Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008) : 77-84
www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

A quartic functional equation and its generalized Hyers-Ulam-Rassias stability

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Abstract : In this paper, we study the general solution of the quartic functional equation

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y)$$

and prove its generalized Hyers-Ulam-Rassias stability.

Keywords : Functional Equation; Quartic Functional Equation; Stability; Hyers-Ulam-Rassias stability
2000 Mathematics Subject Classification : 39B22, 39B52, 39B82

1 Introduction

In 1940, S.M.Ulam [10] proposed the Ulam stability problem of a linear mapping. In the next year, D.H. Hyers [6] considered the case of approximately additive mapping $f : E \to E'$ where E and E' are Banach spaces and f satisfies the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x)-L(x)\| \le \varepsilon$. In recent years, a number of authors [4, 5, 9] have investigated the stability of linear mappings in various forms. In 2005, S.H. Lee, S.M. Im and I.S. Hwang [8] studied the solution of a quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

and proved its stability in the sense of Hyers-Ulam.

In this paper, we use a different approach to study the general solution of the new functional equation

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y) \tag{1.1}$$

and prove its generalized Hyers-Ulam-Rassias stability.

2 The general solution

In this section, we establish the general solution of (1.1). Throughout this section X and Y will be real vector spaces.

We recall the definition of multiadditive functions. Suppose that $n \in \mathbb{N}$. A function $A_n : X^n \to Y$ is called n-additive if for every r, $1 \leq r \leq n$, and for every $x_1, ..., x_n, y_r \in X$,

$$f(x_1, ..., x_{r-1}, x_r + y_r, x_{r+1}, ..., x_n) = f(x_1, ..., x_n) + f(x_1, ..., x_{r-1}, y_r, x_{r+1}, ..., x_n).$$

That is, A_n is additive with respect to each of its variable $x_r \in X$, $r = 1, ..., n$. Given a function $A_n : X^n \to Y$, by the diagonalization of A_n we understand the function $A^n : X \to Y$ given by the formula

$$A^n(x) := A_n(x, ..., x), \quad x \in X.$$

In our studying for the general solution, we use some fact about a polynomial function and an n-additive symmetric function. A function $f : X \to Y$ is a polynomial function of order s ($s \in \mathbb{N}$) if f fulfil the condition $\Delta_h^{s+1} f(x) = 0$ for every $x, h \in X$ where Δ_x is the forward difference operator with the span x defined by $\Delta_x f(y) = f(y+x) - f(y)$ for all $x, y \in X$. Moreover, it was proved that f can be written as $f = \sum_{n=0}^{s} A^n(x)$, $x \in X$ where $A_n : X^n \to Y$ is an n-additive symmetric function and $A^n : X \to Y$ is the diagonalization of A_n, for each $n = 0, ..., s$ (see [2], pp.71-77).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.1) if and only if there exists a 4-additive symmetric function $A_4 : X^4 \to Y$ such that $f(x) = A_4(x, x, x, x)$ for all $x \in X$.*

Proof. Assume that f satisfies the functional equation (1.1).
Putting $x = y = 0$ in (1.1), we have $f(0) = 0$. Replacing x and y by $x + y$ and $x - y$, respectively, in (1.1), we obtain

$$f(4x + 2y) + f(4x - 2y) = 64f(x + y) + 64f(x - y) + 24f(2x) - 6f(2y). \quad (2.1)$$

Replacing y by $-y$ in (2.1), we can see that

$$f(y) = f(-y)$$

for all $y \in X$. That is f is an even function. Replacing y by $-x$ in (1.1) and using the evenness of f, we get

$$f(2x) = 16f(x) \quad (2.2)$$

for all $x \in X$. Applying (2.2) to (2.1), we obtain

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \quad (2.3)$$

Replacing y by $y+3x$ and $y+2x$, respectively, in (2.3), then take the difference of the two newly obtained equations, we get

$$f(5x+y)-5f(4x+y)+10f(3x+y)-10f(2x+y)+5f(x+y)-f(y)=0$$

Hence, f satisfies the difference functional equation $\Delta_x^5 f(y)=0$. Consequently, f is a polynomial function of order 4. Then there exist n-additive symmetric functions $A_n : X^n \to Y$, $n=0,...,4$, such that

$$f(x)=A^0+A^1(x)+A^2(x)+A^3(x)+A^4(x) \tag{2.4}$$

where $A^n : X \to Y$ is the diagonalization of A_n, for each $n=0,...,4$. Since f is an even function, $A^1(x)$ and $A^3(x)$ must vanish. Moreover, since $f(0)=0$, we have $A^0=0$. Then (2.4) is reduced to

$$f(x)=A^2(x)+A^4(x). \tag{2.5}$$

By using the symmetry and the additivity of $A_2(x,y)$, one can verify that

$$A^2(x+y)+A^2(x-y)=2A^2(x)+2A^2(y). \tag{2.6}$$

Substituting (2.5) into (1.1) and using the property (2.6), we obtain $A^2(x)=0$. Hence, we conclude that $f(x)=A^4(x)$ for all $x\in X$.

