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Simple Analytical Formulas for Pricing and Hedging Moment Swaps

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Abstract Moment swaps are essentially forward contracts on realized higher moments of log-returns of a specified underlying asset, which play an important role in protection against different kinds of market shocks, and variance, skewness, and kurtosis swaps are examples of moment swaps currently traded in markets. To facilitate market practitioners, this work provides a simple and easy-to-use pricing formula of moment swaps on discrete sampling under the Black-Scholes model with time-dependent parameters. The formula is investigated for validity and compared with the fair delivery prices of moment swaps. Furthermore, a closed-form formula for hedging moment swaps on futures is deduced. Finally, Monte Carlo simulations are performed to support the accuracy of the pricing formula and numerical examples are provided to check the sensitivity of the parameters and relationships of calculated prices between moment swaps.

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1. Introduction

Moment swaps are essentially forward contracts on the realized higher moments of the log-returns of a specified underlying asset. More specifically, their payoff is a function of powers of the (daily) log-returns of the underlying asset at certain pre-specified discretely sampled points. According to recent studies by Schoutens [12] and Rompolis and Tzavalis

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[7], moment swaps play such an important role in financial markets to cover different kinds of market shocks. Speculators trade variance swaps (second order moment swaps) as an easy way to gain exposure to future levels of variance, and they may need to hedge against their portfolio volatility risk. Skewness swaps (third order moment swaps) provide protection against changes in the symmetry of the underlying distribution. Kurtosis swaps (fourth order moment swaps) provide protection against unexpected occurrences of very large jumps or changes in the tail behavior of the underlying distribution. These studies suggest that using variance and higher-moment swaps to hedge European options gives better performance compared with traditional delta hedging strategies. Therefore, it is meaningful to define and price higher-moment swaps to hedge the existing skewness and kurtosis risks.

As a result of the increasing trading activities of variance swaps, researchers have proposed various types of valuation approaches for pricing variance swaps defined either in terms of continuous sampling or discrete sampling; for example, see Zhu and Lian [16], [17], Rujivan and Zhu [11], [10], Zheng and Kwok [15] and Rujivan [9]. On the other hand, Schoutens [12] defined higher-moment swaps using daily log-returns for the realized moments, and claimed that moment swaps can protect against incorrectly estimated skewness or kurtosis without deriving an exact pricing formula for moment swaps. Recently, tremendous growth in the study of skewness and kurtosis risks has been witnessed, see Neuberger [6], Kozhan et al. [5], Zhao et al. [14], Rompolis and Tzavalis [7], and Zhang et al. [13], due to the launching of CBOE Skew Index (SKEW) to measure the skewness risk in the financial market by the Chicago Board Options Exchange (CBOE) in 2011.

In this paper, an analytical method are derive to price the discretely-sampled moment swaps introduced by Schoutens [12]. The study begins by considering a probability space $(\Omega, \mathcal{F}, Q)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a risk-neutral probability measure Q, for finding the conditional expectation of a random variable X with respect to a filtration $\mathcal{F}_t$, $E_t^Q[X]$. The dynamics of the underlying asset price S_t is assumed to follow the Black-Scholes (BS) model with time-dependent parameters, referred to as the extended Black-Scholes (EBS) model, described by

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t \tag{1.1}$$

where $r(t)$ is the time-dependent risk-free interest rate, $\sigma(t)$ is a deterministic positive function of time interpreting the volatility, and W_t is a one-dimensional Brownian motion. The assumption of the time-dependent parameters provides a more flexible model to describe the potential political or economic events which may occur. Kloeden and Platen [4] proposed the solution of the SDE (1.1) with initial price S_0 in the form

$$S_t = S_0 e^{\int_0^t \left(r(s) - \frac{\sigma^2(s)}{2}\right) ds + \int_0^t \sigma(s) dW_s}. \tag{1.2}$$

Schoutens [12] introduced the annualized realized m-moment, $m \geq 2$, in terms of discrete sampling over the contract life $[0, T]$ for a maturity time $T > 0$ on an underlying asset S_t as

$$MOMS_{\text{stock}}^{(m)} = N' \times \sum_{i=1}^{N} \ln^m \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)$$

where S_{t_i} are the closing prices of the underlying asset observed at times t_i, for $i = 0, 1, ..., N$, and N' is the nominal amount, $N' = \frac{AF}{N}$ when AF is the annualized factor for converting to annualized higher moments. If the sampling frequency is calculated daily, then $AF = 252$, assuming that there are 252 trading days in one year; if weekly, then $AF = 52$; and if monthly, then $AF = 12$. Typically, $T = \frac{N}{AF}$ with equally-spaced discrete observations $\Delta t = t_i - t_{i-1} > 0$, for $i = 1, 2, ..., N$. The annualized factor becomes $AF = \frac{N}{T} = \frac{1}{\Delta t}$, and the typical formula for the measure of realized m-moment is

$$MOMS_{\text{stock}}^{(m)} = \frac{1}{T} \sum_{i=1}^{N} \ln^m \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) = \frac{1}{T} \sum_{i=1}^{N} \left(X_{t_i} - X_{t_{i-1}} \right)^m \tag{1.3}$$

where $X_t := \ln S_t$, a log price process.

In a risk-neutral world, the value V_t of an m-moment swap at time t is the expected present value of the future payoff,

$$V_t = E_t^Q \left[e^{-\int_t^T r(s)ds} (MOMS_{\text{stock}}^{(m)} - K^m) L \right]$$

where K^m is the annualized delivery price for the m-moment swap and L is the notional amount of the swap. The value of V_0 is zero at the beginning of the contract since there is no cost entering into a forward contract, therefore, the fair delivery price of the m-moment swap is $K^m = E_0^Q[MOMS_{\text{stock}}^{(m)}]$. The valuation problem for an m-moment swap is reduced to calculating the conditional expectation of the realized m-moment (1.3) in the risk-neutral world.

The valuation of moment swaps on discrete sampling under the EBS model (1.1) is non-trivial, even though S_t is log-normally distributed, there is no analytical pricing formula for moment swaps available. It is the purpose of this study to provide market practitioners with a simple and easy-to-use pricing formula for moment swaps by deriving an analytical formula of the sum of the conditional expectations $E_0^Q \left[\left(X_{t_i} - X_{t_{i-1}} \right)^m \right], i = 1, ..., N$, in terms of X_t^k which satisfies a nonlinear SDE, for $k = 2, ..., m$.

The steps start with employing the Feynman-Kac theorem to derive solutions of partial differential equations (PDEs) as the the conditional expectations of X_t^k. The PDE is solved analytically using the method of reduction to produce a closed-form formula for the conditional expectations of X_t^k for all $k = 1, ..., m$. The formula is further simplified especially the sum of the conditional expectations, since the number of systems of ordinary differential equations (ODEs) that must be solved will be dramatically increased by the value of m. Fortunately, a simple and easy-to-use analytical formula for the conditional expectations is derived using combinatorial techniques for the fair delivery price of the m-moment swap for all positive integer $m \geq 2$.

The remaining contributions of this paper are followed. Utilizing the pricing formula for moment swaps, the fair delivery prices of any m-moment swap with the futures price as the underlying price is derived in closed form. We also discuss the validity of the solution in the parameter space of the EBS model. This discussion has a practical implication in the market, and practitioners must be aware to ensure that their model parameters, extracted from market data, are in the correct format when the analytical pricing formula is used to calculate the fair delivery price of a discretely-sampled moment swap. Additionally, a comparison result for the fair delivery prices of different moment swaps is obtained, showing that trading variance swaps are more expensive than trading any higher moment swaps.

The paper is organized into six sections. Section 2 presents the analytical approach for obtaining the conditional expectation of the realized m-moment (1.3) in closed-form formula, followed by the analytical formulas for the moment swaps with the stock price as the underlying. Section 3 provides discussion on the validity of the formula and a comparison result for the fair delivery prices of different moment swaps. In Section 4, hedging moment swaps on futures is presented. Section 5 provides some numerical examples to support the obtained results; the correctness of the closed-form formula is confirmed by Monte Carlo (MC) simulations and the comparison of the calculated fair prices of moment swaps. Moreover, the sensitivity for small changes of parameters is displayed in this section. A brief conclusion is finally provided in Section 6.

