Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 715–720

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Property of K_{2n+1} as the Sum of n Spanning Cycles

Hatairat Yingtaweesittikul and Vites Longani*

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : hatairat.y@gmail.com (H. Yingtaweesittikul); vites.l@cmu.ac.th (V. Longani))

Abstract It is known that K_{2n+1} is the sum of n spanning cycles. In this paper we show that when $2n+1$ is prime number we can have additional property that all lines of the first cycle have distances 1, all lines of the second cycle have distances $2, \ldots,$ and all lines of the n-th cycle have distances n. Also, when $2n+1$ is not prime number, this property is not possible.

MSC: 05B99
Keywords: spanning cycles; complete graph

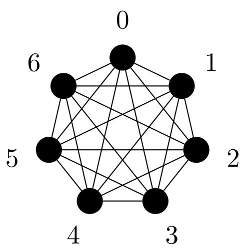
Submission date: 29.09.2019 / Acceptance date: 10.09.2020

1. Introduction

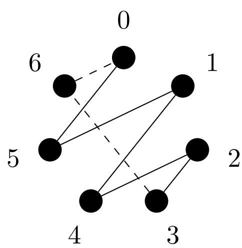
In this paper, unless state otherwise, we shall use definitions from [2]. We are discussing particular cases of complete graph K_{2n+1}. From [2], we have the following theorem.

Theorem 1.1. [see [2]] *The graph K_{2n+1} is the sum of n spanning cycles.*

For example, K_7 is the sum of 3 spanning (hamiltonian) cycles, see Fig. 1.1.

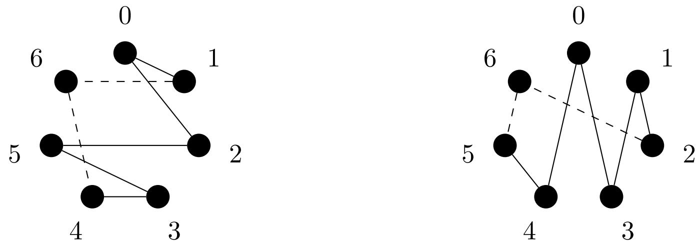


K_7



(a)

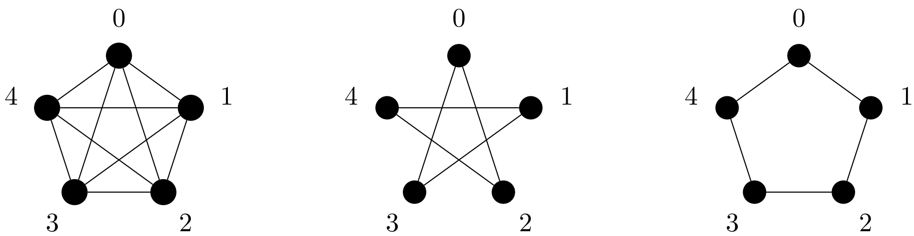
*Corresponding author.



(b) (c)

Fig. 1.1 Three spanning cycles of K_7

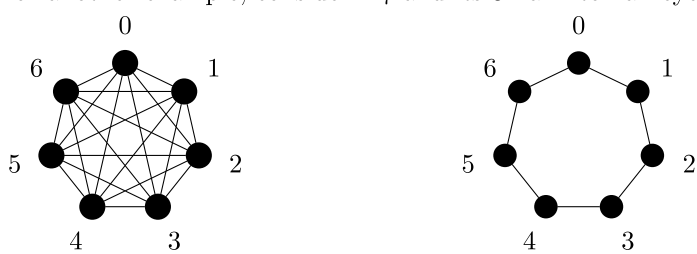
Consider K_5, we can have 2 hamiltonian cycles of K_5 as in Fig. 1.2



K_5 (a) (b)

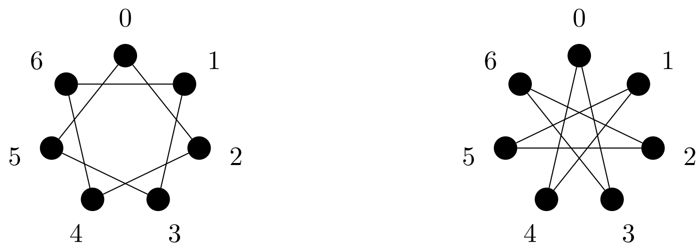
Fig. 1.2

For another example, consider K_7 and its 3 hamiltonian cycles,



K_7 (a)

