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# Quaternary Rectangular Bands and Representations of Ternary Semigroups

**Anak Nongmanee**[1] **and Sorasak Leeratanavalee**[2,*]

[1] *M.S. Program in Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*e-mail : anak_nongmanee*

[2] *Research Center in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*e-mail : sorasak.l@cmu.ac.th*

**Abstract** The article is devoted to investigation of algebraic ternary structures. We start first by recalling the notion of ternary semigroups and its algebraic properties. Analogous to the concept of rectangular bands in ordinary semigroups, we define the new concept of *quaternary rectangular bands*, and investigate some of its algebraic properties. Based on the well-known results on ordinary semigroups, the so-called Cayley's theorem and the full (unary) transformations are defined. These lead us to construct a new algebraic ternary structure and its ternary operation the so-called *the ternary semigroup of all full binary transformations* and *the ternary composition via identity* 1, respectively. Moreover, we prove that every abstract ternary semigroup induced by a semigroup can be represented by full binary transformations. Futhermore, the related algebraic property of the ternary semigroups with an identity element is investigated.

**MSC:** 20M10; 20M20; 20M30
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## 1. Introduction

The definition of ternary semigroups was introduced by D.H. Lehmer [1] in 1932. Previously, algebraic ternary structures were studied by E. Kanser [2] who introduced the concept of $n$-ary algebras in 1904. Moreover, D.H. Lehmer investigated certain ternary algebraic systems called *triplexes* which can be formed as commutative ternary groups. Based on such ternary algebraic structures and its generalization, the so-called $n$-ary structures, there are some applications in Physics. In 1937, Y. Nambu [3] introduced the concept of a new dynamical system based on the canonical triplex called Nambu mechanics (cf. [4]). For more applications in Physics, see [5–7].

*Corresponding author.

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The notion of ternary semigroups was known by S. Banach (cf. [8]) who gave an example of a ternary semigroup which cannot reduced to a semigroup. A ternary semigroup is a nonempty set together with a ternary operation satisfying a ternary associative law. Futhermore, ternary semigroups can be considered as universal algebras (cf. [9]) under one associative ternary operation. On another hand, ternary semigroups can be considered as a special case of $n$-ary semigroups for $n = 3$ (cf. [10]). Based on this knowledge, ternary semigroups and its related algebraic properties are studied by many authors. F.M. Sioson [11] introduced the notions of ideals and radicals of ternary semigroups in 1965. Previously in 1963, he also introduced the concept of regular ternary semigroups which was characterized by ideals see [12]. In 1983, ternary groups were characterized by M.L. Santiago (see [13]), and he also investigated some properties on regular ternary semigroups (see [14]). In addition, M.L. Santiago and S. Sri Bala [15] also characterized regular ternary semigroups by idempotent pairs, and investigated some algebraic properties of ternary semigroups. Furthermore, the concept of congruences on ternary semigroups was defined. There are many mathematicians who published many research works related to congruences on ternary semigroups see [16–18].

In order to obtain the main purpose of this work, let us recall some algebraic structures on group theory and semigroup theory. A semigroup $(S, *)$ is called a *left (right) zero semigroup* if $*(a, b) = a$ $(*(a, b) = b)$ for all $a, b \in S$. A semigroup $S$ is said to be a *rectangular band* if $S$ is isomorphic to the direct product $L \times R$ of a left zero semigroup $L$ and a right zero semigroup $R$. Moreover, a semigroup $S$ is said to be a *rectangular group* if $S$ is isomorphic to the direct product $G \times B$ of a group $G$ and a rectangular band $B$. Based on the knowledge of rectangular bands and rectangular groups, there are many publications related to such structrues. In 2010, S. Panma [19] gave a characterization of digraphs which are the Cayley graphs of rectangular groups; while, a characterization of Cayley graphs of rectangular groups was first gave by B. Khosravi and M. Mahmoudi (see[20]). Moreover, Cayley graphs of rectangular bands relative to Green's equivalence classes were described by L. John and A.N. Kumari in 2011, see [21]. Up to 2013, R. S. Gigoń [22] studied rectangular group congruences on a semigroup. Furthermore, X. Guo and K.P. Shum [23] studied a matrix representation of rectangular bands of semigroups.

Let $X$ be a nonempty set. The set of all permutations on $X$ (i.e. $\mu : X \longrightarrow X$ is both injective and surjective) is denoted by $\mathrm{Sym}(X)$. Then $\mathrm{Sym}(X)$ under the usual (binary) composition of functions forms a group, and it is called, a *symmetric group* on $X$. Every subgroup of symmetric groups is called a permutation group. Based on this knowlegde, an important theory of groups, the so-called *Cayley's theorem for groups*, was introduced by A. Cayley. The theorem is stated that every abstract group is isomorphic to some subgroups of symmetric groups on some set. For each element $a$ in a group $(G, \cdot)$, we define a fuction $L_a$ on $G$ by $L_a(g) = \cdot(a, g)$ for all $g \in G$. We call $L_a$, a *left multiplication map by a*. Then $L_a$ is a permutation on $G$. Let $G' = \{L_a \mid a \in G\}$. Next, we define a function $\pi : G \longrightarrow G'$ by $\pi(a) = L_a$ for all $a \in G$. By Cayley's theorem for groups, $G \cong G'$. Consequently, every element $a \in G$ can be represented by $L_a$. Such representation $\pi$ is called a *left regular representation* (sometimes call, a *permutation representation*). For more information on algebraic group theory see [24].

Similar to group theory, the *Cayley's theorem for semigroups* was investigated. The analogous of the set of all permutations on $X$ is the set of all *full (unary) transformations* on $X$ which is denoted by $\mathcal{T}(X)$. Then $\mathcal{T}(X)$ under the usual (binary) composition of functions forms a semigroup, which is called the *semigroup of all full (unary) transformations*. In general, any semigroup $(S, *)$ does not necessary have identity element. In this case, a new element 1 is adjoined into $S$. Then we construct a new base set $S \cup \{1\}$ together with a binary operation $\diamond$ on $S \cup \{1\}$ which is defined by

$$\diamond(a, b) = *(a, b)\ \forall a, b \in S, \quad \diamond(a, 1) = a = \diamond(1, a)\ \forall a \in S, \quad \text{and} \diamond(1, 1) = 1. \tag{1.1}$$

By directly computation, we conclude that $\diamond$ is associative and hence $(S \cup \{1\}, \diamond)$ forms a semigroup with an identity element 1. For convenience, for every semigroup $(S, *)$, we denote

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element,} \\ S \cup \{1\} & \text{if } S \text{ has no identity element.} \end{cases}$$

Let $a$ be an element of a semigroup $(S, *)$. The inner left translation $\lambda_a : S \longrightarrow S$ is defined by $\lambda_a(x) = *(a, x)$ for all $a \in S$. Next, let $S' = \{\lambda_a \mid a \in S\}$. Hence $S'$ forms a subsemigroup of $\mathcal{T}(S)$. Then we define a function $\phi : S \longrightarrow S'$ by $\phi(a) = \lambda_a$ for all $a \in S$. By Cayley's theorem for semigroups, $S \cong S'$. That is, every element $a \in S$ can be represented by an inner left translation $\lambda_a$ in $S'$. We call such representation $\phi$, a *regular representation*. Similarly, we can extend this concept to a semigroup $S^1$. That is, there exists an isomorphism $\pi$ from $S^1$ onto $\{\lambda_a \mid a \in S^1\}$. We call the representation $\pi$, an *extend regular representation*. For more related information to this topic see [25]. Futhermore, by using the concept of full (unary) transformations, T. Kumduang and S. Leeratanavalee [26] extended to study the left translations for Menger algebras, which are a generalization of arbitrary semigroups in 2020. Moreover, they studied the representations of Menger hyperalgebras,which can be considered as a generalization of arbitrary semihypergroups, see [27].

