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# On the Graded 2-Absorbing Primary Submodules of Graded Multiplication Modules

**Fatemeh Soheilnia**[1] **and Shiroyeh Payrovi**[1,*]

[1] *Department of Mathematics, Imam Khomeini International University, P.O.Box 34149-16818, Qazvin, Iran.*
*e-mail : f.soheilnia@edu.ikiu.ac.ir; shpayrovi@sci.ikiu.ac.ir*

**Abstract** Let $G$ be a multiplicative group, $R$ be a $G$-graded commutative ring and $M$ be a graded $R$-module. A proper graded submodule $N$ of $M$ is called graded 2-absorbing primary, if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in Gr_M(N)$ or $bm \in Gr_M(N)$. Let $M$ be a graded finitely generated multiplication $R$-module. It is shown that $Gr(N :_R M) = \big(Gr_M(N) :_R M\big)$. Furthermore, it is proved that $(N :_R M)$ is a graded 2-absorbing primary ideal of $R$, if $N$ is a graded 2-absorbing primary submdoule of $M$. Moreover, it is generalized some results of graded 2-absorbing ideals over trivial extension of a ring.

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## 1. Introduction

Let $R$ be a commutative ring. A proper ideal $I$ of $R$ is called 2-absorbing, if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $bc \in I$ or $ac \in I$, see [6]. As a generalization of the concept of 2-absorbing ideal the authors in [13, 19], introduced and studied the concept of 2-absorbing submodule of a module. Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is said to be a 2-absorbing submodule, if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Recently, in [7], the concept of primary ideals have studied. A proper ideal $I$ of $R$ is said to be a 2-absorbing primary ideal, if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Some researchers in [11], have generalized the concept of primary ideals for submodules. A proper submodule $N$ of $M$ is called 2-absorbing primary, if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in M$-rad$(N)$ or $bm \in M$-rad$(N)$, where $M$-rad$(N)$ is the intersection of all prime submodules of $M$ containing $N$. If $N$ is not contained in any prime submodule of $M$, then $M$-rad$(N) = M$. All these concepts are introduced and studied in last decade and further on $G$-graded rings and graded $R$-modules, see

*Corresponding author.

[1, 10, 14–17]. Let $G$ be a multiplicative group with identity $e$. Then $R$ is a $G$-graded ring, if there exists a family of additive subgroups $\{R_g\}_{g\in G}$ of $R$ such that $R = \oplus_{g\in G}R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and we write $h(R) = \cup_{g\in G}R_g$. Let $R$ be a $G$-graded ring and $M$ be an $R$-module. Then $M$ is called a graded $R$-module, if there exists a family of subgroups (as abelian groups) $\{M_g\}_{g\in G}$ of $M$ such that $M = \oplus_{g\in G}M_g$ and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. We write $h(M) = \cup_{g\in G}M_g$ and the elements of $h(M)$ are called homogeneous. If $M = \oplus_{g\in G}M_g$ is a graded $R$-module, then for all $g \in G$ the subgroup $M_g$ of $M$ is an $R_e$-module. Let $N$ be a submodule of $M = \oplus_{g\in G}M_g$. Then $N$ is called a graded submodule, if $N = \oplus_{g\in G}N_g$ where $N_g = N \cap M_g$ for $g \in G$, see [12]. Let $R$ be a $G$-graded ring. A proper graded ideal $I$ of $R$ is called graded 2-absorbing, if whenever $a, b, c \in h(R)$ with $abc \in I$, then $ab \in I$ or $bc \in I$ or $ac \in I$. A graded ideal $I$ of $R$ is said to be graded 2-absorbing primary, if whenever $a, b, c \in h(R)$ with $abc \in I$, then $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$, see [15]. The concept of graded 2-absorbing primary submodule of a graded $R$-module was defined in [17]. Here we find more results on the graded 2-absorbing primary submodule of a graded multiplication module. Throughout this work $G$ is a multiplicative group with identity $e$, $R$ is $G$-graded commutative ring with non-zero identity and $M$ is a graded $R$-module.