Conversely, assume that there exists a 4-additive symmetric function $A_4 : X^4 \to Y$ such that $f(x)=A^4(x)$ for all $x\in X$. Note that $\Delta_x^4 A^4(y)=4!A^4(x)$ (see [2], p.74). Thus, we obtain

$$A^4(4x+y)-4A^4(3x+y)+6A^4(2x+y)-4A^4(x+y)+A^4(y)=24A^4(x). \tag{2.7}$$

Replacing y by $y-x$ in (2.7), we obtain

$$A^4(3x+y)-4A^4(2x+y)+6A^4(x+y)-4A^4(y)+A^4(y-x)=24A^4(x). \tag{2.8}$$

Replacing x and y by $x+y$ and $-2y$, respectively, in (2.8), we obtain

$$A^4(3x+y)-4A^4(2x)+6A^4(x-y)-4A^4(-2y)+A^4(-3y-x)=24A^4(x+y). \tag{2.9}$$

On account of the additivity of $A_4(x_1,x_2,x_3,x_4)$, we have $A^4(nx)=n^4A^4(x)$ for all $n\in\mathbb{Z}$. Then we have

$$A^4(3x+y)+A^4(3y+x)=64A^4(x)+64A^4(y)+24A^4(x+y)-6A^4(x-y) \tag{2.10}$$

By the assumption, we arrive at the functional equation (1.1). □

3 The Generalized Hyers-Ulam-Rassias Stability

Throughout this section X and Y will be a real normed space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x,y) := f(3x+y) + f(x+3y) - 64f(x) - 64f(y) - 24f(x+y) + 6f(x-y)$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \to [0,\infty)$ be a function such that*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{81^i} \text{ converges and} \\ \lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{81^n} = 0 \text{ for all } x, y \in X \end{cases} \tag{3.1}$$

or

$$\begin{cases} \sum_{i=1}^{\infty} 81^i \phi(\frac{x}{3^i}, 0) \text{ converges and} \\ \lim_{n\to\infty} 81^n \phi(\frac{x}{3^n}, \frac{y}{3^n}) = 0 \text{ for all } x, y \in X. \end{cases} \tag{3.2}$$

If a function $f : X \to Y$ satisfies

$$\|Df(x,y)\| \le \phi(x,y) \tag{3.3}$$

for all $x, y \in X$ and $f(0) = 0$, then there exists a unique function $T : X \to Y$ which satisfies the equation (1.1) and the inequality

$$\|f(x) - T(x)\| \le \begin{cases} \frac{1}{81} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{81^i} & \text{if (3.1) holds} \\ \frac{1}{81} \sum_{i=1}^{\infty} 81^i \phi(\frac{x}{3^i}, 0) & \text{if (3.2) holds} \end{cases} \tag{3.4}$$

for all $x \in X$. The function T is given by

$$T(x) = \begin{cases} \lim_{n\to\infty} \frac{f(3^n x)}{81^n} & \text{if (3.1) holds} \\ \lim_{n\to\infty} 81^n f(\frac{x}{3^n}) & \text{if (3.2) holds} \end{cases} \tag{3.5}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (3.3) and dividing by 81, we have

$$\|\frac{f(3x)}{81} - f(x)\| \le \frac{1}{81}\phi(x, 0) \tag{3.6}$$

for all $x \in X$. Replacing x by $3x$ in (3.6) and dividing by 81, we obtain

$$\|\frac{f(3^2 x)}{81^2} - \frac{f(3x)}{81}\| \le \frac{1}{81^2}\phi(3x, 0) \tag{3.7}$$

for all $x \in X$. From the equations (3.6) and (3.7) , we have

$$\|\frac{f(3^2x)}{81^2} - f(x)\| \leq \frac{1}{81}\left(\phi(x,0) + \frac{\phi(3x,0)}{81}\right) \tag{3.8}$$

for all $x \in X$. Using a mathematical induction, we can extend (3.8) to

$$\|\frac{f(3^nx)}{81^n} - f(x)\| \leq \frac{1}{81}\sum_{i=0}^{n-1}\frac{\phi(3^ix,0)}{81^i} \leq \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^ix,0)}{81^i} \tag{3.9}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