In this article, our purposes are to construct the well-known algebraic structures on arbitrary semigroups including the rectangular bands and the Cayley's theorem for ternary semigroups induced by a semigroup. In Section 2, we start by recalling some neccessary definitions and theories of ternary semigroups. Moreover, we give several examples of a homomorphism for understanding the structures of two ternary semigroups. In order to obtain the main results, we complete the section by construction of the ternary semigroups with an identity element. Next, in Section 3, we define the so-called *quaternary rectangular bands*, give an example, and characterize it by the specific condition. Furthurmore, we illustrate (by showing a figure) that why we call such ternary semigroups like that. After that, in Section 4, we establish the so-called *full binary transformations*, which can be acted as representations of every element of ternary semigroups induced by a semigroup. Next, we introduce a new algebraic ternary structure the so-called *the ternary semigroups of all full binary transformations* under a specific ternary operation. Such operation is defined by using one specific element. Furthermore, we also give an example of a ternary semigroup induced by a semigroup and its representation. Finally, we complete the article by proving the related algebraic property of the ternary semigroups with an identity element.

## 2. Preliminaries

In this section, we start with recalling some basic concepts and definitions which are the common termimology in the theory of ternary semigroups. Moerover, the ternary semigroup together with an identity element 1, the so-called $T^1$ where $(T, \bullet)$ is a ternary semigroup, is established.

**Definition 2.1.** A nonempty set $T$ together with one ternary operation $\bullet : T^3 \longrightarrow T$ is said to be a *ternary semigroup* if the operation $\bullet$ satisfies the *ternary associative law*, i.e.,

$$\bullet(\bullet(a, b, c), d, e) = \bullet(a, \bullet(b, c, d), e) = \bullet(a, b, \bullet(c, d, e)) \quad \text{for all } a, b, c, d, e \in T.$$

**Definition 2.2.** An element 1 of a ternary semigroup $T$ is called

(i) a left identity (or left unital element) if $\bullet(1, 1, a) = a$ for all $a \in T$;
(ii) a right identity (or right unital element) if $\bullet(a, 1, 1) = a$ for all $a \in T$;
(iii) a lateral identity (or lateral unital element) if $\bullet(1, a, 1) = a$ for all $a \in T$;
(iv) a two-sided identity (or bi-unital element) if $\bullet(1, 1, a) = \bullet(a, 1, 1) = a$ for all $a \in T$;
(v) an identity (or unital element) if $\bullet(1, 1, a) = \bullet(1, a, 1) = \bullet(a, 1, 1) = a$ for all $a \in T$.

**Remark 2.3.** Let $(S, *)$ be an ordinary semigroup together with a binary operation $(a, b) \mapsto *(a, b)$. Then $S$ under a ternary operation $\bullet$, which is defined by

$$\bullet(a, b, c) = *(*(a, b), c) \quad \text{for all } a, b, c \in S, \tag{2.1}$$

forms a ternary semigroup, while a ternary semigroup does not necessary reduce to an ordinary semigroup.

According to Remark 2.3, we call such ternary operation, which is defined as (2.1), the *usual ternary operation.*

**Example 2.4.** (i) Consider on the set of all real numbers $\mathbb{R}$ under a ternary operation $\bullet$ which is defined by

$$\bullet(r, s, t) = r + s + t \qquad \text{for all } r, s, t \in \mathbb{R},$$

where $+$ are the usual (binary) addition. Therefore, $(\mathbb{R}, \bullet)$ is a ternary semigroup induced by the semigroup $(\mathbb{R}, +)$ with an identity element 0.

(ii) Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ under a ternary operation $\bullet$, which is defined by $\bullet(a, b, c) = \max\{a, b, c\}$ for all $a, b, c \in \mathbb{Z}$, forms a ternary semigroup.

(iii) Let $\mathbb{R}$ be the set of all real numbers. We define a ternary semigroup $\bullet$ on $\mathbb{R}$ by

$$\bullet(a, b, c) = a \qquad \text{for all } a, b, c \in \mathbb{R}.$$

It is evidently that $\bullet$ is ternary associative. Consequently, $(\mathbb{R}, \bullet)$ forms a ternary semigroup. Moreover, every element in $\mathbb{R}$ is a right identity element while there has no identity element.

(iv) Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ under the usual ternary multiplication forms a ternary semigroup and both of $-1$ and 1 are identity elements.

(v) Let $T = \{-i, i\}$ be a subset of the set of all complex numbers. Then $T$ under the usual ternary multiplication is a ternary semigroup. However, $T$ under the usual (binary) multiplication $\cdot$ is not a semigroup, because $i \cdot i = -1 \notin T$.

**Remark 2.5.** Let $(T, \bullet)$ be a ternary semigroup and let $m, n \in \mathbb{N}$ be such that $m \leq n$ and for every $i = 1, 2, ..., 2n + 1$, $a_i \in T$. By the ternary associativity, we can write

$$\bullet(a_1, a_2, ..., a_{2n+1}) = \bullet(a_1, ..., a_{m-1}, \bullet(\bullet(a_m, a_{m+1}, a_{m+2}), a_{m+3}, a_{m+4}), a_{m+5}, ..., a_{2n+1}).$$

**Definition 2.6.** Let $(T, \bullet)$ be a ternary semigroup. A nonempty subset $S$ of $T$ is called a *ternary subsemigroup* if $S$ is closed under the ternary operation $\bullet$ of $T$, i.e., $\bullet(s_1, s_2, s_3) \in S$ for all $s_1, s_2, s_3 \in S$.

**Definition 2.7.** Let $(T, \bullet)$ be a ternary semigroup. Then $T$ is said to be a *commutative ternary semigroup* if for each $a_1, a_2, a_3 \in T$, $\bullet(a_1, a_2, a_3) = \bullet(a_{\mu(1)}, a_{\mu(2)}, a_{\mu(3)})$ for every permutation $\mu$ of $\{1, 2, 3\}$, i.e,

$$\bullet(a_1, a_2, a_3) = \bullet(a_1, a_3, a_2) = \bullet(a_2, a_1, a_3) = \bullet(a_2, a_3, a_1) = \bullet(a_3, a_1, a_2) = \bullet(a_3, a_2, a_1).$$

**Definition 2.8.** An element $a$ of a ternary semigroup $(T, \bullet)$ is called an *idempotent element* if $\bullet(a, a, a) = a$.

Note that, a ternary semigroup $(T, \bullet)$ is called a *band* if all elements of $T$ are idempotent.

**Definition 2.9.** Let $(T, \bullet)$ be a ternary semigroup. Then $T$ is called
(i) a left cancellative if $\bullet(a, b, x) = \bullet(a, b, y) \Longrightarrow x = y$ for all $a, b, x, y \in T$
(ii) a right cancellative if $\bullet(x, a, b) = \bullet(y, a, b) \Longrightarrow x = y$ for all $a, b, x, y \in T$
(iii) a lateral cancellative if $\bullet(a, x, b) = \bullet(a, y, b) \Longrightarrow x = y$ for all $a, b, x, y \in T$
(iv) a cancellative if $T$ is left cancellative, right cancellative and lateral cancellative.

Now, let us consider Example 2.4 (ii). Then we have $(\mathbb{Z}, \bullet)$ forms a commutative ternary semigroup with every element of $\mathbb{Z}$ is an idempotent element. Moreover, in Example 2.4 (iii), we get $(\mathbb{R}, \bullet)$ is a left cancellative ternary semigroup.