## 2. Graded 2-absorbing Primary submodule

Let $R$ be a $G$-graded ring and $M$ be a graded $R$-module. As noted in [1], a graded proper submodule $N$ of $M$ is said to be graded 2-absorbing submodule, if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. The graded radical of a graded submodule $N$ of a graded $R$-module $M$, which is denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of $M$ containing $N$. If $N$ is not contained in any graded prime submodule of $M$, then $Gr_M(N) = M$.

**Definition 2.1.** [17] Let $R$ be a $G$-graded ring, $M$ be a graded $R$-module and $N$ be a proper graded submodule of $M$. Then $N$ is called a graded 2-absorbing primary submodule of $M$, if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in Gr_M(N)$ or $bm \in Gr_M(N)$.

In the following we show some straightforward results.

**Lemma 2.2.** *Let $R$ be a $G$-graded ring and $M$ be a graded $R$-module.*

*(i) Every graded primary submodule of $M$ is a graded 2-absorbing primary submodule of $M$.*

*(ii) Every graded 2-absorbing submodule of $M$ is a graded 2-absorbing primary submodule of $M$.*

A graded $R$-module $M$ is called graded multiplication module, if for every graded submodule $N$ of $M$, there exists a graded ideal $I$ of $R$ such that $N = IM$. In this case, it can easily see that if $M$ is a graded multiplication $R$-module, then $N = (N :_R M)M$ for every graded submodule $N$ of $M$.

**Lemma 2.3.** *Let $R$ be a $G$-graded ring, $M$ be a graded finitely generated multiplication $R$-module and $N$ be a graded submodule of $M$. Then $\big(Gr_M(N) :_R M\big) = Gr(N :_R M)$.*

*Proof.* First we show that $(Gr_M(N) :_R M) \subseteq Gr(N :_R M)$. Let $a \in (Gr_M(N) :_R M)$. Then $aM \subseteq Gr_M(N) = Gr(N :_R M)M$, by [10, Theorem 9]. It can be easily to see that $a^t + b \in \mathrm{Ann}(M)$ for some $b \in Gr(N :_R M)$ and $t \in \mathbb{N}$. So $a^t + b \in Gr(N :_R M)$ and $a \in Gr(N :_R M)$. For the reverse inclusion assume that $K$ is a graded prime submodule of $M$ containing $N$. Then $(K :_R M)$ is a graded prime ideal of $R$, by [3, Proposition 2.5]. Thus $(N :_R M) \subseteq (K :_R M)$ and so $Gr(N :_R M) \subseteq (K :_R M)$. Hence, $Gr(N :_R M)M \subseteq (K :_R M)M = K$. Therefore, $Gr(N :_R M)M \subseteq Gr_M(N)$ and $Gr(N :_R M) \subseteq \big(Gr_M(N) :_R M\big)$. ■

**Theorem 2.4.** *Let $R$ be a $G$-graded ring, $M$ be a graded finitely generated multiplication $R$-module and $N$ be a graded proper submodule of $M$. Then $N$ is a graded 2-absorbing primary submodule of $M$ if and only if $(N :_R M)$ is a graded 2-absorbing primary ideal of $R$.*

*Proof.* Assume that $N$ is a graded 2-absorbing primary submodule of $M$ and $abc \in (N :_R M)$ for some $a, b, c \in h(R)$. Let $cM = K$. Then $abK \subseteq N$. By assumption $N$ is a graded 2-absorbing primary submodule, so we conclude that $ab \in (N :_R M)$ or $aK \subseteq Gr_M(N)$ or $bK \subseteq Gr_M(N)$, see [17, Theorem 3]. If $ab \in (N :_R M)$ we are done. Suppose that $aK = acM \subseteq Gr_M(N)$, so by Lemma 2.3, $ac \in (Gr_M(N) :_R M) = Gr(N :_R M)$ as desired. If $bK = bcM \subseteq Gr_M(N)$ by a similar argument one can show $bc \in Gr(N :_R M)$. Thus $(N :_R M)$ is a graded 2-absorbing primary ideal of $R$. The converse follows from [17, Theorem 8]. ■