2. Analytical method for pricing moment swaps

In this section, an analytical formula for pricing discretely-sampled moment swaps under the EBS model (1.1) is derived by applying the method presented by Rujivan and Zhu [10].

2.1. Analytical formula for m-conditional moment

From (1.3), the expectation of $MOMS^{(m)}_{\text{stock}}$ with respect to $\mathcal{F}_0$ is

$$K^m = E_0^Q\left[MOMS^{(m)}_{\text{stock}}\right] = E_0^Q\left[\frac{1}{T}\sum_{i=1}^{N}\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]$$
$$= \frac{1}{T}\sum_{i=1}^{N} E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]. \tag{2.1}$$

Therefore, the problem of pricing moment swaps is reduced to calculating the conditional expectations

$$E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]. \tag{2.2}$$

Using the fact that $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ and $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable, the binomial theorem, and the tower property, the conditional expectation (2.2) becomes

$$E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right] = E_0^Q\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X_{t_{i-1}}^{m-k}E_{t_{i-1}}^Q\left[X_{t_i}^k\right]\right]. \tag{2.3}$$

The conditional expectations with respect to $\mathcal{F}_{t_{i-1}}$ on the right-hand side of (2.3), $E_{t_{i-1}}^Q\left[X_{t_i}^k\right]$, for $1 \le k \le m$, are computed by using the following theorem.

Theorem 2.1. *Suppose that $k \ge 2$ is an integer and S_t follows the EBS model in* (1.1)*. We set $X_t = \ln S_t$ and $\Delta t_i = t_i - t$ for all $i = 1, 2, \ldots, N$. If $r(t)$, $\sigma(t) > 0$ are integrable on $[t_{i-1}, t_i]$ in which $r(t) - \frac{1}{2}\sigma^2(t)$ is not a zero function on $[t_{i-1}, t_i]$ then*

$$E_{t_{i-1}}^Q[X_t^k] = E^Q[X_t^k \mid X_{t_{i-1}} = x] = \sum_{j=0}^{k} x^{k-j}A_j(\Delta t_i; t_i, k) \tag{2.4}$$

for all $t \in [t_{i-1}, t_i]$ *and* $x \in (-\infty, \infty)$*, where we define* $x^0 := 1$ *for all* $x \in (-\infty, \infty)$ *and*

$$A_0(\Delta t_i; t_i, k) = 1, \tag{2.5}$$

$$A_1(\Delta t_i; t_i, k) = k \int_0^{\Delta t_i} \left(r(t_i - \eta) - \frac{1}{2}\sigma^2(t_i - \eta) \right) d\eta, \tag{2.6}$$

$$\begin{aligned} A_j(\Delta t_i; t_i, k) &= (k - (j-1)) \int_0^{\Delta t_i} \left(r(t_i - \eta) - \frac{1}{2}\sigma^2(t_i - \eta) \right) A_{j-1}(\eta; t_i, k) d\eta \\ &\quad + \frac{1}{2} (k - (j-2)) (k - (j-1)) \int_0^{\Delta t_i} \sigma^2(t_i - \eta) A_{j-2}(\eta; t_i, k) d\eta \end{aligned} \tag{2.7}$$

for $j = 2, 3, \ldots, k$*.*

Proof. We first apply Itô's lemma to the transformation $Y_t = X_t^k$. Thus, Y_t follows the SDE

$$dY_t = \left[\left(r(t) - \frac{1}{2}\sigma^2(t) \right) kY_t^{1-\frac{1}{k}} + \frac{1}{2}k(k-1)\sigma^2(t)Y_t^{1-\frac{2}{k}} \right] dt + k\sigma(t)Y_t^{1-\frac{1}{k}} dW_t. \tag{2.8}$$

Consider a real-valued function defined by

$$U_i^{(k)}(y, t) := E^Q[Y_t \mid Y_{t_{i-1}} = y], \tag{2.9}$$

for all $(y, t) \in \mathbb{R} \times [t_{i-1}, t_i)$. Applying the Feynman-Kac formula to (2.8) and (2.9), we have that $U_i^{(k)}$ satisfies the PDE

$$\begin{aligned} &\frac{\partial U_i^{(k)}}{\partial t} + \left[\left(r(t) - \frac{1}{2}\sigma^2(t) \right) ky^{1-\frac{1}{k}} + \frac{1}{2}k(k-1)\sigma^2(t)y^{1-\frac{2}{k}} \right] \frac{\partial U_i^{(k)}}{\partial y} \\ &\quad + \frac{1}{2} \left[k\sigma(t)y^{1-\frac{1}{k}} \right]^2 \frac{\partial^2 U_i^{(k)}}{\partial y^2} = 0 \end{aligned} \tag{2.10}$$

subject to the terminal condition

$$U_i^{(k)}(y, t_i) = y \tag{2.11}$$

for all $(y, t) \in \mathbb{R} \times [t_{i-1}, t_i)$. Let $\tau = t_i - t$. Adopting the solution form of the PDE proposed in Rujivan [8], we solve the PDE (2.10) subject to the terminal condition (2.11) by assuming that the solution can be written in the form

$$U_i^{(k)}(y, t) = \sum_{j=0}^{k} y^{1-\frac{j}{k}} A_j(\tau; t_i, k) \tag{2.12}$$

where $A_j(\tau; t_i, k)$ is the function depend on τ, t_i and k for $j = 0, 1, \ldots, k$. Calculating all partial derivatives of $U_i^{(k)}$ in (2.10) by using the solution form (2.12) yields

$$\frac{\partial U_i^{(k)}}{\partial t} = -\left(\sum_{j=0}^{k} y^{1-\frac{j}{k}} \frac{dA_j}{d\tau}\right), \tag{2.13}$$

$$\frac{\partial U_i^{(k)}}{\partial y} = \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) y^{-\frac{j}{k}} A_j, \tag{2.14}$$

$$\frac{\partial^2 U_i^{(k)}}{\partial y^2} = \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right)\left(-\frac{j}{k}\right) y^{-\frac{j}{k}-1} A_j. \tag{2.15}$$

Inserting (2.13)–(2.15) into (2.10), we can derive a system of ODEs

$$\frac{dA_0}{d\tau} = 0, \tag{2.16}$$

$$\frac{dA_1}{d\tau} = k\left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau)\right) A_0, \tag{2.17}$$

$$\begin{aligned}\frac{dA_j}{d\tau} &= (k - (j-1))\left(r(t_i - \tau) - \frac{1}{2}\sigma^2(t_i - \tau)\right) A_{j-1} \\ &\quad + \frac{1}{2}(k - (j-2))(k - (j-1))\, \sigma^2(t_i - \tau) A_{j-2}\end{aligned} \tag{2.18}$$

for $j = 2, 3, ..., k$, subject to the initial conditions derived from the terminal condition (2.11) as

$$A_0(0; t_i, k) = 1 \quad \text{and} \quad A_j(0; t_i, k) = 0 \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{2.19}$$

The solution of (2.16)–(2.18) subject to the initial conditions (2.19) can be found as expressed in (2.5), (2.6), and (2.7), respectively. This completes the proof of the theorem. ■

In the case that $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function on $[t_{i-1}, t_i]$, an analytical formula for the conditional expectation (2.4) can also be obtained similarly to Theorem (2.1); however, in this case the coefficients A_j become zero when j is odd as described in the following result.