**Definition 2.10.** Let $(T_1, \bullet_1)$ and $(T_2, \bullet_2)$ be two ternary semigroups and $\pi : T_1 \longrightarrow T_2$ be a mapping. Then $\pi$ is said to be a *(ternary) homomorphism* (*(ternary) anti-homomorphism*) of $T_1$ into $T_2$ if

$$\pi(\bullet_1(a, b, c)) = \bullet_2(\pi(a), \pi(b), \pi(c))$$
$$(\ \pi(\bullet_1(a, b, c)) = \bullet_2(\pi(c), \pi(b), \pi(a))\ )$$

for all $a, b, c \in T_1$.

Moreover, if $1_{T_1}$ and $1_{T_2}$ are identity elements of the ternary semigroups $T_1$ and $T_2$ respectively, then $\pi(1_{T_1}) = 1_{T_2}$. Furthermore, the homomorphism $\pi$ is called
(i) a (ternary) monomorphism if it is injective;
(ii) an (ternary) epimorphism if it is surjective;
(iii) an (ternary) isomorphism if it is both injective and surjective. In this case, we call that the ternary semigroups $T_1$ and $T_2$ are *isomorphic*, and denoted by $T_1 \cong T_2$.

**Example 2.11.** Let $\mathbb{Z}$ be the set of all integers. Consider on two ternary semigroups $(\mathbb{Z}, \oplus)$ and $(\mathbb{Z}, \odot)$, where $\oplus$ and $\odot$ are the usual ternary addition and the usual ternary multiplication, respectively. We define a mapping $\pi : (\mathbb{Z}, \oplus) \longrightarrow (\mathbb{Z}, \odot)$ by $\pi(n) = 2^n$ for all $n \in \mathbb{Z}$. Consequently, $\pi$ is a homomorphism. Since for each $n_1, n_2, n_3 \in \mathbb{Z}$,

$$\begin{aligned}\pi(\oplus(n_1, n_2, n_3)) &= 2^{\oplus(n_1, n_2, n_3)} = 2^{(n_1+n_2)+n_3} \\ &= (2^{n_1} \cdot 2^{n_2}) \cdot 2^{n_3} \\ &= \cdot(\cdot(2^{n_1}, 2^{n_2}), 2^{n_3}) \\ &= \cdot(\cdot(\pi(n_1), \pi(n_2)), \pi(n_3)) \\ &= \odot(\pi(n_1), \pi(n_2), \pi(n_3)).\end{aligned}$$

Similarly, it is easy to check that $\pi$ is an anti-homomorphism, because of $\oplus(n_1, n_2, n_3) = \oplus(n_3, n_2, n_1)$ and $\odot(n_1, n_2, n_3) = \odot(n_3, n_2, n_1)$.

**Example 2.12.** Let $\mathbb{Z}^+$ be a set of all non-positive integers and $\mathcal{M}_2(\mathbb{Z}^+)$ be the set of all $2 \times 2$ matrices over $\mathbb{Z}^+$. Consider two ternary semigroups $(\mathbb{Z}^+, \oplus)$ and $(\mathcal{M}_2(\mathbb{Z}^+), \oplus_{\mathcal{M}})$, where $\oplus$ is the usual ternary addition on $\mathbb{Z}^+$ and $\oplus_{\mathcal{M}}$ is the usual ternary matrix addition on $\mathcal{M}_2(\mathbb{Z}^+)$. Next, we define a mapping $\Phi : \mathbb{Z}^+ \longrightarrow \mathcal{M}_2(\mathbb{Z}^+)$ by

$$\Phi(n) = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \quad \text{for all } n \in \mathbb{Z}^+.$$

Hence $\Phi$ is a homomorphism, because for each $k, m, n, \in \mathbb{Z}^+$ we have

$$\begin{aligned}
\Phi(\oplus(k,m,n)) &= \Phi((k+m)+n) \\
&= \begin{pmatrix} 0 & (k+m)+n \\ (k+m)+n & 0 \end{pmatrix} \\
&= \left[\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} +_{\mathcal{M}} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\right] +_{\mathcal{M}} \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} \\
&= [\Phi(k) +_{\mathcal{M}} \Phi(m)] +_{\mathcal{M}} \Phi(n) \\
&= \oplus_{\mathcal{M}}(\Phi(k), \Phi(m), \Phi(n)).
\end{aligned}$$

Similary, it is easily seen that $\Phi$ is an anti-homomorphism, which implies by $\oplus(k,m,n) = \oplus(n,m,k)$ and $\oplus_{\mathcal{M}}(k,m,n) = \oplus_{\mathcal{M}}(n,m,k)$.

Similar to the concept of the construction of the semigroup $(S^1, \diamond)$ with an identity element 1, the following result is important to construct ternary semigroups with an identity element from some ternary semigroups.

**Proposition 2.13** ([28]). *An identity element can be added to the ternary semigroups which are induced by a binary semigroup.*

*Proof.* Let $(T, \bullet)$ be a ternary semigroup induced by a binary semigroup $(T, *)$. In the semigroup $(T, *)$, we can add an element 1 into $T$ and then $(T \cup \{1\}, \diamond)$ forms a semigroup with an identity element 1 under the binary operation $\diamond$ defined as (1.1). By Remark 2.3, we immediately obtain that $(T \cup \{1\}, \star)$ is a ternary semigroup induced by the semigroup $(T \cup \{1\}, \diamond)$. In the ternary semigroup $(T \cup \{1\}, \star)$, the element 1 is an identity element and $\bullet(x,y,z) = \star(x,y,z)$ for all $x, y, z \in T$. Therefore we can add an identity element $1 \notin T$ into $(T, \bullet)$. ∎

According to the proof of Proposition 2.13, we conclude that in case of the ternary semigroup induced by a binary semigroup we can add an identity element $1 \notin T$ into $(T, \bullet)$. Moreover, $(T \cup \{1\}, \star)$, which is induced by the binary semigroup $(T \cup \{1\}, \diamond)$, forms a semigroup with an identity element 1.

For convenience, for every ternary semigroup $(T, \bullet)$ induced by a binary semigroup, we denote

$$T^1 = \begin{cases} T & \text{if } T \text{ has an identity element,} \\ T \cup \{1\} & \text{if } T \text{ has no identity element.} \end{cases}$$

Consequently, $(T^1, \star)$ forms a ternary semigroup and the element 1 acts as an identity element on $T^1$. Furthermore, the restriction of the ternary operation $\star$ to the base set $T$ coincides with the original ternary operation $\bullet$, and hence $(T, \bullet)$ can be considered as a ternary subsemigroup of $(T^1, \star)$.

## 3. Quaternary Rectangular Bands

In this section, we show the relationship between the so-called *quaternary rectangular bands*, a left zero ternary semigroup and a right zero ternary semigroup. Finally, we complete this section by giving a characterization of quaternary rectangular bands.

**Definition 3.1.** Let $(T, \bullet)$ be a ternary semigroup. An element $0 \in T$ is said to be
(i) a left zero element if $\bullet(0, x, y) = 0$ for all $x, y \in T$;
(ii) a right zero element if $\bullet(x, y, 0) = 0$ for all $x, y \in T$;
(iii) a lateral zero element if $\bullet(x, 0, y) = 0$ for all $x, y \in T$;
(iv) a two-sided zero element if $\bullet(0, x, y) = \bullet(x, y, 0) = 0$ for all $x, y \in T$;
(v) a zero element if $\bullet(0, x, y) = \bullet(x, 0, y) = \bullet(x, y, 0) = 0$ for all $x, y \in T$;

**Definition 3.2.** Let $(T, \bullet)$ be a ternary semigroup. $T$ is called a *zero (left, right, lateral, or two-sided zero)* if for all elements in $T$ are zero (left, right, lateral, or two-sided zero, respectively) element.