**Proposition 2.5.** *Let $R$ be a $G$-graded ring, $M$ be a graded $R$-module and $N$ be a graded 2-absorbing primary submodule of $M$. Then $Gr(N :_R m) \subseteq \big(Gr_M(N) :_R m\big)$, for every $m \in h(M) \setminus N$.*

*Proof.* Assume that $a \in Gr(N :_R m)$ and $a \in h(R)$. If $a \in Gr(N :_R M)$, then there exists some positive integer $n$ such that $a^n M \subseteq N \subseteq Gr_M(N)$. So $aM \subseteq Gr_M(N)$ since $Gr_M(N)$ is the intersection of prime submodules of $M$ contain $N$. Thus $a \in (Gr_M(N) :_R m)$. Now, assume that $a \in Gr(N :_R m) \setminus Gr(N :_R M)$. Then there exists some positive integer $n$ such that $a^n m \in N$. Since $N$ is a graded 2-absorbing primary submodule of $M$, we conclude that $am \in Gr_M(N)$ or $a^{n-1}m \in Gr_M(N)$. If $am \in Gr_M(N)$, we are done. If $a^{n-1}m \in Gr_M(N)$, then $am \in Gr_M(N)$ since $Gr_M(N)$ is the intersection of prime submodules of $M$ contain $N$. ■

**Theorem 2.6.** *Let $R$ be a $G$-graded ring, $M$ be a graded finitely generated multiplication $R$-module. If $N$ is a graded 2-absorbing primary submodule of $M$, then $Gr_M(N)$ is a graded 2-absorbing submodule of $M$.*

*Proof.* Assume that $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in Gr_M(N)$ with $ab \notin \big(Gr_M(N) :_R M\big) = I$. Since $M$ is a graded multiplication $R$-module so there exists a graded ideal $J$ of $R$ such that $Rm = JM$. Thus $abJ \subseteq I$ and $ab \notin I$. We claim that $aJ \subseteq I$ or $bJ \subseteq I$. Suppose in the contrary, $aJ \nsubseteq I$ and $bJ \nsubseteq I$. Hence, there exist $i_1, i_2 \in I$ such that $ai_1 \notin I$ and $bi_2 \notin I$. Since $abi_1 \in I$ but $ab, ai_1 \notin I$ and $I = \big(Gr_M(N) :_R M\big) = Gr\big(N :_R M\big)$ is a graded 2-absorbing ideal, see [15, Theorem 2.10], we conclude that $bi_1 \in I$. By a similar argument one can show that $ai_2 \in I$. Now, $ab(i_1 + i_2) \in I$, $ab \notin I$ and $I$ is a graded 2-absorbing ideal, we conclude that either $a(i_1 + i_2) \in I$ or $b(i_1 + i_2) \in I$. If $a(i_1 + i_2) \in I$, then $ai_1 \in I$, which is a contradiction. Similarly, by $b(i_1 + i_2) \in I$ we get a contradiction. Hence, either $aJ \subseteq I$ or $bJ \subseteq I$. Then $am \in Gr_M(N)$ or $bm \in Gr_M(N)$, as needed. Therefore, $Gr_M(N)$ is graded 2-absorbing. ■

**Lemma 2.7.** *Let $R$ be a $G$-graded ring, $M$ be a graded multiplication $R$-module and $N$ be a graded 2-absorbing primary submodule of $M$ such that $Gr_M(N) = N$. Then $(N :_R m)$ is graded 2-absorbing primary, for every $m \in h(M) \backslash N$.*

*Proof.* Assume that $m \in h(M) \setminus N, a, b, c \in h(R)$ and $abc \in (N :_R m)$. By hypothesis, $N$ is graded 2- absorbing primary, so from $abcm \in N$, we conclude that $ab \in (N :_R M)$ or $bcm \in Gr_M(N)$ or $acm \in Gr_M(N)$. Then $ab \in (N :_R m)$ or $bc \in (Gr_M(N) :_R m) = (N :_R m) \subseteq Gr(N :_R m)$ or $ac \in (Gr_M(N) :_R m) = (N :_R m) \subseteq Gr(N :_R m)$, as desired. ∎

**Theorem 2.8.** *Let $R$ be a $G$-graded ring, $M$ be a graded multiplication $R$-module and $N$ be a graded 2-absorbing submodule of $M$ such that $Gr_M(N) = N$ and $Gr(N :_R M)$ is a prime ideal of $R$. Then the following statements hold:*

