{k=1}^{n}|\lambda_r-\lambda|q_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]<\left(\frac{\varepsilon}{2}\right)^M \tag{4.4}$$

Finally,

$$\left(g_X\left[\left(\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}}$$
$$=\left(g_X\left[\sum_{n=1}^{\infty}\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]\right)^{\frac{1}{M}}$$
$$\Rightarrow\left(g_X\left[\left(\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}}$$
$$\leqslant\left(g_X\left[\sum_{n=1}^{m_0}\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]\right)^{\frac{1}{M}}$$
$$+\left(g_X\left[\sum_{n=m_0+1}^{\infty}\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}e_n\right]\right)^{\frac{1}{M}}$$
$$\Rightarrow\left(g_X\left[\left(\left\{f\left(|\lambda_r-\lambda|n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

for all $r\geqslant r_0=\max(r_1,r_2)$, by inequalities (4.3) and (4.4). From inequality (4.1), we have $g(\lambda_r x^r-\lambda x)\to 0$ as $r\to\infty$. Hence $X(E,f,\Delta^m,p)$ is a paranormed space. ■

Theorem 4.4. *$X(E,f,\Delta^m,p)$ is a complete space under paranorm g if (E_k,q_k) is a sequence of complete seminormed spaces.*

Proof. Let (x^r) be a Cauchy sequence in $X(E,f,\Delta^m,p)$. Then

$$g(x^r-x^s)\to 0 \text{ as } r,s\to\infty, \quad \text{i.e.,}$$

$$\sum_{i=1}^{m}f(q_i(x_i^r-x_i^s))+\left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m(x_k^r-x_k^s))\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}}\to 0 \text{ as } r,s\to\infty.$$

Since each term in the above expression is non-negative, so

$$f(q_i(x_i^r - x_i^s)) \to 0 \text{ as } r, s \to \infty \tag{4.5}$$

and

$$g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k^s))\right)\right\}^{p_n}\right)\right] \to 0 \text{ as } r, s \to \infty. \tag{4.6}$$

As f is a continuous function, condition (4.5) implies sequence (x_i^r) in E_i is Cauchy sequence for each $i = 1, 2, \cdots, m$. Since X is K-space, so condition (4.6) gives

$$\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k^s))\right)\right\}^{p_n} \to 0 \text{ as } r, s \to \infty, \text{ for each } n \in \mathbb{N},$$

i.e., $f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k^s))\right) \to 0$ as $r, s \to \infty$, for each $n \in \mathbb{N}$.
By continuity of function f, we have $q_k(\Delta^m(x_k^r - x_k^s)) \to 0$ as $r, s \to \infty$. This means that $(\Delta^m x_k^r)$ is Cauchy sequence in E_k for each k. As (x_i^r) is Cauchy sequence for $1 \leqslant i \leqslant m$ implies that (x_k^r) is Cauchy sequence in E_k for each k. Completeness of E_k implies (x_k^r) converges for each k. Let $x_k \in E_k$ be limit of (x_k^r). Then $q_k(x_k^r - x_k) \to 0$ $(r \to \infty)$ for each k. Again, by the continuity of f,

$$\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right)\right\}^{p_n} \to 0 \text{ as } r \to \infty, \text{ for each } n \in \mathbb{N},$$

which implies that we can choose a sequence η_n^r $(0 < \eta_n^r \leqslant 1)$ such that

$$\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right)\right\}^{p_n} \leqslant \eta_n^r \left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k^r)\right)\right\}^{p_n} \tag{4.7}$$

Now, by property of modulus function and inequality (3.1), we have, for each n

$$\begin{aligned}\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n} &\leqslant \left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right) + f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k^r)\right)\right\}^{p_n} \\ &\leqslant T\left[\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right)\right\}^{p_n} + \left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r)\right)\right\}^{p_n}\right] \\ &\leqslant T(1+\eta_n^r)\left[\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r)\right)\right\}^{p_n}\right] \\ &\leqslant 2T\left[\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r)\right)\right\}^{p_n}\right], \text{ by inequality (4.7).}\end{aligned}$$

Since X is a normal space, so $\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n}\right) \in X$.
Thus $x \in X(E, f, \Delta^m, p)$.
Now, it remains to show that $g(x^r - x) \to 0$ as $r \to \infty$. For every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that $g(x^r - x^s) < \varepsilon$ for all $r, s \geqslant r_1$.