Corollary 2.2. *Suppose that $k \geq 2$ is an integer and S_t follows the EBS model in (1.1). We set $X_t = \ln S_t$ and $\Delta t_i = t_i - t$ for all $i = 1, 2, \ldots, N$. If $r(t)$, $\sigma(t) > 0$ are integrable on $[t_{i-1}, t_i]$ in which $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function on $[t_{i-1}, t_i]$ then*

$$E^Q_{t_{i-1}}[X_t^k] = E^Q[X_t^k \mid X_{t_{i-1}} = x] = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} x^{k-2j} A_{2j}(\triangle t_i; t_i, k)$$

for all $t \in [t_{i-1}, t_i]$ *and* $x \in (-\infty, \infty)$*, where we define* $x^0 := 1$ *for all* $x \in (-\infty, \infty)$ *and*

$$
\begin{aligned}
A_0(\triangle t_i; t_i, k) &= 1, \\
A_2(\triangle t_i; t_i, k) &= \frac{1}{2} k(k-1) \int_0^{\triangle t_i} \sigma^2(t_i - \eta) d\eta, \\
A_{2j}(\triangle t_i; t_i, k) &= \frac{1}{2^j} \left(\prod_{r=0}^{2j-1} (k-r) \right) \\
&\qquad \int_0^{\triangle t_i} \int_0^{\eta_j} \cdots \int_0^{\eta_2} \sigma^2(t_i - \eta_1) \cdots \sigma^2(t_i - \eta_j) d\eta_1 \cdots d\eta_j
\end{aligned}
$$

for $j = 2, 3, \ldots, \lfloor \frac{k}{2} \rfloor$.

Proof. Since $r(t) - \frac{1}{2}\sigma^2(t)$ is a zero function $[t_{i-1}, t_i]$, we can reduce $A_1(\triangle t_i; t_i, k)$ and $A_j(\triangle t_i; t_i, k)$ defined as (2.6) and (2.7) to the form

$$
\begin{aligned}
&A_j(\triangle t_i; t_i, k) \\
&= \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \dfrac{1}{2} \left(k - (j-2)\right) \left(k - (j-1)\right) \displaystyle\int_0^{\triangle t_i} \sigma^2(t_i - \eta) A_{j-2}(\eta; t_i, k) d\eta & \text{if } j \text{ is even} \end{cases}
\end{aligned}
$$

for $j = 1, 2, \ldots, k$. This proof is complete. ∎

2.2. Analytical pricing formula for m-moment swap under EBS model

The following lemma will be used to derive the fair delivery price of moment swaps under the EBS model (1.1).

Lemma 2.3. *Let* $\tau, \zeta \in \mathbb{R}$ *and* $j \in \mathbb{N}$*. Then,*

$$
A_j(\tau; \zeta, k_1) = \frac{k_1!}{(k_1 - j)!} \frac{(k_2 - j)!}{k_2!} A_j(\tau; \zeta, k_2) \tag{2.20}
$$

for all $k_1, k_2 \in \{j, j+1, \ldots\}$.

Proof. We shall prove the lemma by using the strong induction principle. It easy to show that (2.20) holds for $j = 1, 2$. Let $n \in \mathbb{N}$. We assume that (2.20) holds for $j = 1, 2, \ldots, n$. From (2.7), we write

$$
A_{n+1}(\tau; \zeta, k_1) = A'_{n+1}(\tau; \zeta, k_1) + A''_{n+1}(\tau; \zeta, k_1),
$$

where

$$
\begin{aligned}
A'_{n+1}(\tau; \zeta, k_1) &= (k_1 - n) \int_0^{\tau} \left(r(\zeta - \eta) - \frac{1}{2}\sigma^2(\zeta - \eta) \right) A_n(\eta; \zeta, k_1) d\eta, \\
A''_{n+1}(\tau; \zeta, k_1) &= \frac{1}{2}(k_1 - (n-1))(k_1 - n) \int_0^{\tau} \sigma^2(\zeta - \eta) A_{n-1}(\eta; \zeta, k_1) d\eta.
\end{aligned}
$$

By the hypothesis for $k_1, k_2 \geq n$, using (2.20) with $j = n$ and $j = n-1$ gives

$$A'_{n+1}(\tau;\zeta,k_1) = \frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}(k_2-n) \int_0^\tau \left(r(\zeta-\eta) - \frac{1}{2}\sigma^2(\zeta-\eta)\right) A_n(\eta;\zeta,k_2)d\eta,$$

$$A''_{n+1}(\tau;\zeta,k_1) = \frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}\frac{1}{2}(k_2-(n-1))(k_2-n) \int_0^\tau \sigma^2(\zeta-\eta)A_{n-1}(\eta;\zeta,k_2)d\eta,$$

respectively. Therefore, from (2.7),

$$A_{n+1}(\tau;\zeta,k_1) = \frac{k_1!}{(k_1-(n+1))!}\frac{(k_2-(n+1))!}{k_2!}A_{n+1}(\tau;\zeta,k_2).$$

This show that (2.20) holds for $j = n+1$, hence, it is true for all $j \in \mathbb{N}$. ■

In the following theorem, we derive the fair delivery price of the m-moment swap under the EBS model (1.1) by utilizing Theorem 2.1 and Lemma 2.3.

Theorem 2.4. *Suppose that S_t follows the EBS model (1.1) and $m \geq 2$ is an integer. Then, the fair delivery price of the m-moment swap under the EBS model (1.1), denoted by K^m_{EBS}, can be expressed as*

$$K^m_{\text{EBS}}(T,N) = \frac{1}{T}\sum_{i=1}^{N} A_m(\Delta t; t_i, m) \tag{2.21}$$

where $\Delta t = \frac{T}{N}, t_i = i\Delta t, i = 0,1,...,N,$ and $A_m(\Delta t;t_i,m)$ are defined in (2.5)–(2.7).

Proof. From (2.1) and (2.3), we have

$$\begin{aligned} K^m_{\text{EBS}} &= \frac{1}{T}\sum_{i=1}^{N} E^Q_0\left[(X_{t_i} - X_{t_{i-1}})^m\right] \\ &= \frac{1}{T}\sum_{i=1}^{N} E^Q_0\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X^{m-k}_{t_{i-1}}E^Q_{t_{i-1}}\left[X^k_{t_i}\right]\right]. \end{aligned} \tag{2.22}$$

Utilizing Theorem 2.1, the conditional expectations with respect to $\mathcal{F}_{t_{i-1}}$ on the right-hand side of (2.22) can be written as

$$E^Q_{t_{i-1}}\left[X^k_{t_i}\right] = \sum_{j=0}^{k} A_j(\Delta t;t_i,k)X^{k-j}_{t_{i-1}} \tag{2.23}$$

where $A_j(\Delta t;t_i,k), j = 0,1,...,k$, are defined in (2.5)–(2.7). This implies

$$\begin{aligned} &\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X^{m-k}_{t_{i-1}}E^Q_{t_{i-1}}\left[X^k_{t_i}\right] \\ &\quad = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X^{m-k}_{t_{i-1}}\sum_{j=0}^{k}A_j(\Delta t;t_i,k)X^{k-j}_{t_{i-1}}. \end{aligned} \tag{2.24}$$

Next, we rearrange the terms in the summations on the right-hand side of (2.24) to obtain

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}X_{t_{i-1}}^{m-k}\sum_{j=0}^{k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{k-j}$$
$$=\sum_{j=0}^{m}\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{m-j}. \qquad (2.25)$$

Applying Lemma 2.3 to $A_j(\Delta t;t_i,k)$ gives us

$$A_j(\Delta t;t_i,k)=\frac{k!}{(k-j)!j!}A_j(\Delta t;t_i,j). \qquad (2.26)$$

Inserting (2.26) into (2.25), we arrive

$$\sum_{j=0}^{m}\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}A_j(\Delta t;t_i,k)X_{t_{i-1}}^{m-j}$$
$$=\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}\right)\frac{1}{j!}A_j(\Delta t;t_i,j)X_{t_{i-1}}^{m-j}. \qquad (2.27)$$

The following identity is useful to reduce the summation terms on the right-hand side of (2.27)

$$\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}=0 \qquad \text{for} \quad 0\le j<m. \qquad (2.28)$$

Applying the identity (2.28) to the right-hand side of (2.27) gives us

$$\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\binom{m}{k}(-1)^{m-k}\frac{k!}{(k-j)!}\right)\frac{1}{j!}A_j(\Delta t;t_i,j)X_{t_{i-1}}^{m-j}=A_m(\Delta t;t_i,m). \qquad (2.29)$$

Utilizing (2.24)-(2.29), we have that the conditional expectation $E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]$ can be simplified to a simple form as

$$E_0^Q\left[\left(X_{t_i}-X_{t_{i-1}}\right)^m\right]=E_0^Q\left[A_m(\Delta t;t_i,m)\right]=A_m(\Delta t;t_i,m). \qquad (2.30)$$

We insert (2.30) into the right-hand side of (2.22) to complete the proof. ■

2.3. Analytical pricing formula for m-moment swap under BS model

Next, we consider the BS model described by the SDE as

$$dS_t=rS_tdt+\sigma S_tdW_t \qquad (2.31)$$

where r and $\sigma>0$ are constants. From Theorem 2.1, the ODEs (2.16)-(2.18) subject to the initial conditions (2.19) can be solved analytically as proposed in the following lemma.