*(i) If $m \in h(M) \setminus N$, then $(N :_R m)$ is a graded prime ideal.*
*(ii) If $m, m' \in h(M) \setminus N$, then $(N :_R m) \subseteq (N :_R m')$ or $(N :_R m') \subseteq (N :_R m)$.*

*Proof.* (i) Assume that $m \in h(M) \setminus N$, $a, b \in h(R)$ and $ab \in (N :_R m)$. So $abm \in N$. Thus $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ since $N$ is graded 2-absorbing. If $am \in N$ or $bm \in N$, we are done. Now suppose that $ab \in (N :_R M)$. By [16, Theorem 2.2], $(N :_R M)$ is a graded 2-absorbing ideal of $R$ and by hypotheses $Gr(N :_R M)$ is a graded prime ideal, see [5, Theorrem 2.2]. Hence, $ab \in Gr(N :_R M)$ which implies that $a \in Gr(N :_R M)$ or $b \in Gr(N :_R M)$. If $a \in Gr(N :_R M)$, then $a \in (N :_R m)$ since $Gr(N :_R m) \subseteq (Gr_M(N) :_R m) = (N :_R m)$, by Proposition 2.5. If $b \in Gr(N :_R M)$, then $b \in (N :_R m)$, as needed.

(ii) Assume that $(N :_R m') \nsubseteq (N :_R m)$. We have to show that $(N :_R m) \subseteq (N :_R m')$. Let $a \in (N :_R m)$ and $b \in (N :_R m') \setminus (N :_R m)$. If $a(m + m') \in N$, then $am' \in N$ and we are done. Suppose that $a(m + m') \notin N$. Then by hypotheses and $b(m + m') \notin N$ it follows that $ab \in (N :_R M)$. If $b \in Gr(N :_R M)$, then $b \in Gr(N :_R m) \subseteq (Gr_M(N) :_R m) = (N :_R m)$ which is a contradiction. Hence, $a \in Gr(N :_R M) \subseteq (Gr_M(N) :_R m') = (N :_R m')$ as needed. ∎

**Theorem 2.9.** *Let $R$ be a $G$-graded Noetherian ring and $I$ be a graded 2-absorbing (res., 2-absorbing primary) ideal of $R$. Let $M$ be a finitely generated faithful multiplication graded $R$-module such that $Ass_G(M/Gr(I)M)$ be a totally ordered set. Then $IM$ is a graded 2-absorbing (res., 2-absorbing primary) submodule of $M$.*

*Proof.* Assume that $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in IM$. By a similar argument to that of the proof of [11, Theorem 2.12] we get that $am \in IM$ or $bm \in IM$ or $ab \in I$. Now, the result follows by [4, Lemma 3.10]. For the second part, we need to show that $Gr_M(IM) = Gr(I)M$. By [10, Theorem 8(i)] we have

$$Gr(I)M = \Big( \bigcap_{\substack{I \subseteq \mathfrak{p} \\ \mathfrak{p} \text{ is prime}}} \mathfrak{p} \Big) M = \bigcap_{IM \subseteq pM} \mathfrak{p}M \supseteq Gr_M(IM).$$

On the other hand,

$$Gr_M(IM) = \bigcap_{\substack{IM \subseteq Q \\ Q \text{ is prime}}} Q = \bigcap_{IM \subseteq Q} (Q :_R M)M = \Big( \bigcap_{I \subseteq (Q :_R M)} (Q :_R M) \Big) M.$$

So $Gr_M(IM) \supseteq Gr(I)M$ and $Gr_M(IM) = Gr(I)M$. Now, the result follows by [11, Theorem 2.12]. ■

Let $I$ be a graded ideal of $R$ and $N$ be a graded submodule of a graded $R$-module $M$. The graded residual of $N$ by $I$ is defined $(N :_M I) = \{m \in h(M) | mI \subseteq N\}$. In the following we show more results on graded residual submodule $(N :_M I)$.