**Example 3.3.** (i) Let $\mathbb{N}$ be a set of all natural numbers. Then $(\mathbb{N}, \bullet)$ is a ternary semigroup where the operation $\bullet$ is define by

$$\bullet(x, y, z) = x \qquad \text{for all } x, y, z \in \mathbb{N}.$$

Hence every element on $T$ is a left zero element while there has no a right (lateral) zero element. Therefore $(T, \bullet)$ is a left zero ternary semigroup. A right (lateral) zero ternary semigroup is defined in an anlogous way.

(ii) Consider on the closed interval $I = [0, 1]$. We define a ternary operation $\bullet$ on $I$ by

$$\bullet(x, y, z) = \min\{x, y, z\} \qquad \text{for all } x, y, z \in I.$$

It is clear that $\bullet$ is ternary associative. Moreover, we have 0 is a zero element and 1 is an identity element.

**Remark 3.4.** Let $(S, \bullet_1)$ and $(T, \bullet_2)$ be two ternary semigroups. Then $(S \times T, \otimes)$ forms a ternary semigroup under the ternary operation $\otimes : (S \times T)^3 \longrightarrow S \times T$ defined by

$$\otimes((a, b), (c, d), (e, f)) = (\bullet_1(a, c, e), \bullet_2(b, d, f)) \qquad \text{for all } a, c, e \in S,\ b, d, f \in T.$$

We call $(S \times T, \bullet)$, the *direct product* of $S$ and $T$.

Now, let $(T, \bullet)$ be a ternary semigroup and $X, Y, Z \subseteq T$. We set

$$\bullet(X, Y, Z) \text{ (or a short form } XYZ) = \{\bullet(x, y, z) \mid x \in X, y \in Y, z \in Z\}.$$

In case of the singleton sets, we write $Xyz$ insteads of $X\{y\}\{z\}$. Moreover, it is easy to check that, for each subset $X_i$, $i = 1, ..., 5$ of $T$,

$$(X_1X_2X_3)X_4X_5 = X_1(X_2X_3X_4)X_5 = X_1X_2(X_3X_4X_5).$$

**Definition 3.5.** Let $(T, \bullet)$ be a ternary semigroup. $T$ is called a *quaternary rectangular band* if $\bullet(x, y, x) = x$ for all $x, y \in T$.

According to Example 3.3 (i), we conclude that $(\mathbb{N}, \bullet)$ forms a ternary semigroup under the ternary operation $\bullet$. Moreover, it is easily seen that $\bullet(m, n, m) = m$ for all $m, n \in \mathbb{N}$. Therefore $(\mathbb{N}, \bullet)$ is a quaternary rectangular band.

Finally, we complete this section by showing the relationship between the quaternary rectangular bands and the direct product of a left zero ternary semigroup and a right zero ternary semigroup in the following theorem.

**Theorem 3.6.** *Let $(T, \bullet)$ be a ternary semigroup. The following statements are equivalent:*

*(i) $T$ is a quaternary rectangular band;*

*(ii) every element of $T$ is an idempotent element, and*

$$\bullet(a, \bullet(b, c, d), e) = \bullet(a, b, e) = \bullet(a, d, e) \quad \text{for all } a, b, c, d, e \in T;$$

*(iii) there exist a left zero ternary semigroup $(L, \bullet)$ and a right zero ternary semigroup $(R, \bullet)$ such that $T \cong L \times R$;*

*(iv) $T$ is isomorphic to a ternary semigroup $(X \times Y, \boxtimes)$, where $X$ and $Y$ are nonempty sets and the ternary operation $\boxtimes$ is defined by*

$$\boxtimes((a, b), (c, d), (e, f)) = (a, f) \quad \text{for all } a, c, e \in X,\ b, d, f \in Y.$$

*Proof.* (i) $\Longrightarrow$ (ii) Assume that $T$ is a quaternary rectangular band. Let $a \in T$. By our assumption, we have $\bullet(a, a, a) = a$ and hence $a$ is an idempotent element. So, every element in $T$ is an idempotent element. Next, let $a, b, c, d, e \in T$. Again, by our assumption we have $\bullet(b, c, b) = b, \bullet(c, d, c) = c$ and $\bullet(e, d, e) = e$. Moreover, since $\bullet(a, b, e) \in T$, we have $\bullet(d, \bullet(a, b, e), d) = d$. By ternary associativity of the operation $\bullet$, we have

$$\begin{aligned}
\bullet(a, b, e) &= \bullet(a, \bullet(b, c, b), e) \\
&= \bullet(a, \bullet(b, \bullet(c, d, c), b), \bullet(e, d, e)) \\
&= \bullet(a, b, \bullet(\bullet(c, d, c), b, \bullet(e, d, e))) \\
&= \bullet(a, b, \bullet(c, \bullet(d, c, b), \bullet(e, d, e))) \\
&= \bullet(a, b, \bullet(c, d, \bullet(c, b, \bullet(e, d, e)))) \\
&= \bullet(a, b, \bullet(c, d, \bullet(\bullet(c, b, e), d, e))) \\
&= \bullet(a, b, \bullet(c, \bullet(d, \bullet(c, b, e), d), e)) \\
&= \bullet(a, b, \bullet(c, d, e)) \\
&= \bullet(a, \bullet(b, c, d), e).
\end{aligned}$$

Similar to the above argument, we conclude that $\bullet(a, \bullet(b, c, d), e) = \bullet(a, d, e)$ which implies by using the facts that $\bullet(a, b, a) = a, \bullet(c, b, c) = c, \bullet(d, c, d) = d$ and $\bullet(b, \bullet(a, d, c), b) = b$. Therefore $\bullet(a, \bullet(b, c, d), e) = \bullet(a, b, e) = \bullet(a, d, e)$.

(ii) $\Longrightarrow$ (iii) Suppose the condition (ii) holds. Firstly, we fix the element $c$ of $T$. Let $L = Tcc$ and $R = ccT$. Indeed, for each $x, y, z \in L$ there are $k, m, n \in T$ such that $x = \bullet(k, c, c)$, $y = \bullet(m, c, c)$ and $z = \bullet(n, c, c)$. Then by using (ii), we have

$$\begin{aligned}
\bullet(x, y, z) &= \bullet(\bullet(k, c, c), \bullet(m, c, c), \bullet(n, c, c)) \\
&= \bullet(k, c, \bullet(c, \bullet(m, c, c), \bullet(n, c, c))) \\
&= \bullet(k, c, \bullet(c, c, \bullet(n, c, c))) \\
&= \bullet(k, c, \bullet(c, \bullet(c, n, c), c)) \\
&= \bullet(k, c, \bullet(c, c, c)) \\
&= \bullet(k, c, c) \\
&= x.
\end{aligned}$$

Similarly, we can show that $\bullet(x', y', z') = z'$ for all $x', y', z' \in R$. Hence $(L, \bullet)$ and $(R, \bullet)$ are a left zero ternary semigroup and a right ternary semigroup, respectively.