Lemma 2.5. *Suppose that S_t follows the BS model (2.31) such that $r\neq\frac{1}{2}\sigma^2$ and $k\ge 1$ is an integer. Then, the solutions of ODEs (2.16)-(2.18) subject to the initial conditions (2.19) can be expressed as*

$$A_j(\tau;t_i,k)=\frac{k!}{(k-j)!}\sum_{n=0}^{\lfloor\frac{j}{2}\rfloor}\frac{\sigma^{2n}}{2^n n!(j-2n)!}\left(r-\frac{1}{2}\sigma^2\right)^{j-2n}\tau^{j-n} \qquad (2.32)$$

for $\tau \geq 0$ and $j = 0, 1, ..., k$.

Proof. following the same proof of Theorem2.1 that $A_j(\tau; t_i, k)$, it is easy to see that (2.32) satisfies the ODEs (2.16)-(2.18) subject to the initial conditions (2.19) when the parameter functions are constant. ∎

Applying Theorem 2.4 and Lemma 2.5, the fair delivery price of moment swaps under the BS model (2.31) can be deduced as follows.

Theorem 2.6. *Suppose that S_t follows the BS model (2.31) such that $r \neq \frac{1}{2}\sigma^2$ and $m \geq 2$ is an integer. Then, the fair delivery price of the m-moment swap under the BS model (2.31), denoted by K^m_{BS}, can be expressed as*

$$K^m_{\mathrm{BS}}(T, N) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{2^n n!(m-2n)!} \sigma^{2n} \left(r - \frac{1}{2}\sigma^2\right)^{m-2n} (\Delta t)^{m-n-1} \tag{2.33}$$

where $\Delta t = \frac{T}{N}$. In particular, the fair delivery prices of variance, skewness, and kurtosis swaps under the BS model (2.31) can be expressed as

$$K^2_{\mathrm{BS}}(T, N) = \left(r - \frac{\sigma^2}{2}\right)^2 \frac{T}{N} + \sigma^2, \tag{2.34}$$

$$K^3_{\mathrm{BS}}(T, N) = \left(r - \frac{\sigma^2}{2}\right)^3 \frac{T^2}{N^2} + 3\sigma^2 \left(r - \frac{\sigma^2}{2}\right) \frac{T}{N}, \tag{2.35}$$

$$K^4_{\mathrm{BS}}(T, N) = \left(r - \frac{\sigma^2}{2}\right)^4 \frac{T^3}{N^3} + 6\sigma^2 \left(r - \frac{\sigma^2}{2}\right)^2 \frac{T^2}{N^2} + 3\sigma^4 \frac{T}{N}, \tag{2.36}$$

respectively.

Proof. Utilizing (2.32) in Lemma 2.5, we have

$$A_m(\Delta t; t_i, m) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\sigma^{2n}}{2^n n!(m-2n)!} \left(r - \frac{1}{2}\sigma^2\right)^{m-2n} (\Delta t)^{m-n}. \tag{2.37}$$

Inserting (2.37) into the right-hand side of (2.21), we immediately obtain (2.33). ∎

3. Validity of the Solution

3.1. Positivity of the Solution

The construction of the formula for pricing moment swaps under the EBS model (2.21) presents some interesting discussions in terms of the validity of the solution. The purpose of such an examination is to ensure that one of the fundamental assumptions that the fair delivery price of a moment swap is finite and strictly positive for a given set of parameters determined from market data.

Theorem 3.1. *According to Theorem 2.4, if the parameter functions $r(t)$, $\sigma(t) > 0$ are integrable on $[0, T]$ and satisfy*

$$\int_{t_{i-1}}^{t_i} \left(r(t) - \frac{1}{2}\sigma^2(t)\right) dt > 0 \tag{3.1}$$

for all $i = 1, ..., N$. *Then,*

$$0 < K^m_{\mathrm{EBS}}(T, N) < \infty \tag{3.2}$$

for all integer $m \geq 2$. *In particular, if* m *is even then the inequality* (3.2) *holds without assuming the condition* (3.1).

Proof. The integrability of $r(t) - \frac{1}{2}\sigma^2(t)$ on $[0, T]$ implies that the coefficient functions $A_j(t; t_i, m), i = 1, ..., N$, and $j = 1, 2, .., m$, which can be computed by using (2.6)-(2.7), are bounded on $[0, T]$ and so is $K^m_{\mathrm{EBS}}(T, N)$. In addition the condition (3.1) implies $A_j(t; t_i, m) > 0$ for all $t \in [t_{i-1}, t_i], i = 1, ..., N$, and $j = 1, 2, .., m$. Hence, we immediately obtain that $0 < K^m_{\mathrm{EBS}}(T, N)$. Moreover, when m is even, the RHS of (2.7) is always positive without assuming the condition (3.1). This result yields (3.2). ∎

3.2. Comparisons of fair delivery prices of moment swaps

This section provides a comparison theorem for the fair delivery prices of different moment swaps under the BS model. The following theorem demonstrates that trading variance swaps are more expensive than trading any higher moment swaps.

Theorem 3.2. *According to Theorem 2.6 and 3.1, we suppose that* $r > \frac{1}{2}\sigma^2$ *and* m, n *are integers such that* $2 \leq n < m - 1$. *Then,*

$$K^m_{\mathrm{BS}}(T, N) < K^n_{\mathrm{BS}}(T, N) \tag{3.3}$$

for $\frac{T}{N} \in (0, \tau^*_{m,n})$ *where* $\tau^*_{m,n}$ *is the smallest positive root of a polynomial function of degree* $m - n + \left\lfloor \frac{n}{2} \right\rfloor$ *with respect to* τ *defined by*

$$P_{m-n+\left\lfloor \frac{n}{2} \right\rfloor}(s) := \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} C_{m,j} s^{m-(n+j)+\left\lfloor \frac{n}{2} \right\rfloor} - \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} C_{n,j} s^{\left\lfloor \frac{n}{2} \right\rfloor - j} \tag{3.4}$$

$$C_{l,j} := \frac{l!}{2^j j!(l-2j)!} \sigma^{2j} \left(r - \frac{1}{2}\sigma^2 \right)^{l-2j} \tag{3.5}$$

for $l = m, n$. *In particular,*

$$K^m_{\mathrm{BS}}(T, N) = K^n_{\mathrm{BS}}(T, N) \tag{3.6}$$

when $\frac{T}{N} = \tau^*_{m,n}$.