**Lemma 2.10.** *Let $R$ be a $G$-graded ring, $M$ be a graded $R$-module and $N$ be a graded submodule of $M$. Then the following statements hold:*

*(i) $(N :_M I)$ is a graded submodule of $M$.*
*(ii) If $M$ is a graded multiplication $R$-module, then*

$$(N :_M I) = (N :_R IM)M = ((N :_R M) :_R I)M.$$

*Proof.* (i) Let $m = \sum_{g\in G} m_g \in (N :_M I)$. Without loss of generality we may assume that $m = \sum_{i=1}^{t} m_{g_i}$, where $m_{g_i} \neq 0$ for all $1 \leq i \leq t$. Thus we conclude that $Im = \sum_{i=1}^{t} Im_{g_i} \subseteq N$ and so $Im_{g_i} \subseteq N$, for all $1 \leq i \leq t$. Hence, $m_{g_i} \in (N :_M I)$, for all $1 \leq i \leq t$.

(ii) Obviously, $(N :_R IM)M = ((N :_R M) :_R I)M$. Now, we have to show that $(N :_M I) = (N :_R IM)M$. Since $M$ is a graded multiplication $R$-module, we have $(N :_R IM)IM = (N :_R IM)(IM :_R M)M \subseteq (N :_R M)M = N$ and hence $(N :_R IM)M \subseteq (N :_M I)$. For the reverse inclusion suppose that $m \in (N :_M I)$. Thus $I(Rm :_R M) \subseteq (Im :_R M) \subseteq (N :_R M)$. Hence $(Rm :_R M) \subseteq ((N :_R M) :_R I)$ and $Rm = (Rm :_R M)M \subseteq ((N :_R M) :_R I)M = (N :_R IM)M$. Thus $(N :_M I) \subseteq (N :_R IM)M$, as desired. ■

**Proposition 2.11.** *Let $R$ be a $G$-graded ring, $M$ be a graded $R$-module and $N$ be a graded submodule of $M$. Then the following statements are equivalent:*

*(i) $N$ is a graded 2-absorbing primary submodule of $M$;*
*(ii) $(N :_M ab) \subseteq (Gr_M(N) :_M a) \cup (Gr_M(N) :_M b)$, for every $a, b \in h(R)$ and $m \in h(M)$ with $ab \notin (N :_R M)$.*

*Proof.* $(i) \Rightarrow (ii)$ Assume that $m \in (N :_M ab)$ for some $m \in h(M)$. Thus $abm \in N$. Since $N$ is graded 2-absorbing primary and $ab \notin (N :_R M)$, we get that $am \in Gr_M(N)$ or $bm \in Gr_M(N)$. Hence, $m \in (Gr_M(N) :_R a)$ or $m \in (Gr_M(N) :_R b)$. Therefore, $(N :_M ab) \subseteq (Gr_M(N) :_M a) \cup (Gr_M(N) :_M b)$, as required.

$(ii) \Rightarrow (i)$ Assume that $a, b \in h(R)$, $m \in h(M)$ and $abm \in N$. Assume that $ab \notin (N :_R M)$. By hypotheses $(N :_M ab) \subseteq \big(Gr_M(N) :_M a\big) \cup \big(Gr_M(N) :_M b\big)$, we conclude that $am \in Gr_M(N)$ or $bm \in Gr_M(N)$, as needed. ■

**Proposition 2.12.** *Let $R$ be a $G$-graded ring, $M$ be a graded multiplication $R$-module and $N$ be a graded 2-absorbing primary submodule of $M$ such that $Gr_M(N) = N$. Then $(N :_M I)$ is a graded 2-absorbing primary submodule of $M$.*

*Proof.* Assume that $a, b \in h(R)$, $m \in h(M)$ and $abm \in (N :_M I)$ with $abM \nsubseteq (N :_M I)$. So from $abmI \subseteq N$, we obtain $abI \subseteq (N :_R m)$. By Lemma 2.7, $(N :_R m)$ is a graded 2-absorbing primary ideal. Suppose that $ab \notin (N :_R m)$. Hence $aI \subseteq Gr(N :_R m)$ or $bI \subseteq Gr(N :_R m)$, by [15, Proposition 2.28]. If $aI \subseteq Gr(N :_R m) \subseteq (Gr_M(N) :_R m) = (N :_R m)$, then $am \in (N :_M I)$, by Proposition 2.5. If $bI \subseteq Gr(N :_R m)$, then by a similar argument we get that $bm \in (N :_M I)$. Therefore, the proof is complete. ■