By inequality (4.1), we obtained $\sum_{i=1}^{m} f(q_i(x_i^r - x_i^s)) < \varepsilon$ for all $r, s \geqslant r_1$ and

$$g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k^s))\right)\right\}^{p_n}\right)\right] < \varepsilon^M \text{ for all } r, s \geqslant r_2.$$

By taking $s \to \infty$, we get $\sum_{i=1}^{m} f(q_i(x_i^r - x_i)) < \varepsilon$ for all $r \geqslant r_1$ and

$$g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right)\right\}^{p_n}\right)\right] < \varepsilon^M \text{ for all } r \geqslant r_2.$$

Taking $r_0 = \max(r_1, r_2)$, we have

$$\begin{aligned} g(x^r - x) &= \sum_{i=1}^{m} f(q_i(x_i^r - x_i)) + \left(g_X\left[\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m(x_k^r - x_k))\right)\right\}^{p_n}\right)\right]\right)^{\frac{1}{M}} \\ &< 2\varepsilon \text{ for all } r \geqslant r_0. \end{aligned}$$

■

Theorem 4.5. Space $X(E, f, \Delta^m, p)$ is a normal space if $m = 0$.

Proof. For $m = 0$, the space $X(E, f, \Delta^m, p)$ is denoted by $X(E, f, p)$. Let (λ_k) be a sequence of scalars such that $|\lambda_k| \leqslant 1$ for each $k \in \mathbb{N}$ and x be any arbitrary element of $X(E, f, p)$. Then $\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(x_k)\right)\right\}^{p_n}\right) \in X$ and for each $k \in \mathbb{N}$,

$$q_k(\lambda_k x_k) = |\lambda_k| q_k(x_k) \leqslant q_k(x_k).$$

Now, by using property of modulus function f, we have

$$\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\lambda_k x_k)\right)\right\}^{p_n} \leqslant \left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(x_k)\right)\right\}^{p_n}, \text{ for each } n \in \mathbb{N}$$

As X is a normal space, so $\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(\lambda_k x_k)\right)\right\}^{p_n}\right) \in X$.

This means that $\lambda x \in X(E, f, p)$. ■

5. Inclusion Relations on $X(E, f, \Delta^m, p)$

Theorem 5.1. (i) $X(E, f_1, \Delta^m, p) \cap X(E, f_2, \Delta^m, p) \subseteq X(E, f_1 + f_2, \Delta^m, p)$, where f_1 and f_2 are modulus functions.
(ii) If $X_1 \subseteq X_2$, then $X_1(E, f, \Delta^m, p) \subseteq X_2(E, f, \Delta^m, p)$.
(iii) $X(E, f_1, \Delta^m, p) \subseteq X(E, f \circ f_1, \Delta^m, p)$, where f_1 and f are modulus functions.

Proof. (i) Let $x \in X(E, f_1, \Delta^m, p) \cap X(E, f_2, \Delta^m, p)$. Then

$$\left(\left\{f_1\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n}\right) \in X \text{ and } \left(\left\{f_2\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n}\right) \in X.$$

By using inequality (3.1), we have

$$\left\{(f_1+f_2)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n} \leqslant T\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n} + T\left\{f_2\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n},$$

where $H=\sup_k p_k$, $T=\max(1,2^{H-1})$. Since X is a normal space, so

$\left(\left\{(f_1+f_2)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right) \in X$. Consequently, $x\in X(E,f_1+f_2,\Delta^m,p)$.