Proof. Using (2.33), one can derive the following relation

$$K^m_{\mathrm{BS}}(T, N) - K^n_{\mathrm{BS}}(T, N) = \tau^{n - \left\lfloor \frac{n}{2} \right\rfloor - 1} P_{m-n+\left\lfloor \frac{n}{2} \right\rfloor}(\tau) \tag{3.7}$$

for $\tau = \frac{T}{N}$. To obtain (3.3), we shall show that $\lim_{s \to 0^+} P_{m-n+\left\lfloor \frac{n}{2} \right\rfloor}(s) < 0$. Since $2 \leq n < m - 1$ and $r > \frac{1}{2}\sigma^2$, the limit can be deduced from (3.4) and (3.5) that

$$\lim_{s \to 0^+} P_{m-n+\left\lfloor \frac{n}{2} \right\rfloor}(s) = -C_{n,\left\lfloor \frac{n}{2} \right\rfloor} = -\frac{n!}{2^{\left\lfloor \frac{n}{2} \right\rfloor} \left\lfloor \frac{n}{2} \right\rfloor! \left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)!} \sigma^{2\left\lfloor \frac{n}{2} \right\rfloor} \left(r - \frac{1}{2}\sigma^2 \right)^{n - 2\left\lfloor \frac{n}{2} \right\rfloor} < 0. \tag{3.8}$$

Next, we consider the coefficient of $s^{m-(n+j)+\lfloor \frac{n}{2} \rfloor}$ for $j=0$ in (3.4). We note from (3.5) that $C_{m,0} = (r - \frac{1}{2}\sigma^2)^m > 0$ and this implies

$$\lim_{s\to\infty} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) = \infty. \tag{3.9}$$

From (3.8) and (3.9), we immediately obtain that $P_{m-n+\lfloor \frac{n}{2} \rfloor}(s)$ has at least one positive root. We let $\tau^*_{m,n}$ be the smallest positive root. Therefore, (3.3) and (3.6) hold for $\frac{T}{N} \in (0, \tau^*_{m,n})$ and $\frac{T}{N} = \tau^*_{m,n}$, respectively. ■

Corollary 3.3. *According to Theorem 3.2, if $r > \frac{1}{2}\sigma^2$ then (3.3) and (3.6) hold for all integers m, n such that m is odd and $2 \leq n < m$.*

Proof. The proof is complete following the fact that when m is odd, (3.8) and (3.9) hold for $2 \leq n < m$. ■

Corollary 3.4. *According to Theorem 3.2, if $r > \frac{3}{2}\sigma^2$ then (3.3) and (3.6) hold for all integers m, n such that $2 \leq n < m$.*

Proof. Since $r > \frac{3}{2}\sigma^2 > \frac{1}{2}\sigma^2$. Thus, we have the following facts: (i) (3.8) and (3.9) hold for $2 \leq n < m-1$ and (ii) (3.8) and (3.9) hold for m is odd and $2 \leq n < m$. Next, we consider $\lim_{s\to 0^+} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s)$ under the case that m is even and $n = m-1$. The limit can be deduced from (3.4) and (3.5) that

$$\begin{aligned}
\lim_{s\to 0^+} P_{m-n+\lfloor \frac{n}{2} \rfloor}(s) &= C_{m,\lfloor \frac{m}{2} \rfloor} - C_{n,\lfloor \frac{n}{2} \rfloor} \\
&= C_{2h,h} - C_{2h-1,h-1} \\
&= -\frac{(2h)!}{2^h h!}\sigma^{2h-2}\left(r - \frac{3}{2}\sigma^2\right) \\
&< 0
\end{aligned} \tag{3.10}$$

where $m = 2h$ for some positive integer h. Using (3.9) and (3.10), we now obtain (3.3) and (3.6) for $\frac{T}{N} \in (0, \tau^*_{m,n})$ and $\frac{T}{N} = \tau^*_{m,n}$, respectively. ■

4. Hedging moment swaps

The present section considers moment swaps with the futures price as the underlying price. We assume that futures which expire at time T are available on the underlying asset where the price S_t follows the EBS model (1.1). Under the risk-neutral valuation, the price process of the futures is given by

$$F_t = E_t^Q[S_T] = S_t e^{\int_t^T r(s)ds} \tag{4.1}$$

for $0 \leq t \leq T$. According to Schoutens [12] and (4.1), we introduce the realized m-moment on futures as

$$MOMS^{(m)}_{\text{futures}} = \frac{1}{T}\sum_{i=1}^{N} \ln^m\left(\frac{F_{t_i}}{F_{t_{i-1}}}\right) = \frac{1}{T}\sum_{i=1}^{N}\left((X_{t_i} - X_{t_{i-1}}) + R_i\right)^m \tag{4.2}$$

where we define

$$R_i := -\int_{t_{i-1}}^{t_i} r(s)ds. \tag{4.3}$$

In the following theorem, we demonstrate that the fair delivery price of a moment swap on futures can be expressed in terms of a linear combination of the fair delivery prices of moment swaps on its corresponding underlying stock.

Theorem 4.1. *Suppose that S_t follows the EBS model (1.1) and $m \geq 1$ is a positive integer. Then, the fair delivery price of the m-moment swap on futures which expire at time T under the EBS model (1.1), defined by*

$$F_{\text{EBS}}^m := E_0^Q\left[MOMS_{\text{futures}}^{(m)}\right] \tag{4.4}$$

can be expressed as

$$\begin{aligned} &F_{\text{EBS}}^m(T,N) \\ &= \sum_{n=0}^{m}\binom{m}{n}K_{\text{EBS}}^{m-n}(T,N) + \sum_{n=1}^{m}\binom{m}{n}\sum_{j=1}^{n}\binom{n}{j}\frac{1}{T}\sum_{i=1}^{N}(-M_i)^j A_{m-n}(\Delta t; t_i, m-n) \end{aligned} \tag{4.5}$$

where we define $K_{\text{EBS}}^0(T,N) := 1$ and $M_i := 1 - R_i = 1 + \int_{t_{i-1}}^{t_i} r(s)ds$, and let $\Delta t = \frac{T}{N}, t_i = i\Delta t, i = 0, 1, ..., N$, and $A_{m-n}(\Delta t; t_i, m-n)$ can be computed using (2.5)–(2.7).

Proof. From (4.2)-(4.4), we apply the Binomial theorem to obtain

$$\begin{aligned} F_{\text{EBS}}^m &= \frac{1}{T}\sum_{i=1}^{N}E_0^Q\left[((X_{t_i} - X_{t_{i-1}}) + R_i)^m\right] \\ &= \frac{1}{T}\sum_{i=1}^{N}\sum_{n=0}^{m}\binom{m}{n}(R_i)^n E_0^Q\left[(X_{t_i} - X_{t_{i-1}})^{m-n}\right] \\ &= \frac{1}{T}\sum_{i=1}^{N}\sum_{n=0}^{m}\binom{m}{n}(1-M_i)^n E_0^Q\left[(X_{t_i} - X_{t_{i-1}})^{m-n}\right] \\ &= \frac{1}{T}\sum_{i=1}^{N}\sum_{n=0}^{m}\binom{m}{n}\left(1 + \sum_{j=1}^{n}\binom{n}{j}(-M_i)^j\right)E_0^Q\left[(X_{t_i} - X_{t_{i-1}})^{m-n}\right]. \end{aligned} \tag{4.6}$$

Using (2.30) gives $E_0^Q\left[(X_{t_i} - X_{t_{i-1}})^{m-n}\right] = A_{m-n}(\Delta t; t_i, m-n)$ and

$$\begin{aligned} F_{\text{EBS}}^m &= \frac{1}{T}\sum_{i=1}^{N}\sum_{n=0}^{m}\binom{m}{n}\left(1 + \sum_{j=1}^{n}\binom{n}{j}(-M_i)^j\right)A_{m-n}(\Delta t; t_i, m-n) \\ &= \sum_{n=0}^{m}\binom{m}{n}\frac{1}{T}\sum_{i=1}^{N}A_{m-n}(\Delta t; t_i, m-n) \\ &\quad + \sum_{n=1}^{m}\binom{m}{n}\sum_{j=1}^{n}\binom{n}{j}\frac{1}{T}\sum_{i=1}^{N}(-M_i)^j A_{m-n}(\Delta t; t_i, m-n). \end{aligned} \tag{4.7}$$

Utilizing (2.21) in Theorem 2.4, we replace $\frac{1}{T}\sum_{i=1}^{N}A_{m-n}(\Delta t; t_i, m-n)$ in (4.7) by $K_{\text{EBS}}^{m-n}(T,N)$ to obtain (4.5). ∎

Next, we consider the fair delivery price of the m-moment swap on futures when r and $\sigma > 0$ are constants.