**Theorem 2.13.** *Let $R$ be a $G$-graded ring, $I$ be a graded multiplication ideal and $M$ be a finitely generated graded multiplication $R$-module. Then $N$ is a graded 2-absorbing primary submodule of $IM$ if and only if $(N :_M I)$ is a graded 2-absorbing primary submodule of $M$.*

*Proof.* By [8, Corollary 2.8], $IM$ is a graded multiplication $R$-module. By Lemma 2.10(ii), we get that $(N :_R IM) = ((N :_M I) :_R M)$. Then $N$ is a graded 2-absorbing primary submodule of $IM$ if and only if $(N :_M I)$ is a graded 2-absorbing primary submodule of $M$, by Theorem 2.4. ∎

## 3. Graded 2-absorbing ideals of trivial extension of a ring

Let $R$ be a ring with identity and $M$ be an $R$-module. Then $R(+)M$ with addition $(a, m) + (b, n) = (a + b, m + n)$ and multiplication $(a, m)(b, n) = (ab, an + bm)$ is a commutative ring. The ring $R(+)M$ is said to be trivial extension of $R$ by $M$ or the idealization of $M$. We view $R$ as a subring of $R(+)M$ via $r \to (r, 0)$.

Let $R = \oplus_{g\in G} R_g$ be a $G$-graded commutative ring and $M = \oplus_{g\in G} M_g$ be a graded $R$-module. Then $R(+)M = \oplus_{g\in G}(R(+)M)_g$ is a graded ring, denoted by $GR(M)$, where $(R(+)M)_g = R_g \oplus M_g$ and $(R(+)M)_g(R(+)M)_h = (R_g \oplus M_g)(R_h \oplus M_h) = R_gR_h \oplus (R_gM_h + R_hM_g) \subseteq R_{gh} \oplus M_{gh}$ for all $g, h \in G$, see [9, 18].

**Theorem 3.1.** *Let $R$ be a $G$-graded ring, $I$ be a graded proper ideal and $M$ be a graded $R$-module. Then the following statement are equivalent:*

*(i) $I$ is a graded 2-absorbing ideal of $R$;*
*(ii) $I(+)M$ is a graded 2-absorbing ideal of $GR(M)$.*

*Proof.* (i)⇒ (ii) Assume that $(a_1, m_1)(a_2, m_2)(a_3, m_3) \in I(+)M$ for some $(a_1, m_1)$, $(a_2, m_2), (a_3, m_3) \in h(R(+)M)$. Thus $a_1a_2a_3 \in I$, where $a, b, c \in h(R)$. Since $I$ is graded 2-absorbing, we conclude that $a_1a_2 \in I$ or $a_2a_3 \in I$ or $a_1a_3 \in I$. Hence, $(a_1, m_1)(a_2, m_2) \in I(+)M$ or $(a_2, m_2)(a_3, m_3) \in I(+)M$ or $(a_1, m_1)(a_3, m_3) \in I(+)M$, as needed.

(ii)⇒ (i) Assume that $abc \in I$ for some $a, b, c \in h(R)$. Then $(a, 0)(b, 0)(c, 0) \in I(+)M$. Since $I(+)M$ is a graded 2-absorbing ideal, we get that $(a_1, 0)(a_2, 0) \in I(+)M$ or $(a_2, 0)(a_3, 0) \in I(+)M$ or $(a_1, 0)(a_3, 0) \in I(+)M$ and hence $ab \in I$ or $bc \in I$ or $ac \in I$. Then $I$ is a graded 2-absorbing ideal of $R$. ∎

**Example 3.2.** Let $R = \mathbb{Z}\oplus\mathbb{Z}$ be a $\mathbb{Z}_2$-graded ring and $M = \mathbb{Z}\oplus\mathbb{Z}$ be a graded $R$-module. Suppose that $I = 15\mathbb{Z} \oplus \{0\}$ and $N = 12\mathbb{Z} \oplus \{0\}$. Then $I(+)N$ is a graded ideal of $GR(M)$ but is not 2-absorbing. Since $(3, 2)(3, 2)(5, 4) \in I(+)N$, but $(3, 2)(5, 4) \notin I(+)N$ and $(3, 2)(3, 2)I(+)M \nsubseteq I(+)N$. Notice to the fact that $I$ is a 2-absorbing ideal of $R$ and $N$ is not a 2-absorbing submodule of $M$.