All lines have distances 1



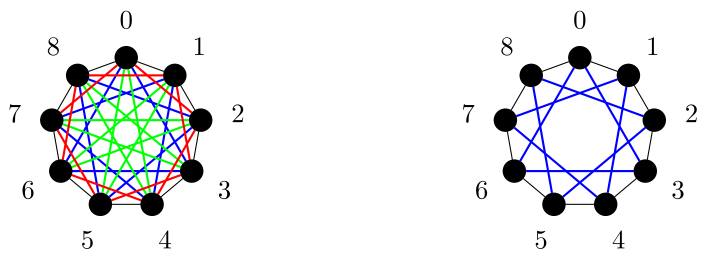
(*b*) All lines have distances 2

(*c*) All lines have distances 3

Fig. 1.3

Note that K_5 can have 2 hamiltonian cycles of which all lines have distances 1 for the first hamiltonian cycle and all lines have distances 2 for the second cycle. For K_7, it has 3 hamiltonian cycles of which the first, second, and third cycles have all of their lines of distances 1, 2, and 3 respectively.

Now look at K_9 (where $n = 4$) in Fig. 1.4



K_9

No hamiltonian cycle of distance 3

Fig. 1.4

From Fig.1.4, we can see that K_9 has no hamiltonian cycle of length 3. The reader can verify, for example, that K_{15} (where $n = 7$) can not have hamiltonian cycle of length 5.

Definition 1.2. K_{2n+1} is called n sequentially hamiltonian if K_{2n+1} can have n spanning cycles of which all lines of the first cycles, all lines of the second cycle, all lines of the third cycles, ..., all lines of the n-th cycle have distances 1, 2, 3,..., n respectively.

Therefore, according to Definition 1.2 , K_5 and K_7 are n sequentially hamiltonian, while K_9 is not. Next section, we apply Theorem 1.1 to enable us to know when K_p is n sequently hamiltonian.

2. Seat Arrangement Problems

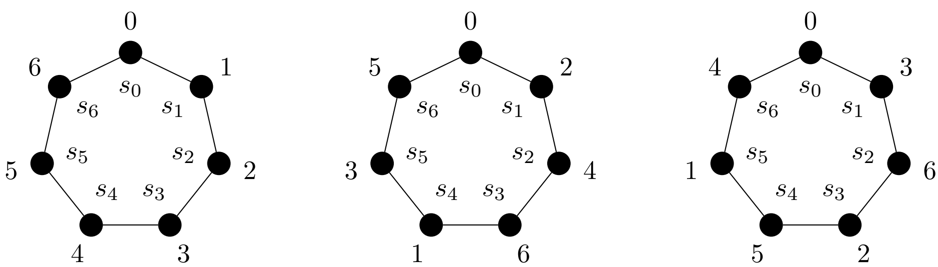
We shall apply a theorem on Seat Arrangement Problems (SAP) in [1] to explain Theorem 3.1 which is a theorem on n sequentially hamiltonian.

First, we shall briefly describe SAP and the corresponding results in [1]. Let there be n students and n row seats. For n days, a seat is arranged for each student on each day, and each student required to sit on different seat on each of the n days. Also, for these n days, it is required that each student shall has one chance to sit next to every other $(n-1)$ students on one of his side, and shall has one chance to sit next to every other $(n-1)$ students on the other side. In [1], we provide an algorithm Seat Arrangement Algorithm (SAA) that can arrange seats for students on each day, and also provide a theorem related to the algorithm.