Corollary 4.2. *Suppose that S_t follows the BS model (2.31) and $m \geq 1$ is a positive integer. Then, the fair delivery price of the m-moment swap on futures which expire at time T under the BS model (2.31), defined by*

$$F_{\mathrm{BS}}^{m} := E_0^Q \left[MOMS_{\text{futures}}^{(m)} \right]$$

can be expressed as

$$F_{\mathrm{BS}}^{m}(T,N) = K_{\mathrm{BS}}^{m}(T,N) + \sum_{n=1}^{m} \binom{m}{n} (-r\Delta t)^n K_{\mathrm{BS}}^{m-n}(T,N) \tag{4.8}$$

where we define $K_{\mathrm{BS}}^{0}(T,N) := 1$.

Proof. Under the BS model (2.31), we have $R_i = -r\Delta t$ in (4.3) and

$$\begin{aligned} F_{\mathrm{BS}}^{m} &= \frac{1}{T} \sum_{i=1}^{N} \sum_{n=0}^{m} \binom{m}{n} (R_i)^n E_0^Q \left[\left(X_{t_i} - X_{t_{i-1}} \right)^{m-n} \right] \\ &= \sum_{n=0}^{m} \binom{m}{n} (-r\Delta t)^n \frac{1}{T} \sum_{i=1}^{N} A_{m-n}(\Delta t; t_i, m-n). \end{aligned} \tag{4.9}$$

The proof is complete by replacing $\frac{1}{T}\sum_{i=1}^{N} A_{m-n}(\Delta t; t_i, m-n)$ in (4.9) by $K_{\mathrm{BS}}^{m-n}(T,N)$ to obtain (4.8). ■

The following theorem is very useful for hedging moment swaps on futures.

Theorem 4.3. *According to Theorem 4.1, the fair delivery price of the m-moment swap on futures which expire at time T under the EBS model (1.1) can be approximated as*

$$F_{\mathrm{EBS}}^{m}(T,N,m) = \sum_{j=1, j\neq m}^{m+1} \left(-\tfrac{m!}{j!} \right) F_{\mathrm{EBS}}^{j}(T,N,j) + R_{\mathrm{EBS}}^{(m)}(T,N) \tag{4.10}$$

for $m \geq 1$, where we define the remainder term as

$$R_{\mathrm{EBS}}^{(m)}(T,N) := -\sum_{j=m+2}^{\infty} \left(\tfrac{m!}{j!} \right) \frac{1}{T} \sum_{i=1}^{N} \sum_{n=0}^{j} \binom{j}{n} (-1)^{j-n} e^{\frac{1}{2}(n-1)n \int_{t_{i-1}}^{t_i} \sigma^2(s) ds}. \tag{4.11}$$

Moreover, $\lim_{N\to\infty} R_{\mathrm{EBS}}^{(m)}(T,N) = 0$ for $m \geq 2$.

Proof. We first consider the power series representation of the exponential function as

$$e^y = 1 + \sum_{j=1}^{\infty} \frac{y^j}{j!} \tag{4.12}$$

for $y \in \mathbb{R}$. Substituting y in (4.12) by $\ln\left(\frac{F_{t_i}}{F_{t_{i-1}}}\right)$ leads to

$$\left(\frac{F_{t_i} - F_{t_{i-1}}}{F_{t_{i-1}}} \right) = \sum_{j=1}^{m+1} \frac{1}{j!} \ln^j \left(\frac{F_{t_i}}{F_{t_{i-1}}} \right) + \sum_{j=m+2}^{\infty} \frac{1}{j!} \left(\frac{F_{t_i} - F_{t_{i-1}}}{F_{t_{i-1}}} \right)^j \tag{4.13}$$

where we approximate $\ln^j\left(\frac{F_{t_i}}{F_{t_{i-1}}}\right)$ by $\left(\frac{F_{t_i}-F_{t_{i-1}}}{F_{t_{i-1}}}\right)^j$ for $j \geq m+2$. From (4.2) and (4.13),

$$\frac{1}{T}\sum_{i=1}^{N}\left(\frac{F_{t_i}-F_{t_{i-1}}}{F_{t_{i-1}}}\right) = \sum_{j=1}^{m+1}\tfrac{1}{j!}MOMS^{(j)}_{\text{futures}} + \frac{1}{T}\sum_{i=1}^{N}\sum_{j=m+2}^{\infty}\frac{1}{j!}\left(\frac{F_{t_i}-F_{t_{i-1}}}{F_{t_{i-1}}}\right)^j. \tag{4.14}$$

Applying the binomial theorem to give

$$\left(\frac{F_{t_i}-F_{t_{i-1}}}{F_{t_{i-1}}}\right)^j = \left(\frac{S_{t_i}}{S_{t_{i-1}}}e^{R_i}-1\right)^j = \sum_{n=0}^{j}\binom{j}{n}(-1)^{j-n}e^{nR_i}\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^n. \tag{4.15}$$

From the solution of the SDE (1.1) as written in (1.2), we have

$$\begin{aligned}\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^n &= e^{n\int_{t_{i-1}}^{t_i}\left(r(s)-\frac{1}{2}\sigma^2(s)\right)ds+n\int_{t_{i-1}}^{t_i}\sigma(s)dW_s} \\ &= e^{\int_{t_{i-1}}^{t_i}\left(nr(s)+\frac{1}{2}(n-1)n\sigma^2(s)\right)ds}\times e^{M_{t_i}-\frac{1}{2}\langle M\rangle_{t_i}}\end{aligned} \tag{4.16}$$

where $M_t := n\int_{t_{i-1}}^{t}\sigma(s)dW_s$ and its quadratic variation $\langle M\rangle_t := n^2\int_{t_{i-1}}^{t}\sigma^2(s)ds$. Applying the proposition proposed by Karatzas and Shreve [2] (see on page 198), we have

$$E_0^Q\left[e^{M_{t_i}-\frac{1}{2}\langle M\rangle_{t_i}}\right] = E_0^Q\left[E_{t_{i-1}}^Q\left[e^{M_{t_i}-\frac{1}{2}\langle M\rangle_{t_i}}\right]\right] = E_0^Q[1] = 1. \tag{4.17}$$

Applying (4.17) to (4.16) and (4.15), respectively, leads to

$$E_0^Q\left[\left(\frac{F_{t_i}-F_{t_{i-1}}}{F_{t_{i-1}}}\right)^j\right] = \sum_{n=0}^{j}\binom{j}{n}(-1)^{j-n}e^{\frac{1}{2}(n-1)n\int_{t_{i-1}}^{t_i}\sigma^2(s)ds} \tag{4.18}$$

for $j \in \mathbb{N}$. Taking the conditional expectation with respect to $\mathcal{F}_0$ to both side of (4.14) and using (4.18) give (4.10) and (4.11). To show that $\lim_{N\to\infty} R^{(m)}_{\text{EBS}}(T,N) = 0$, we set $t_i - t_{i-1} = \frac{T}{N}$ for all $i = 1, ..., N$. From (4.11), we have $e^{\frac{1}{2}(n-1)n\int_{t_{i-1}}^{t_i}\sigma^2(s)ds} = e^{\frac{1}{2}(n-1)n\int_0^{\frac{T}{N}}\sigma^2(t_i-s)ds}$ converges to 1 for large N. Using this result with the identity $\sum_{n=0}^{j}\binom{j}{n}(-1)^{j-n} = 0$ for $j \geq 1$, the limit is obtained as desired. ■

5. Numerical results

For the purpose of demonstrating the accuracy of the closed-form formulas (2.21) and (2.33), we present some numerical examples in this section. We compare the results obtained from the formulas and those from MC simulations. Although theoretically there would be no need to discuss the accuracy of the closed-form formulas and present numerical results, some comparisons with the MC simulations may give readers a sense of verification for the newly found solutions. This is particularly so for some market practitioners who are very used to MC simulations and would not trust analytical solutions that may contain algebraic errors unless they have seen numerical evidence of such a comparison.