For integers $m, n > 0$, we have

$$\begin{aligned}\|\frac{f(3^n3^mx)}{81^{n+m}} - \frac{f(3^mx)}{81^m}\| &= \frac{1}{81^m}\|\frac{f(3^n3^mx)}{81^n} - f(3^mx)\| \\ &\leq \frac{1}{81^m}\cdot\frac{1}{81}\sum_{i=0}^{n-1}\frac{\phi(3^i3^mx,0)}{81^i} \\ &\leq \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i3^mx,0)}{81^{i+m}}\end{aligned}$$

Since the right-hand side of the inequality tends to 0 as $m \to \infty$, the sequence $\{81^{-n}f(3^nx)\}$ is a Cauchy sequence. Since Y is complete, there exists the limit function $T(x) = lim_{n\to\infty}81^{-n}f(3^nx)$ for all $x \in X$. By letting $n \to \infty$ in (3.9), we arrive at the formula (3.4). To show that T satisfies the equation (1.1), replace x and y by 3^nx and 3^ny, respectively, in (3.3) and divide by 81^n, then it follows that

$$\begin{aligned}81^{-n}\|f(3^n(3x+y)) + f(3^n(x+3y)) - 64f(3^nx) - 64f(3^ny) - 24f(3^n(x+y))\\ +6f(3^n(x-y))\| \leq 81^{-n}\phi(3^nx,3^ny)\end{aligned}$$

Taking the limit as $n \to \infty$, we find that T satisfies (1.1) for all $x, y \in X$.

To prove the uniqueness of quartic function T subject to (3.4), assume that there exists a function $S : X \to Y$ which satisfies (1.1) and (3.4) with T replaced by S. Note that Theorem 2.1 gives us $T(3^nx) = 81^nT(x)$ and $S(3^nx) = 81^nS(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\|T(x) - S(x)\| &= \frac{1}{81^n}\|T(3^nx) - S(3^nx)\| \\ &\leq \frac{1}{81^n}\left(\|T(3^nx) - f(3^nx)\| + \|f(3^nx) - S(3^nx)\|\right) \\ &\leq \frac{1}{81^n}\left(\frac{2}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i3^nx,0)}{81^i}\right) \\ &= \frac{2}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i3^nx,0)}{81^{i+n}}\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T. This completes the proof of the theorem. □

Remark 3.2. *In case of condition (3.1) a function f which satisfies the inequality (3.3) needs not to be zero at $x = 0$. By using the same argument, we can find a unique quartic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} 81^{-n} f(3^n x)$ which satisfies the equation (1.1) and the inequality*

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \leq \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i x, 0)}{81^i} \tag{3.10}$$

for all $x \in X$.

Corollary 3.3. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \varepsilon \tag{3.11}$$

for all $x, y \in X$ for some real number $\varepsilon > 0$, then there exists a unique function $T : X \to Y$ such that T satisfies (1.1) and

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \leq \frac{\varepsilon}{80}$$

for all $x \in X$. The function T is given by $T(x) = \lim_{n\to\infty} 81^{-n} f(3^n x)$ for all $x \in X$.

Proof. Taking $\phi(x, y) = \varepsilon$ for all $x, y \in X$. Being in accordance with (3.1) in Remark of Theorem 3.1, we obtain

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \leq \frac{1}{81}\sum_{i=0}^{\infty}\frac{\varepsilon}{81^i} = \frac{\varepsilon}{80}$$

for all $x \in X$, as desired. □

Corollary 3.4. *Given positive real number ε and p with $p \neq 4$. If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \varepsilon\big(\|x\|^p + \|y\|^p\big) \tag{3.12}$$

for all $x, y \in X$, then there exists a unique function $T : X \to Y$ such that T satisfies (1.1) and

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{|3^4 - 3^p|}\|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x,y) = \varepsilon\big(\|x\|^p + \|y\|^p\big)$ for all $x, y \in X$.
Putting $x = y = 0$ in (3.12), we obtain $\|f(0)\| \leq 0$. Hence,we have $f(0) = 0$.
If $0 < p < 4$, then the condition (3.1) in Theorem 3.1 holds. It follows that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\varepsilon}{81}\sum_{i=0}^{\infty}\frac{\big(3^{ip}\|x\|^p\big)}{81^i}\\ &= \frac{\varepsilon}{3^4 - 3^p}\|x\|^p\end{aligned}$$

for all $x \in X$. If $p > 4$, then the condition (3.2) in Theorem 3.1 holds. It follows that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\varepsilon}{81}\sum_{i=1}^{\infty}81^i \cdot \frac{\|x\|^p}{3^{ip}}\\ &= \frac{\varepsilon}{3^p - 3^4}\|x\|^p\end{aligned}$$

for all $x \in X$. □

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(Received 15 May 2008)

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