Secondly, we define $\phi : (T, \bullet) \longrightarrow (L \times R, \otimes)$ by

$$\phi(x) = (\bullet(x,c,c), \bullet(c,c,x)) \quad \text{for all } x \in T.$$

Next, we show that $\phi$ is both injective and surjective. Suppose that $\phi(x) = \phi(y)$. Then $(\bullet(x,c,c), \bullet(c,c,x)) = (\bullet(y,c,c), \bullet(c,c,y))$. Consequently, $\bullet(x,c,c) = \bullet(y,c,c)$ and $\bullet(c,c,x) = \bullet(c,c,y)$. Consider,

$$\begin{aligned}
x &= \bullet(x,x,x)\\
&= \bullet(\bullet(x,c,c),x,x) = \bullet(\bullet(y,c,c),x,x) = \bullet(y,\bullet(c,c,x),x)\\
&= \bullet(y,\bullet(c,c,y),x) = \bullet(y,y,x) = \bullet(y,\bullet(y,c,c),x)\\
&= \bullet(y,y,\bullet(c,c,x)) = \bullet(y,y,\bullet(c,c,y)) = \bullet(y,\bullet(y,c,c),y)\\
&= \bullet(y,y,y)\\
&= y.
\end{aligned}$$

Hence $\phi$ is injective. Moreover, for each $(\bullet(a,c,c), \bullet(c,c,b)) \in L \times R$, we have $\bullet(a,c,b) \in T$ such that

$$\begin{aligned}
\phi(\bullet(a,c,b)) &= (\bullet(\bullet(a,c,b),c,c), \bullet(c,c,\bullet(a,c,b)))\\
&= (\bullet(a,\bullet(c,b,c),c), \bullet(c,\bullet(c,a,c),b))\\
&= (\bullet(a,c,c), \bullet(c,c,b)).
\end{aligned}$$

So, $\phi$ is surjective.

Finally, we show that $\phi$ is a homomorphism. Indeed, for each $x,y,z \in T$ we have

$$\begin{aligned}
\phi(\bullet(x,y,z)) &= (\bullet(\bullet(x,y,z),c,c), \bullet(c,c,\bullet(x,y,z)))\\
&= (\bullet(x,\bullet(y,z,c),c), \bullet(c,\bullet(c,x,y),z))\\
&= (\bullet(x,c,c), \bullet(c,c,z))\\
&= (\bullet(x,c,\bullet(c,c,c)), \bullet(c,\bullet(c,c,c),z))\\
&= (\bullet(x,c,\bullet(c,\bullet(c,z,c),c)), \bullet(c,\bullet(c,\bullet(c,y,c),c),z))\\
&= (\bullet(x,\bullet(c,c,\bullet(c,z,c)),c), \bullet(\bullet(c,c,\bullet(c,y,c)),c,z))\\
&= (\bullet(x,\bullet(\bullet(c,c,c),z,c),c), \bullet(\bullet(\bullet(c,c,c),y,c),c,z))\\
&= (\bullet(x,\bullet(\bullet(c,\bullet(c,y,c),c),z,c),c), \bullet(\bullet(\bullet(c,\bullet(c,x,c),c),y,c),c,z))\\
&= (\bullet(\bullet(x,\bullet(c,c,\bullet(y,c,c),z),c,c), \bullet(\bullet(\bullet(\bullet(c,c,x),c,c),y,c),c,z))\\
&= (\bullet(x,\bullet(c,c,\bullet(y,c,c)),\bullet(z,c,c)), \bullet(\bullet(\bullet(c,c,x),\bullet(c,c,y),c),c,z))\\
&= (\bullet(\bullet(x,c,c),\bullet(y,c,c),\bullet(z,c,c)), \bullet(\bullet(c,c,x),\bullet(c,c,y),\bullet(c,c,z)))\\
&= \otimes((\bullet(x,c,c),\bullet(c,c,x)),(\bullet(y,c,c),\bullet(c,c,y)),(\bullet(z,c,c),\bullet(c,c,z)))\\
&= \otimes(\phi(x),\phi(y),\phi(z)).
\end{aligned}$$

Hence $\phi$ is a homomorphism. Therefore $\phi$ is an isomorphism and hence $T \cong L \times R$.

(iii) $\Longrightarrow$ (iv) Assume the condition (iii) holds. First, we suppose that $T = L \times R$, where $(L, \bullet)$ and $(R, \bullet)$ are a left zero ternary semigroup and a right ternary semigroup, respectively. Next, we define a ternary operation $\boxtimes$ by, for all $(a,b),(c,d),(e,f) \in T$,

$$\boxtimes((a,b),(c,d),(e,f)) = (a,f).$$

It is easily to check that the operation $\boxtimes$ is ternary asscociative. Then we only take $L = X$ and $R = Y$, and hence $(X \times Y, \boxtimes)$ forms a ternary semigroup. Consequently, $T \cong X \times Y$.

(iv) $\Longrightarrow$ (i) Let $T = X \times Y$, together with the given ternary operation $\boxtimes$. Indeed, for each $a = (x, y)$ and $b = (x', y')$ we obtain

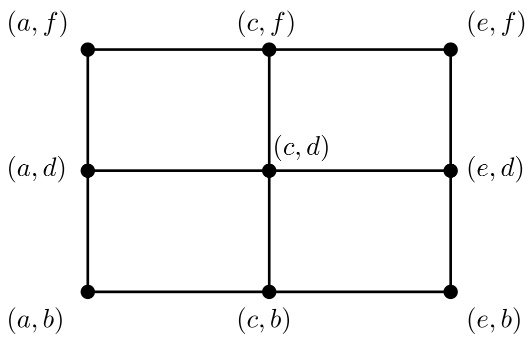
$$\bullet(a, b, a) = \boxtimes((x, y), (x', y'), (x, y)) = (x, y) = a.$$

Therefore $T$ is a quaternary rectangular band. ∎

As the condition (iv) of Theorem 3.6, we obtain the term *quaternary rectangular*. That is, if we consider $(a, b), (c, d), (e, f)$ as the three vertices of the Cartesian plane, then we have

$$\begin{aligned} &\boxtimes((a, b), (c, d), (e, f)) = (a, f), \quad \boxtimes((a, b), (e, f), (c, d)) = (a, d),\\ &\boxtimes((c, d), (a, b), (e, f)) = (c, f), \quad \boxtimes((c, d), (e, f), (a, b)) = (c, d),\\ &\boxtimes((e, f), (a, b), (c, d)) = (e, d), \quad \boxtimes((e, f), (c, d), (a, b)) = (e, b) \end{aligned}$$

are placed by the vertices of the following figure.




Furthermore, the term *band* is referred to the ternary semigroups such that all elements are idempotent elements.

**Theorem 3.7.** *Let $(T, \bullet)$ be a ternary semigroup. $T$ is a quaternary rectangular band if and only if*

$$\bullet(x, x, y) = \bullet(x, y, x) = \bullet(y, x, x) \Longrightarrow x = y \quad \textit{for all } x, y \in T$$

*Proof.* ($\Longrightarrow$) Assume that $(T, \bullet)$ is a quaternary rectangular band. By Theorem 3.6 (i) implies (ii), we have all elements of $T$ are idempotent elements and

$$\bullet(a, \bullet(b, c, d), e) = \bullet(a, b, e) = \bullet(a, d, e) \quad \text{for all } a, b, c, d, e \in T.$$

Next, let $x, y \in T$ be such that $\bullet(x, x, y) = \bullet(x, y, x) = \bullet(y, x, x)$. Then

$$\begin{aligned} x &= \bullet(x, x, x) = \bullet(x, \bullet(x, y, y), x) = \bullet(x, \bullet(y, x, y), x)\\ &= \bullet(x, y, \bullet(x, y, x)) = \bullet(x, y, \bullet(x, x, y)) = \bullet(\bullet(x, y, x), x, y)\\ &= \bullet(\bullet(y, x, x), x, y) = \bullet(y, \bullet(x, x, x), y) = \bullet(y, x, y) = y. \end{aligned}$$