In the MC simulations, we consider the dynamics of the log price process $X_t = \ln S_t$ derived by using (2.8) with $k = 1$ as

$$dX_t = \left(r(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t) dW_t. \tag{5.1}$$

We employ the simple Euler-Maruyama discretization for the log price process (5.1) on the time interval $[0, T]$ as

$$X_{t_j}(\omega) = X_{t_{j-1}}(\omega) + \left(r(t_j) - \frac{1}{2}\sigma^2(t_j) \right) \overline{\Delta t} + \sigma(t_j)\sqrt{\overline{\Delta t}} Z_{t_j}(\omega), \tag{5.2}$$

for $\omega \in \Omega, \overline{\Delta t} = \frac{T}{M_e}$ and $t_j = j\overline{\Delta t}, j = 0, 1, ..., M_e$ where M_e is a positive integer representing the number of time steps used in the discretization and Z_{t_j} is the standard normal random variable. For simplicity, we set $M_e = N$ and this gives us the approximate of X_{t_i} at the observation time $t_i, i = 1, 2, ..., N$, used to compute the realized m-moment defined in (1.3).

Next, we introduce an approximate of $K^m_{\text{EBS}}(T, N)$ obtained by MC simulations as

$$K^m_{\text{MC}}(T, N; N_p) := \frac{\sum_{p=1}^{N_p} \left(\frac{1}{T} \sum_{i=1}^{N} \left(X_{t_i}(\omega_p) - X_{t_{i-1}}(\omega_p) \right)^m \right)}{N_p}, \tag{5.3}$$

for $\omega_p \in \Omega$ and $p = 1, 2, ..., N_p$ where N_p is the number of sample paths used in MC simulations. Moreover, we shall construct a standard method in order to measure the level of accuracy of the closed-form formulas (2.21) and (5.3). Define the percentage relative error (ε^m) from using MC simulations by

$$\varepsilon^m(T, N; N_p) := \left| \frac{K^m_{\text{EBS}}(T, N) - K^m_{MC}(T, N; N_p)}{K^m_{\text{EBS}}(T, N)} \right| \times 100\%.$$

The presented numerical examples are performed on a quad-processor Intel Core i7 3.4 GHz with 32 GB of main memory using Mathematica V9.0 under Microsoft Windows 10 64-bit.

Example 5.1. (Comparison to MC simulations) In this example, we confirm the closed-form formula (2.21) by comparing with MC simulations. The parameters used in the experiment are $N = 252$, and for various $T = 0.1, 0.2, \ldots, 1.0$. The testing is taken on the EBS with the parameter functions $r(t) = 0.075 + 0.05t$ and $\sigma(t) = \sqrt{0.03 + 0.02t}$ satisfying the condition (3.1). The comparisons for $m = 2, 3, 4$ as displayed in Figure 1.

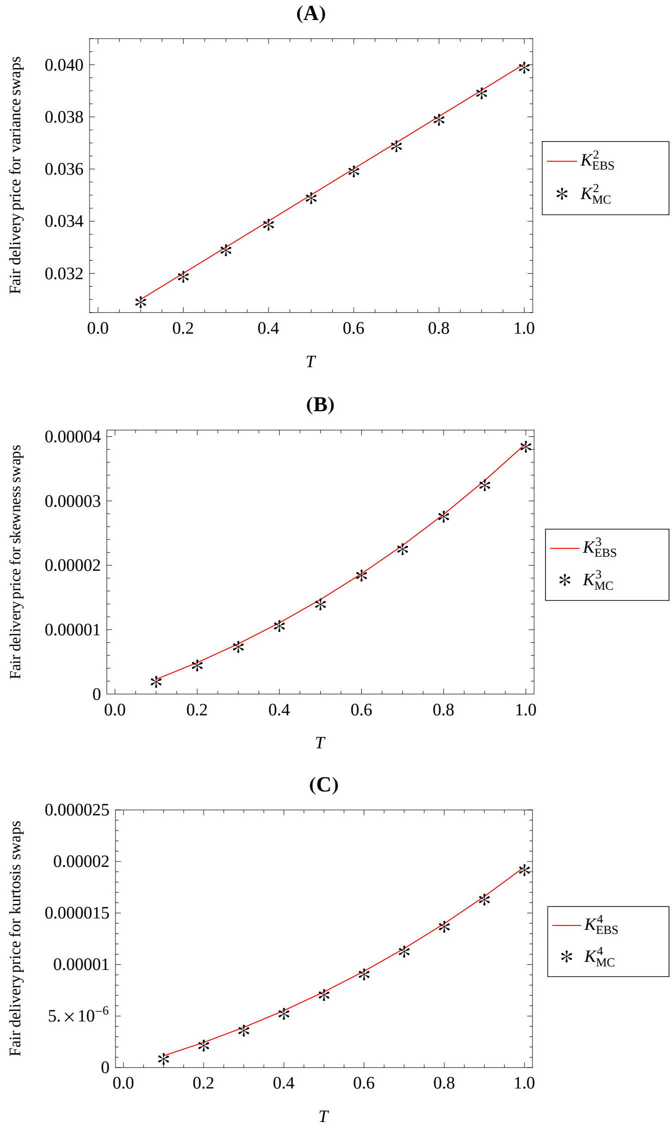

FIGURE 1. Comparisons of fair delivery prices from the closed-form solution K^m_{EBS} and the MC simulations for pricing K^m_{MC}: (A) variance swaps, (B) skewness swaps, and (C) kurtosis swaps

Figure 1 shows that the results from the closed-form solution and the MC simulations perfectly match, illustrating that the closed-form formula does not contain any algebraic errors and practitioners can confidently use the formula for pricing moment swaps.

In addition to the comparisons in Figure 1, the levels of accuracy, measured in terms of ε^m, is shown in the Table 1 for $N_p = 10,000, 30,000, 50,000$, and $T = 1$.

m^{th} moment	K^m_{MC} N_p	$\varepsilon^m(\%)$	Comp. (s)	K^m_{EBS} Comp. (s)
	$10,000$	0.074	6403.919	
$m=2$	$30,000$	0.053	19840.584	0.406
	$50,000$	0.033	34406.980	
	$10,000$	4.421	6861.916	
$m=3$	$30,000$	1.970	20313.791	3.609
	$50,000$	1.024	33784.318	
	$10,000$	0.169	6314.332	
$m=4$	$30,000$	0.087	18831.897	10.297
	$50,000$	0.050	31651.802	

TABLE 1. Percentage relative errors ε^m and computational times (Comp.) of MC simulations for pricing variance swaps ($m=2$), skewness swaps ($m=3$) and kurtosis swaps ($m=4$) for $N_p = 10,000, 30,000$, and $50,000$, comparing with computational times of the closed-form formula

Table 1 confirms in addition that the results from the closed-form formula and the MC simulations match with high accuracy with very small ε^m for all cases of m and N_p, the highest ε^m is 4.4% when $m=3$ and $N_p = 10,000$. Moreover, the accuracy for MC simulations is improved when N_p increases, trade-off with increasing in computational times. The experiment showed that the computational time from closed-form formula is extremely faster than that from MC simulations, around 600 times faster.

Example 5.2. (Sensitivity of parameters)In this study, we investigate the sensitivity of fair prices for moment swaps ($m = 2, 3, 4$) based on small changes of parameters $r(t) = r_0 + r_1 t$ and $\sigma(t) = \sqrt{\sigma_0 + \sigma_1 t}$ in the EBS. Here, we use the same parameters provided in Example 5.1 with $r_0 = 0.075$, $r_1 = 0.050$, $\sigma_0 = 0.030$, and $\sigma_1 = 0.020$. To check the sensitivity of each parameter separately, the change of fair price is computed corresponding to the change of one parameter while the other three parameters are fixed. The sensitivity is measured based on the percentage relative errors of the fair price K^m_{EBS} and parameter ΔP, defined by

$$\Delta P := \left|\frac{P-P'}{P}\right| \times 100\%, \Delta K^m_{\text{EBS}}(P,P') := \left|\frac{K^m_{\text{EBS}}(P) - K^m_{\text{EBS}}(P')}{K^m_{\text{EBS}}(P)}\right| \times 100\%,$$

with fixed $T = 1$ and $N = 252$. The results are shown in Tables 2–3.