(ii) Let $x\in X_1(E,f,\Delta^m,p)$. Then $\left(\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\in X_1$. But $X_1\subseteq X_2$, so $\left(\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\in X_2$ which implies $x\in X_2(E,f,\Delta^m,p)$.

(iii) Let $x\in X(E,f_1,\Delta^m,p)$. Then $\left(\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\in X$.

Construct sets J_1 and J_2 as follows:

for any δ satisfying $0<\delta<1$, $J_1=\{n\in\mathbb{N}:\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}>\delta\}$ and $J_2=\{n\in\mathbb{N}:\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\leqslant\delta\}$.

If $n\in J_2$, then

$$\left\{(f\circ f_1)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\leqslant\{f(\delta)\}^{p_n},\ \text{due to function } f \text{ is increasing.}$$

Also, if $n\in J_1$, then by Lemma 3.12,

$$(f\circ f_1)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right) < \left(\frac{2f(1)}{\delta}\right)\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}$$

$$\Rightarrow\left\{(f\circ f_1)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n} < \max\left(1,\left(\frac{2f(1)}{\delta}\right)^H\right)\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}.$$

Thus, for any $n\in\mathbb{N}$

$$\left\{(f\circ f_1)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n} < \{f(\delta)\}^{p_n} + \max\left(1,\left(\frac{2f(1)}{\delta}\right)^H\right)\left\{f_1\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}.$$

But, X is a normal space, so $\left(\left\{(f\circ f_1)\left(n^{-1}\sum_{k=1}^{n}q_k(\Delta^m x_k)\right)\right\}^{p_n}\right)\in X$. ■

6. The Space $X(E, f^v, \Delta^m, p)$ Using Composite of Modulus Functions

For a fixed positive integer v, we define

$$X(E, f^v, \Delta^m, p) = \left\{ x \in W(E) : \left(\left\{ f^v \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \in X \right\}.$$

It is easy to check that $X(E, f^v, \Delta^m, p)$ is a linear space over $\mathbb{C}$. Also, we denote $X(E, \Delta^m, p)$ in place of $X(E, f^v, \Delta^m, p)$ if $f(x) = x$.

Theorem 6.1. The space $X(E, f^v, \Delta^m, p)$ is a *paranormed space* under paranorm g given by

$$g(x) = \sum_{i=1}^{m} f^v(q_i(x_i)) + \left(g_X \left[\left(\left\{ f^v \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \right] \right)^{\frac{1}{M}}.$$

Proof. The proof of this theorem is same as proof of Theorem 4.3. ∎

Theorem 6.2. (i) For any modulus function f and $v \in \mathbb{N}$,

$$X(E, f^v, \Delta^m, p) \subseteq X(E, \Delta^m, p) \text{ if } \lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0.$$

(ii) Let f be a modulus function such that $f(t) \leqslant \beta t$ for all $t \geqslant 0$ and v, l be positive integers with $l < v$. Then

$$X(E, \Delta^m, p) \subseteq X(E, f^l, \Delta^m, p) \subseteq X(E, f^v, \Delta^m, p).$$

Proof. (i) By Lemma 3.13, we have $\alpha = \inf \left\{ \frac{f(t)}{t} : t > 0 \right\}$, which implies that $\alpha t \leqslant f(t)$ for all $t \geqslant 0$. As f is increasing function, we get $\alpha^2 t \leqslant f^2(t)$. By induction, inequality $\alpha^v t \leqslant f^v(t)$ holds for any $v \in \mathbb{N}$. Let $x \in X(E, f^v, \Delta^m, p)$. Then

$$\left(\left\{ f^v \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \right) \in X.$$

Now,

$$\begin{aligned} \left\{ n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right\}^{p_n} &\leqslant \left\{ \alpha^{-v} f^v \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n} \\ &\leqslant \max(1, \alpha^{-vH}) \left\{ f^v \left(n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right) \right\}^{p_n}, \end{aligned}$$

using inequality (3.2), where $H = \sup_k p_k$. But, X is a normal space, so

$$\left(\left\{ n^{-1} \sum_{k=1}^{n} q_k(\Delta^m x_k) \right\}^{p_n} \right) \in X. \text{ This means that } x \in X(E, \Delta^m, p).$$