($\Longleftarrow$) Assume that the condition holds. Note that, $\bullet(x, x, \bullet(x, x, x)) = \bullet(x, \bullet(x, x, x), x) = \bullet(\bullet(x, x, x), x, x)$. By our assumption, we have $\bullet(x, x, x) = x$. Indeed, for each $x, y \in T$

we get

$$\bullet(\bullet(x,y,x),x,x) = \bullet(x,\bullet(y,x,x),x) = \bullet(x,\bullet(x,y,x),x) \quad \text{and}$$
$$\bullet(\bullet(x,y,x),x,x) = \bullet(x,\bullet(y,x,x),x) = \bullet(x,\bullet(x,x,y),x) = \bullet(x,x,\bullet(x,y,x)).$$

Hence $\bullet(\bullet(x,y,x),x,x) = \bullet(x,\bullet(x,y,x),x) = \bullet(x,x,\bullet(x,y,x))$. By our assumption, we conclude that $\bullet(x,y,x) = x$. Therefore $(T,\bullet)$ is a quaternary rectangular band. ∎

## 4. Representations of Ternary Semigroups

In this section, we introduce the notion of the ternary semigroups of full binary transformations, give its example, and investigate its related algebraic properties. Finally, we prove that every ternary semigroup which is induced by a binary semigroup is isomorphic with some ternary subsemigroups of ternary semigroups of all full binary transformations on some sets.

### 4.1. Ternary Semigroups of Full Binary Transformations

From now on, we suppose that $(T,\bullet)$ is a ternary semigroup which is induced by a semigroup $(T,*)$ and the ternary operation $\bullet$ is induced by the binary opertaion $*$ on $T$, i.e., the operation $\bullet$ is defined as (2.1). Moreover, by using the proof of Proposition 2.13 we have $(T^1,\star)$ is a ternary semigroup with an identity element 1.

Now, we denote $T^1 \times T^1$ as the Cartesian product of $T^1$. For the set $\mathcal{T}(T^1 \times T^1, T^1)$ of all *full binary transformations* $f : T^1 \times T^1 \longrightarrow T^1$, we define the following ternary operation $\circ^1 : \mathcal{T}(T^1 \times T^1, T^1)^3 \longrightarrow \mathcal{T}(T^1 \times T^1, T^1)$ by

$$\circ^1(f,g,h)(x,y) = f(1,g(1,h(x,y))) \quad \text{for all } f,g,h \in \mathcal{T}(T^1 \times T^1, T^1) \tag{4.1}$$

where 1 is an identity element of $T^1$. We call the ternary operation $\circ^1$ on $\mathcal{T}(T^1 \times T^1, T^1)$, the *ternary composition via identity* 1 or *ternary composition.*

For convenience, in case of a finite set $T^1$, we denote the image under a full binary transformation $f$ by the following table: for all $x_1, x_2, ..., x_n \in T^1$,

| $f$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ |
|---|---|---|---|---|
| $x_1$ | $f(x_1,x_1)$ | $f(x_1,x_2)$ | $\cdots$ | $f(x_1,x_n)$ |
| $x_2$ | $f(x_2,x_1)$ | $f(x_2,x_2)$ | $\cdots$ | $f(x_2,x_n)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $x_n$ | $f(x_n,x_1)$ | $f(x_n,x_2)$ | $\cdots$ | $f(x_n,x_n)$ |

.

Moreover, we call $f(x,y)$, the image of $(x,y)$ under a full binary transformation $f$.

For instance, let us consider on a set $T = \{a,b,c\}$ under a ternary operation $\bullet$ defined by $\bullet(x,y,z) = *(*(x,y),z)$ for all $x,y,z \in T$, where the binary operation $*$ is defined by $*(x,y) = y$ for all $x,y \in T$. Then $(T,\bullet)$ is a ternary semigroup such that it is induced by a semigroup $(T,*)$. Furthermore, $(T,\bullet)$ has no an identity element. Next, we define three full binary transformations $\alpha,\beta,\gamma : T^1 \times T^1 \longrightarrow T^1$ by

$$\alpha(x,y) = x, \quad \beta(x,y) = y \text{ and } \gamma(x,y) = a \quad \text{for all } x,y \in T^1.$$

From this, we obtain the following table which shows all images under $\alpha,\beta,\gamma,\circ^1(\alpha,\beta,\gamma)$ and $\circ^1(\gamma,\beta,\alpha)$, where $\circ^1$ is the ternary composition via identity 1.

| $\alpha$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |
| $1$ | $1$ | $1$ | $1$ | $1$ |

| $\beta$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $1$ |
| $b$ | $a$ | $b$ | $c$ | $1$ |
| $c$ | $a$ | $b$ | $c$ | $1$ |
| $1$ | $a$ | $b$ | $c$ | $1$ |

| $\gamma$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $1$ | $a$ | $a$ | $a$ | $a$ |

| $\circ^1(\alpha,\beta,\gamma)$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $1$ | $1$ | $1$ | $1$ |
| $b$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ |

| $\circ^1(\gamma,\beta,\alpha)$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $1$ | $a$ | $a$ | $a$ | $a$ |

From these tables, we conclude that $\circ^1(\alpha,\beta,\gamma) \neq \circ^1(\gamma,\beta,\alpha)$, which means that the ternary composition $\circ^1$ need not be commutative.

According to the definition of the ternary composition via identity 1 (4.1), we immediately obtain the following result.

**Theorem 4.1.** *Let $(T,\bullet)$ be a ternary semigroup induced by a semigroup. The ternary composition via identity* 1 *of full binary transformations is ternary associative, i.e., if $f_1, f_2, ..., f_5$ are full binary transformations, then*

$$\circ^1(\circ^1(f_1,f_2,f_3),f_4,f_5) = \circ^1(f_1,\circ(f_2,f_3,f_4),f_5) = \circ^1(f_1,f_2,\circ^1(f_3,f_4,f_5)).$$

*Proof.* It is easily seen that $\circ^1$ is a ternary operation on the set $\mathcal{T}(T^1\times T^1, T^1)$. Indeed, for each $f_1, f_2, ..., f_5 \in \mathcal{T}(T^1\times T^1, T^1)$ and $x, y \in T^1$ we have

$$\begin{aligned}
\circ^1(\circ^1(f_1,f_2,f_3),f_4,f_5)(x,y) &= \circ^1(f_1,f_2,f_3)(1,f_4(1,f_5(x,y)))\\
&= f_1(1,f_2(1,f_3(1,f_4(1,f_5(x,y)))))\\
&= f_1(1,\circ^1(f_2,f_3,f_4)(1,f_5(x,y)))\\
&= \circ^1(f_1,\circ^1(f_2,f_3,f_4),f_5)(x,y) \quad \text{and}\\
\circ^1(f_1,f_2,\circ^1(f_3,f_4,f_5))(x,y) &= f_1(1,f_2(1,\circ^1(f_3,f_4,f_5)(x,y)))\\
&= f_1(1,f_2(1,f_3(1,f_4(1,f_5(x,y)))))\\
&= \circ^1(f_1,f_2,f_3)(1,f_4(1,f_5(x,y)))\\
&= \circ^1(\circ^1(f_1,f_2,f_3),f_4,f_5)(x,y),
\end{aligned}$$

which mean that the ternary composition via identity 1 of full binary transformations is a ternary associative operation on $\mathcal{T}(T^1\times T^1, T^1)$. This completes the theorem. ∎

By Theorem 4.1, we obtain the following important corollary.