P	P'	$\Delta P(\%)$	$\Delta K^2_{\text{EBS}}(P,P')(\%)$	$\Delta K^3_{\text{EBS}}(P,P')(\%)$	$\Delta K^4_{\text{EBS}}(P,P')(\%)$
r_0	$r'_0 = 1.02r_0$	2	2.402×10^{-3}	1.838	4.800×10^{-3}
	$r'_0 = 1.04r_0$	4	4.848×10^{-3}	3.675	9.688×10^{-3}
	$r'_0 = 1.06r_0$	6	7.339×10^{-3}	5.513	1.463×10^{-2}
	$r'_0 = 1.08r_0$	8	9.875×10^{-3}	7.350	1.973×10^{-2}
	$r'_0 = 1.10r_0$	10	1.246×10^{-2}	9.188	2.488×10^{-2}
r_1	$r'_1 = 1.02r_1$	2	8.625×10^{-4}	0.664	1.852×10^{-3}
	$r'_1 = 1.04r_1$	4	1.732×10^{-3}	1.327	3.718×10^{-3}
	$r'_1 = 1.06r_1$	6	2.607×10^{-3}	1.991	5.600×10^{-3}
	$r'_1 = 1.08r_1$	8	3.490×10^{-3}	2.654	7.494×10^{-3}
	$r'_1 = 1.10r_1$	10	4.379×10^{-3}	3.318	9.404×10^{-3}
σ_0	$\sigma'_0 = 1.02\sigma_0$	2	1.499	1.096	2.958
	$\sigma'_0 = 1.04\sigma_0$	4	2.997	2.181	5.960
	$\sigma'_0 = 1.06\sigma_0$	6	4.496	3.255	9.006
	$\sigma'_0 = 1.08\sigma_0$	8	5.994	4.318	12.095
	$\sigma'_0 = 1.10\sigma_0$	10	7.493	5.370	15.229
σ_1	$\sigma'_1 = 1.02\sigma_1$	2	0.500	0.397	1.063
	$\sigma'_1 = 1.04\sigma_1$	4	0.999	0.792	2.133
	$\sigma'_1 = 1.06\sigma_1$	6	1.499	1.186	3.210
	$\sigma'_1 = 1.08\sigma_1$	8	1.998	1.578	4.293
	$\sigma'_1 = 1.10\sigma_1$	10	2.498	1.968	5.382

TABLE 2. The percentage relative errors of the fair prices of moment swaps $\Delta K^m_{\text{EBS}}(m = 2,3,4)$ for $\Delta P = 2,4,6,8,10\%$ of parameters r_0, r_1, σ_0 and σ_1

Moreover, since Table 2 shows that ΔK^m_{EBS} depends linearly on ΔP, the order of sensitivity S^m_p of each parameter is computed as the average of $\frac{\Delta K^m_{\text{EBS}}}{\Delta P}$,

$$S^m_P := \frac{1}{n}\sum_{i=1}^{n} \frac{\Delta K^m_{\text{EBS}}(P_i, P'_i)}{\Delta P_i},$$

shown in Table 3.

Moment swaps	$S^m_{r_0}$	$S^m_{r_1}$	$S^m_{\sigma_0}$	$S^m_{\sigma_1}$
$m = 2$	1.223×10^{-3}	4.346×10^{-4}	0.749	0.250
$m = 3$	0.919	0.332	0.543	0.198
$m = 4$	2.443×10^{-3}	9.331×10^{-4}	1.501	0.535

TABLE 3. The orders of sensitivity of fair prices for $m = 2,3,4$ corresponding to parameters $r_0, r_1, \sigma_0, \sigma_1$

Table 2 shows that ΔK^m_{EBS} depends linearly on ΔP for all cases ($m = 2,3,4$ and all parameters). The results show that K^m_{EBS} is more sensitive to the parameter σ_0 than the others. When comparing using the orders of sensitivity, the results display that when $m = 2,4$, K^m_{EBS} is more sensitive to the volatility $\sigma(t)$ than interest rate $r(t)$, which is not the case when $m = 3$.

Example 5.3. (Comparison fair prices) In this example, we compare the fair prices K^m_{BS} to illustrate Corollary 3.4 for the BS model. The fair prices $K^m_{\text{BS}}, K^n_{\text{BS}}$ are compared based on two sets of parameters for various pairs (m, n) with $m > n$. The first set (I) of parameters is from Broadie and Jain [1], $r = 0.0319$ and $\sigma = 0.1326$. The second set (II) is from Khaled and Samai [3], $r = 0.0013$ and $\sigma = \sqrt{0.0009}$, which were used in the likelihood function for the share price of gold for the period from April 2–December 31, 2007. The evaluation is performed with $T = 1$ and $N = 252$ to find $\tau^*_{m,n}$, the smallest positive root defined in Theorem 3.2, for each pair of K^m_{BS} and K^n_{BS}, where the existing of $\tau^*_{m,n}$ implies the order $K^m_{\text{BS}}(T, N) < K^n_{\text{BS}}(T, N)$ for all $\frac{T}{N} \in (0, \tau^*_{m,n})$. Note that the first set of parameters satisfies $r > \frac{3}{2}\sigma^2$, while the second set $\frac{1}{2}\sigma^2 < r < \frac{3}{2}\sigma^2$. The results of $\tau^*_{m,n}$ for several (m, n) pairs are shown in Table 4.

	(m, n)	$(3, 2)$	$(4, 2)$	$(4, 3)$	$(6, 2)$	$(6, 3)$	$(6, 4)$	$(6, 5)$
$\tau^*_{m,n}$	I	19.10	14.12	6.36	11.56	8.72	9.38	4.59
	II	481.71	305.35	–	245.78	156.90	196.34	–

TABLE 4. The $\tau^*_{m,n}$ of various pairs of K^m_{BS} and K^n_{BS} for the two sets of parameters

The results from Table 4 show that for the set I of parameters, $r > \frac{3}{2}\sigma^2$, the $\tau^*_{m,n}$ exists for all (m, n) pairs, which supports Corollary 3.4 that $\tau^*_{m,n}$ always exists in this case. However, for the set II of parameters, $\frac{1}{2}\sigma^2 < r < \frac{3}{2}\sigma^2$, the $\tau^*_{m,n}$ exists for all pairs (m, n) except for the pairs $(4, 3)$ and $(6, 5)$, where $n = m - 1$ is odd. This illustrates that when the set of parameters does not satisfy the condition of Corollary 3.4, the existence of $\tau^*_{m,n}$ depends on (m, n) according to Theorem 3.2 and Corollary 3.3, namely, the $\tau^*_{m,n}$ exists for all (m, n) except when $n = m - 1$ is odd.

6. Conclusions

This study presented a simple and easy-to-use pricing formula for discretely-sampled moment swaps when the realized higher moments defined in terms of m^{th}-moment of the log-returns of a specified underlying asset described by BS model with time-dependent parameters. The obtained analytical method is developed based on the Feynman-Kac theorem, where the PDE is solved analytically, and some combinatorial techniques are used to simplify the sum of the conditional expectations. In terms of validation purposes, we have demonstrated that a pricing formula is financially meaningful, whilst also showing that the fair prices for moment swaps are always finite and positive in the parameter space. A comparison theorem has been proved to show that trading variance swaps are more expensive than trading any higher moment swaps under the BS model. Furthermore, we have presented a particularly useful formula for hedging moment swaps on futures expressed in terms of a linear combination of the fair delivery prices of moment swaps on its corresponding underlying stock. The first and third numerical examples support the validity of the results. Namely, the first experiment shows that MC simulations produced the same results as our formula, while the third experiment illustrates the comparison results of moments for BS model. Moreover, the second example provides the sensitivity of the fair prices with respect to the parameters, with the results showing that the fair price is more sensitive to the volatility parameters when $m = 2, 4$ (even).

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