An ideal $H$ of $R(+)M$ is said to be homogeneous, if $H = I(+)N$ for some ideal $I$ of $R$ and some submodule $N$ of $M$ and $IM \subseteq N$, see [2].

**Theorem 3.3.** *Let $R$ be a $G$-graded ring and $I$ be a graded ideal of $R$, let $N$ be a graded submodule of $M$. If $I(+)N$ is a graded homogeneous 2-absorbing ideal of $GR(M)$, then $I$ and $N$ are graded 2-absorbing too.*

*Proof.* Assume that $I(+)N$ is a graded 2-absorbing ideal of $GR(M)$. Let $a, b, c \in h(R)$ such that $abc \in I$. Then $(a, 0)(b, 0)(c, 0) \in I(+)N$. Since $I(+)N$ is a graded 2-absorbing ideal, we conclude that $(a, 0)(b, 0) \in I(+)N$ or $(b, 0)(c, 0) \in I(+)N$ or $(a, 0)(c, 0) \in I(+)N$.

So $ab \in I$ or $bc \in I$ or $ac \in I$. Hence, $I$ is a graded 2-absorbing ideal of $R$. Now, suppose that $abm \in N$ for some $a, b \in h(R)$ and $m \in h(M)$. Since $I(+)N$ is a graded homogenous 2-absorbing ideal, we have $(a,0)(b,0)(0,m) \in I(+)N$. Then $(a,0)(b,0) \in I(+)N$ or $(a,0)(0,m) \in I(+)N$ or $(b,0)(0,m) \in I(+)N$. Thus $ab \in I \subseteq (N :_R M)$ and hence $am \in N$ or $bm \in N$, as needed. ■

**Proposition 3.4.** *Let $R$ be a $G$-graded ring and $I$ be a graded ideal of $R$, let $N$ be a graded submodule of $M$. Then the following statements hold:*

*(i) $I$ is a prime ideal if and only if $I(+)N$ is a graded prime ideal of $GR(M)$.*
*(ii) If $I(+)N$ is a graded homogeneous ideal of $GR(M)$, then*

$$Gr(I(+)N) = Gr(I)(+)M.$$

*Proof.* (i) The proof is satisfy by [18, Proposition 3.1].

(ii) Let $(a,m) \in Gr(I(+)N)$. Then there exists positive integer $n$ such that $(a,m)^n \in I(+)M$. Thus $(a,m)^n = (a^n, na^{n-1}m) \in I(+)M$. Hence, $Gr(I(+)N) \subseteq Gr(I)(+)M$.

For the reverse inclusion, suppose that $(a,m) \in Gr(I)(+)M$. Thus $a^n \in I$, for some positive integer $n$. Consider that $(a,m)^{n+1} = (a^{n+1}, (n+1)a^n m)$ and $(a,m)^{n+1} \in I(+)IM \subseteq I(+)N$ since $N$ is a graded homogeneous submodule of $M$. Thus $(a,m) \in Gr(I(+)N)$ and so $Gr(I)(+)M \subseteq Gr(I(+)N)$. ■

**Theorem 3.5.** *Let $R$ be a $G$-graded ring, $I$ be a graded proper ideal of $R$ and $M$ be a graded $R$-module. Then the following statement are equivalent:*

*(i) $I$ is a graded 2-absorbing primary ideal of $R$;*
*(ii) $I(+)M$ is a graded 2-absorbing primary ideal of $GR(M)$.*

*Proof.* By Proposition 3.4 we have $Gr(I(+)M) = Gr(I)(+)M$, now the complete proof is satisfy with similar way such as Theorem 3.1. ■

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