**Corollary 4.2.** *The set $\mathcal{T}(T^1\times T^1, T^1)$ of all full binary transformations under the ternary composition via identity* 1 *forms a ternary semigroup.*

The set $\mathcal{T}(T^1\times T^1, T^1)$ of all full binary transformations, which is defined on a ternary semigroup $(T,\bullet)$ induced by a semigroup, is closed with respect to the ternary composition via identity 1 of full binary transformations. Consequently, it forms a ternary semigroup $(\mathcal{T}(T^1\times T^1, T^1),\circ^1)$. We call such structure, a *ternary semigroup of all full binary transformations.* For convenience, a *ternary semigroup of full binary transformations*, we mean each ternary subsemigroup of $(\mathcal{T}(T^1\times T^1, T^1),\circ^1)$.

### 4.2. Representations of Ternary Semigroups by Full Binary Transformations

In Section 4.1, the concept of the ternary semigroup of all full binary transformations is already established. By using this concept, we conclude the main result by showing that every ternary semigroup induced by a semigroup is isomorphic to a ternary semigroup of full binary transformations.

Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup. Based on the knowledge in previous sections, we have $(T^1, \star)$ and $(\mathcal{T}(T^1 \times T^1, T^1), \circ^1)$ form ternary semigroups under ternary operations $\star$ and $\circ^1$, respectively.

For each element $a \in T$, we define a binary mapping $\lambda_a : T^1 \times T^1 \longrightarrow T^1$ by

$$\lambda_a(x, y) = \star(a, x, y) \quad \text{for all } x, y \in T^1. \tag{4.2}$$

Hence, the binary mapping $\lambda_a$ is an element of $\mathcal{T}(T^1 \times T^1, T^1)$ which is called an *inner left translation* of $T^1$ corresponding to the element $a$ of $T^1$.

Similar to (4.2), we can define other binary mappings which are elements in $\mathcal{T}(T^1 \times T^1, T^1)$. That is, for each element $a \in T$, we define two binary mappings $\rho_a, \sigma_a : T^1 \times T^1 \longrightarrow T^1$ by

$$\rho_a(x, y) = \star(x, y, a) \text{ and } \sigma_a(x, y) = \star(x, a, y) \quad \text{for all } x, y \in T^1,$$

which implies that $\rho_a, \sigma_a \in \mathcal{T}(T^1 \times T^1, T^1)$. We call $\rho_a$ ($\sigma_a$), an *inner right (lateral) translation* of $T^1$ corresponding to the element $a$ of $T^1$. Based on this knowledge, it is immediately that if $(T, \bullet)$ is a commutative ternary semigroup and $a$ is a fixed element of $T$, then $\lambda_a, \rho_a$ and $\sigma_a$ are the same.

In order to conclude our main results of this article, the following lemmas are important.

**Lemma 4.3.** *Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup. Then $\lambda_{\star(a,b,c)} = \circ^1(\lambda_a, \lambda_b, \lambda_c)$ for all $a, b, c \in T^1$, where $\circ^1$ is the ternary composition via identity* 1.

*Proof.* Indeed, for each $a, b, c, x, y \in T^1$ we get

$$\begin{aligned}
\lambda_{\star(a,b,c)}(x, y) &= \star(\star(a, b, c), x, y) \\
&= \star(a, b, \star(c, x, y)) \\
&= \star(a, \star(1, b, 1), \star(c, x, y)) \\
&= \star(a, 1, \star(b, 1, \star(c, x, y))) \\
&= \lambda_a(1, \star(b, 1, \star(c, x, y))) \\
&= \lambda_a(1, \lambda_b(1, \star(c, x, y))) \\
&= \lambda_a(1, \lambda_b(1, \lambda_c(x, y))) \\
&= \circ^1(\lambda_a, \lambda_b, \lambda_c)(x, y),
\end{aligned}$$

which means that $\lambda_{\star(a,b,c)} = \circ^1(\lambda_a, \lambda_b, \lambda_c)$, and this completes the proof. ∎

**Lemma 4.4.** *Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup. Then $\rho_{\star(a,b,c)} = \circ^1(\rho_c, \rho_b, \rho_a)$ for all $a, b, c \in T^1$, where $\circ^1$ is the ternary composition via identity* 1.

*Proof.* The proof is similar to Lemma 4.3. ∎

For every ternary semigroup $(T, \bullet)$ induced by a semigroup, let $T' = \{\lambda_a \mid a \in T\}$ and $T'' = \{\rho_a \mid a \in T\}$. In order to the next aim, we show that $(T', \circ^1)$ and $(T'', \circ^1)$ are ternary subsemigroups of $(\mathcal{T}(T^1 \times T^1, T^1), \circ^1)$ where $\circ^1$ is the ternary composition via identity 1.

**Lemma 4.5.** *Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup. Then $(T', \circ^1)$ forms a ternary subsemigroup of $(\mathcal{T}(T^1 \times T^1, T^1), \circ^1)$, and it is a ternary semigroup of full binary transformations.*

*Proof.* By the definition of $T'$, we have $T' \subseteq \mathcal{T}(T^1 \times T^1, T^1)$. Since $T \neq \emptyset$, we have $\lambda_a \in T'$ for all $a \in T$. So, $T' \neq \emptyset$. By Lemma 4.3, for each $\lambda_a, \lambda_b, \lambda_c \in T'$ we have $\circ^1(\lambda_a, \lambda_b, \lambda_c) = \lambda_{\star(a,b,c)} \in T'$. Hence, $(T', \circ^1)$ is a ternary subsemigroup of $(\mathcal{T}(T^1 \times T^1, T^1), \circ^1)$. ∎

**Lemma 4.6.** *Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup. Then $(T'', \circ^1)$ forms a ternary subsemigroup of $(\mathcal{T}(T^1 \times T^1, T^1), \circ^1)$, and it is a ternary semigroup of full binary transformations.*

*Proof.* The proof is similar to Lemma 4.5 and it follows from Lemma 4.4. ∎

**Theorem 4.7.** *(Cayley's theorem for ternary semigroups induced by a semigroup) Every ternary semigroup $(T, \bullet)$ induced by a semigroup is isomorphic to some ternary semigroups of full binary transformations.*

*Proof.* Let $(T, \bullet)$ be a ternary semigroup induced by a semigroup $(T, *)$. We show that $(T, \bullet)$ and the ternary semigroup of full binary transformations $(T', \circ^1)$, where $T' = \{\lambda_a \mid a \in T\}$, are isomorphic. Since $(T, *)$ is a semigroup, we obtain that $(T^1, \diamond)$ forms a semigroup with an identity element 1 under the binary operation $\diamond$ defined as (1.1).