(ii) By given condition $f(t) \leqslant \beta t$, we have $f(t) < (1+[\beta])t$. By increasing property of modulus function f, we obtained $f^2(t) < (1+[\beta])^2 t$. Now, by induction for any $l, v \in \mathbb{N}$, we have

$$f^l(t) < (1+[\beta])^l t \text{ and } f^v(t) < (1+[\beta])^v t \tag{6.1}$$

which implies

$$f^v(t) = f^s(f^l(t)) < (1+[\beta])^s f^l(t), \text{ where } s = v - l \in \mathbb{N}. \tag{6.2}$$

Let $x = (x_k) \in X(E, \Delta^m, p)$. Then $\left(\left\{n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right\}^{p_n}\right) \in X$.

By inequality (6.1), we can write

$$\left\{f^l\left(n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right)\right\}^{p_n} < (1+[\beta])^{lH}\left\{n^{-1}\sum_{k=1}^{n} q_k(\Delta^m x_k)\right\}^{p_n}.$$

As X is a normal space, so $x \in X(E, f^l, \Delta^m, p)$. Thus $X(E, \Delta^m, p) \subseteq X(E, f^l, \Delta^m, p)$. Again, if $x \in X(E, f^l, \Delta^m, p)$, then by inequality (6.2) and proceeding same as above, we have $x \in X(E, f^v, \Delta^m, p)$. Hence $X(E, f^l, \Delta^m, p) \subseteq X(E, f^v, \Delta^m, p)$. ■

Example 6.3. For $t \geqslant 0$, functions f_1 and f_2 defined by

$$f_1(t) = t + t^{\frac{1}{2}} \text{ and } f_2(t) = \log(1+t)$$

satisfy conditions given in parts (i) and (ii) of Theorem 6.2, respectively.

7. Multiplier of the Sequence Space $X[E, f, p]$

In this section, we assume that (E_k, q_k) is seminormed algebra for each $k \in \mathbb{N}$. The multiplier set of $X[E, f, p]$ is denoted by $M(X[E, f, p])$, which is defined as follows:

$$M(X[E,f,p]) = \left\{(a_k) \in W(E) : \left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(a_k x_k)\right)\right\}^{p_n}\right) \in X, \text{ for all } x \in X[E,f,p]\right\}.$$

Theorem 7.1. For any modulus function f,

$$\ell_\infty(E) \subseteq M(X[E_k, f, p]), \text{ where } \ell_\infty(E) = \left\{x = (x_k) \in W(E) : \sup_k q_k(x_k) < \infty\right\}.$$

Proof. Let $a = (a_k) \in \ell_\infty(E)$. Then

$$q_k(a_k) < 1 + [H_1] < \infty, \text{ where } H_1 = \sup_k q_k(a_k). \tag{7.1}$$

Let $x = (x_k)$ be any arbitrary element in $X[E, f, p]$. Since (E_k, q_k) is seminormed algebra, for each $k \in \mathbb{N}$, so

$$q_k(a_k x_k) \leqslant q_k(a_k) q_k(x_k), \text{ for each } k \in \mathbb{N}. \tag{7.2}$$

By property of modulus function and above inequalities (7.1) and (7.2), we get

$$\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(a_k x_k)\right)\right\}^{p_n} \leqslant (1+[H_1])^H \left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(x_k)\right)\right\}^{p_n}, \text{ for each } n \in \mathbb{N}.$$

As X is a normal space, so $\left(\left\{f\left(n^{-1}\sum_{k=1}^{n} q_k(a_kx_k)\right)\right\}^{p_n}\right) \in X$.
Consequently, $a \in M(X[E, f, p])$. ■

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