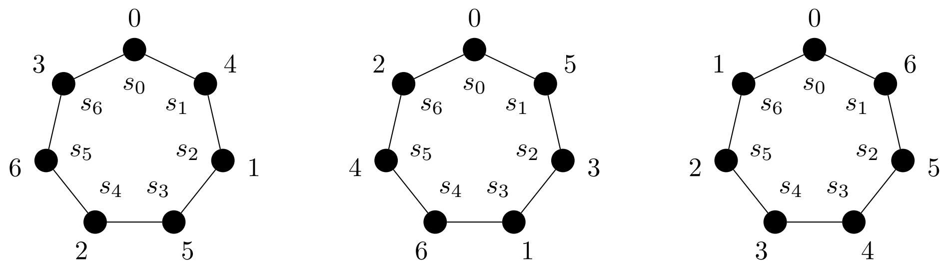
Theorem 2.1. *[see [1]] The SAA are possible when the number of students $n = p - 1$ for any given prime number $p \geq 3$.*

Here, we only need to know how to use the algorithm. Readers who are interested in the proof for the algorithm can find details in the paper. For example, suppose there are 6 students 1, 2, 3, 4, 5, 6 and there are 6 seats $s_1, s_2, s_3, s_4, s_5, s_6$. According to the algorithm, we let 0 to represent the teacher who shall arrange seats for students on each day, and let s_0 to represent the seat for the teacher.

To serve the proof of Theorem 2.1, we fix the seats $s_0, s_1, s_2, s_3, s_4, s_5$, and s_6 in circular form. On the first day the teacher and students 0,1,2,3,4,5,6 shall sit on seats $s_0, s_1, s_2, s_3, s_4, s_5, s_6$ respectively. Every day the teacher shall sit on s_0 but students shall be assigned by the teacher to sit on different seats every day. See Fig. 2.1 for the arrangement for 6 days.



Day 1 Day 2 Day 3



Day 4 Day 5 Day 6

Fig. 2.1

In linear form, the above arrangement can be written as

	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_0
Day 1	0	1	2	3	4	5	6	0
Day 2	0	2	4	6	1	3	5	0
Day 3	0	3	6	2	5	1	4	0
Day 4	0	4	1	5	2	6	3	0
Day 5	0	5	3	1	6	4	2	0
Day 6	0	6	5	4	3	2	1	0

Fig. 2.2

In section 3, we shall apply the Seat Arrangement Algorithm in [1] for explaining Theorem 3.1.

3. K_p THAT ARE n SEQUENTIALLY HAMILTONIAN

From Fig. 2.2, we can see that each of the seat arrangement Day 1, Day 2, Day 3 represent each of the three hamiltonian cycles in Fig. 1.3. The arrangement Day 4, Day 5, Day 6 also represent the three cycles in Fig. 1.3, but in opposite directions.

So, for K_7, we can see that we can find 3 hamiltonian cycles so that K_7 is 3 sequentially hamiltonian.

For general case, we fix the seats $s_0, s_1, s_2, \dots, s_{p-1}$ in circular form. We obtain the seat arrangement for Day 1, Day 2, Day 3, ..., Day $p-1$ by using the Seat Arrangement Algorithm (SAA), see [1]. The SAA is quoted as follow:

(1) Everyday the teacher sits at s_0.

(2) For Day 1 arrangement, teacher shall assign students $1, 2, 3, \dots, p-1$ to sit on $s_1, s_2, s_3, \dots, s_{p-1}$ respectively. Therefore, corresponding to positions of seats $s_0, s_1, s_2, \dots, s_{p-1}$, the Day 1 arrangement is

$$\text{Day 1} \qquad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad p-1 \quad 0.$$

(3) For Day k arrangements ($k = 2, 3, 4, \ldots, p-1$), we have the arrangements as

$$\text{Day } k \qquad 0 \quad k \quad 2k \quad 3k \quad 4k \quad \ldots \quad (p-1)k \quad 0.$$

Next, we represent teacher 0, students $1, 2, 3, \ldots, p-1$ by p points of graph. We then draw lines of graph according to seat arrangement for each day and obtain $p-1$ hamiltonian cycles. All lines of the 1st, 2nd, 3rd, $\ldots, (p-1)$-th cycles have distances $1, 2, 3, \ldots, p-1$ respectively.

Now, we can have Theorem 3.1

Theorem 3.1. *For every $p \geq 3$, K_p is n sequentially hamiltonian if and only if p is prime number.*

References

[1] V. Longani, H. Yingtaweesittikul, Seat Arrangement Problems, Thai Journal of Mathematics 14(2) (2016) 383–390.

[2] F. Harary, "Graph Theory", Addison-Wesley Publishing Company, Boston, 1969.