By Remark 2.3, we obtain that $(T, \star)$ is a ternary semigroup induced by the semigroup $(T^1, \diamond)$ under a ternary operation $\star$ defined as (2.1), i.e.,

$$\star(x, y, z) = \diamond(\diamond(x, y), z) \quad \text{for all } x, y, z \in T.$$

Indeed, for each $a \in T$ we define a mapping $\varphi : T \longrightarrow T'$ by

$$\varphi(a) = \lambda_a \quad \text{for all } a \in T. \tag{4.3}$$

Please note that, by the definition of $\star$ on $T^1$ we have

$$\star(a, b, c) = \diamond(\diamond(a, b), c) = *(*(a, b), c) = \bullet(a, b, c) \quad \text{for all } a, b, c \in T.$$

By Lemma 4.3, we have

$$\varphi(\bullet(a, b, c)) = \lambda_{\bullet(a,b,c)} = \lambda_{\star(a,b,c)} = \circ^1(\lambda_a, \lambda_b, \lambda_c) = \circ^1(\varphi_a, \varphi_b, \varphi_c), \quad \text{for all } a, b, c \in T.$$

It implies that $\varphi$ is a homomorphism. Moreover, if $\lambda_a = \lambda_b$, then $\lambda_a(x, y) = \lambda_b(x, y)$ for all $x, y \in T$ and hence $\lambda_a(1, 1) = \lambda_b(1, 1)$. This implies $\star(a, 1, 1) = \star(b, 1, 1)$ and hence $a = b$. So, $\varphi$ is injective. Finally, let $\lambda_a \in T'$, then there is an element $a \in T$ such that $\varphi(a) = \lambda_a$, which implies that $\varphi$ is surjective. Consequently, $\varphi$ is an isomorphism and hence $T \cong T'$. ∎

According to Theorem 4.7, we notice that for every element $a$ of a ternary semigroup $(T, \bullet)$ induced by a semigroup, $a$ can be represented by a full binary transformation $\lambda_a$ of a ternary semigroup $(T', \circ^1)$ which is a ternary semigroup of full binary tranformations. Consequently, we call the representation $\varphi$, which is defined as (4.3), an *extended regular representation.*

Futhermore, we can define a mapping $\phi$ from $(T, \bullet)$ into $(T'', \circ^1)$ by $\phi(a) = \rho_a$ for all $a \in T$, where $\rho_a$ is an inner right translation. Then $\phi$ is an anti-homomorphism which follows from Lemma 4.4, and we also have $\phi$ is injective and surjective. Therefore, $T \cong T''$. Consequently, we call the representation $\phi$, an *extended regular anti-representation.*

**Example 4.8.** Consider a semigroup $(T = \{a, b, 0\}, *)$ under a binary operation $*$ defined as in the following table.

| $*$ | $a$ | $b$ | $0$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $0$ |
| $b$ | $b$ | $b$ | $0$ |
| $0$ | $0$ | $0$ | $0$ |

Now, we see that the semigroup $(T, *)$ does not contain any identity element. Then we extend the semigroup $(T, *)$ to a semigroup $(T^1 = \{a, b, 0, 1\}, \diamond)$ with identity 1 under a binary operation $\diamond$ defined as (1.1). All images of $(T^1, \diamond)$ are shown in the following table.

| $\diamond$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $0$ | $a$ |
| $b$ | $b$ | $b$ | $0$ | $b$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $a$ | $b$ | $0$ | $1$ |

By Remark 2.3, we obtain that $(T, \star)$ is a ternary semigroup induced by the semigroup $(T^1, \diamond)$ under a ternary operation $\star$ defined as (2.1), i.e., $\star(x, y, z) = \diamond(\diamond(x, y), z)$ for all $x, y, z \in T$.

Next, we illustate that $T$ is isomorphic to a ternary semigroup of full binary transformations. In order to this aim, let us consider an inner left tranlation $\lambda_x : T \longrightarrow \mathcal{T}(T^1 \times T^1, T^1)$ for all $x \in T$. Indeed, for $a, b, 0 \in T$ we have

| $\lambda_a$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $0$ | $a$ |
| $b$ | $a$ | $a$ | $0$ | $a$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $a$ | $a$ | $0$ | $a$ |

| $\lambda_b$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $0$ | $b$ |
| $b$ | $b$ | $b$ | $0$ | $b$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $b$ | $b$ | $0$ | $b$ |

| $\lambda_0$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $0$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $0$ | $0$ | $0$ |

From this, we define $a \mapsto \lambda_a, b \mapsto \lambda_b$ and $0 \mapsto \lambda_0$. By Theorem 4.7, this implies that $T \cong \{\lambda_a, \lambda_b, \lambda_0\}$. Furthermore, the image of all representations are shown by the following tables.

| $\circ^1(\lambda_a, \lambda_a, \lambda_a)$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $0$ | $a$ |
| $b$ | $a$ | $a$ | $0$ | $a$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $a$ | $a$ | $0$ | $a$ |

| $\circ^1(\lambda_b, \lambda_b, \lambda_b)$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $0$ | $b$ |
| $b$ | $b$ | $b$ | $0$ | $b$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $b$ | $b$ | $0$ | $b$ |

| $\circ^1(\lambda_0, \lambda_0, \lambda_0)$ | $a$ | $b$ | $0$ | $1$ |
|---|---|---|---|---|
| $a$ | $0$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $0$ | $0$ | $0$ |

By directly computation, we have

$$\begin{aligned}
&\circ^1(\lambda_a,\lambda_a,\lambda_a)=\circ^1(\lambda_a,\lambda_b,\lambda_a)=\circ^1(\lambda_a,\lambda_a,\lambda_b)=\circ^1(\lambda_a,\lambda_b,\lambda_b),\\
&\circ^1(\lambda_b,\lambda_b,\lambda_b)=\circ^1(\lambda_b,\lambda_a,\lambda_b)=\circ^1(\lambda_b,\lambda_b,\lambda_a)=\circ^1(\lambda_b,\lambda_a,\lambda_a), \quad \text{and}\\
&\circ^1(\lambda_0,\lambda_0,\lambda_0)=\circ^1(\lambda_0,\lambda_0,\lambda_a)=\circ^1(\lambda_0,\lambda_a,\lambda_0)=\circ^1(\lambda_a,\lambda_0,\lambda_0)=\circ^1(\lambda_0,\lambda_0,\lambda_b)\\
&\quad=\circ^1(\lambda_0,\lambda_b,\lambda_0)=\circ^1(\lambda_b,\lambda_0,\lambda_0)=\circ^1(\lambda_0,\lambda_a,\lambda_a)=\circ^1(\lambda_a,\lambda_0,\lambda_a)\\
&\quad=\circ^1(\lambda_a,\lambda_a,\lambda_0)=\circ^1(\lambda_0,\lambda_b,\lambda_b)=\circ^1(\lambda_b,\lambda_0,\lambda_b)=\circ^1(\lambda_b,\lambda_b,\lambda_0)\\
&\quad=\circ^1(\lambda_a,\lambda_b,\lambda_0)=\circ^1(\lambda_a,\lambda_0,\lambda_b)=\circ^1(\lambda_0,\lambda_a,\lambda_b)=\circ^1(\lambda_b,\lambda_a,\lambda_0)\\
&\quad=\circ^1(\lambda_b,\lambda_0,\lambda_a)=\circ^1(\lambda_0,\lambda_b,\lambda_a).
\end{aligned}$$

Finally, we complete this work by investigation of a related property of the ternary semigroups with an identity element. Now, we observe that if $(T,\bullet)$ a ternary semigroup with an identity element 1, then 1 is an idempotent element too. In this work, we will call $T$ has no idempotent element except 1 if every element $a\in T$ $(a\neq 1)$ is not idempotent element.

**Theorem 4.9.** *Let $(T,\bullet)$ be a ternary semigroup induced by a semigroup, $T$ has the right cancellative property and every element $a\in T$ $(a\neq 1)$ is not an idempotent element. If $a,b\in T$ such that $\bullet(a,a,b)=b$, then $a=1$. (i.e. $T=T^1$).*

*Proof.* Let $a,b,c\in T$ be such that $\bullet(a,a,b)=b$. Then $\bullet(a,\bullet(a,a,b),c)=\bullet(a,b,c)$, and hence $\bullet(\bullet(a,a,a),b,c)=\bullet(a,b,c)$. By the right cancellative property, we conclude that $\bullet(a,a,a)=a$ and hence $a$ is an idempotent element. Since $T$ has no an idempotent element except 1, we have $a=1$ and hence $T=T^1$. ∎

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