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Non-Polynomial Cubic Spline Method for Solving Singularly Perturbed Delay Reaction-Diffusion Equations

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Abstract This paper presents a non-polynomial cubic spline method for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behavior. In this method, the second-order singularly perturbed delay reaction-diffusion equation transformed into an asymptotically equivalent singularly perturbed two-point boundary value problem using Taylor series expansion. Then, non-polynomial cubic spline approximations are developed into a three-term recurrence relation, solved using Thomas Algorithm. The stability and convergence of the method have been established. The applicability of the proposed method is validated by implementing it by four model examples without exact solutions for different values of the perturbation parameter ε and delay parameter δ.

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1. Introduction

Singularly perturbed ordinary differential equation with a delay is ordinary differential equations in which the highest derivative is multiplied by a small parameter and involves at least one delay term. Such type of equations frequently arises from the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex [14], the study of bi stable devices [3], and vibrational problems in control theory [7], etc. The analytic solution of some related problems to the problem under consideration were discussed by by the authors [10]-[13]. When the perturbation parameter is very small, most numerical methods for solving such problems may become unstable

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and give inaccurate results. So, it is essential to develop suitable numerical methods to solve singularly perturbed delay differential equations. Hence, many researchers have been trying to develop numerical methods for solving singularly perturbed delay differential equations in recent times. For example, [18] proposed computational method of first order for singularly perturbed delay reaction-diffusion equations with layer or oscillatory behavior.[15] presented fourth order finite difference scheme for second order singularly perturbed differential-difference equation with negative shift. [2] presented exponentially fitted second order finite difference scheme for a class of singularly perturbed delay differential equations with large delay.[17] proposed the numerical solution of singularly perturbed differential-difference equations with dual layer.[6] presented fourth order finite difference scheme for singularly perturbed delay differential reaction diffusion equations.[5] proposed fourth order finite difference method for solving singularly perturbed delay reaction diffusion equations. In [19], it is presented computational method for solving singularly perturbed delay differential equation with twin layers or oscillatory behavior. But, still there is a lack of accuracy because of the treatment of singularly perturbed problems is not trivial, and the solution depends on perturbation parameter ε and mesh size h [4], [9] and [16]. Due to this, numerical treatment of singularly perturbed delay differential equations needs improvement. Therefore, it is essential to develop more accurate and convergent numerical method for solving singularly perturbed delay differential equations. Thus, the purpose of this study is to develop a stable, convergent and more accurate numerical method for solving singularly perturbed delay reaction-diffusion equations.

2. Description of the method

To describe the method, we first consider a linear singularly perturbed delay reaction - diffusion equation of the form

$$\varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x), 0 < x < 1, \tag{2.1}$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), \delta \le x \le 0, y(1) = \varphi, \tag{2.2}$$

where ε is perturbation parameter, $\epsilon(0 < \epsilon << 1)$ and δ is a small delay parameter of $o(\varepsilon)$, $(0 < \delta << 1)$. Also $a(x), b(x)$, $f(x)$ and $\phi(x)$ are bounded smooth functions and φ is a given constant. The layer or oscillatory behavior of the problem under consideration is maintained for $\delta \neq 0$ but sufficiently small, depending on the sign of $a(x)+b(x)$, for all $x \in (0,1)$. If $a(x)+b(x) < 0$ the solution of the problem in Eqs. (2.1) and (2.2) exhibits layer behavior and if $a(x)+b(x) > 0$ it exhibits oscillatory behavior.
Therefore, if the solution exhibits layer behavior, there will be two boundary layers which will occur at both end points $x = 0$ and $x = 1$ [6]. By using Taylor series expansion in the neighborhood of x we have

$$y(x-\delta) = y(x) - \delta y'(x) + O(\delta^2). \tag{2.3}$$

Substituting Eq.(2.3) into Eq.(2.1), we obtain an asymptotically equivalent SPTBVPs of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \tag{2.4}$$

under boundary conditions $y(0) = \phi_0, y(1) = \varphi$, where $p(x) = \frac{-\delta a(x)}{\varepsilon}, q(x) = \frac{a(x)+b(x)}{\varepsilon}$ and $r(x) = \frac{f(x)}{\varepsilon}$.
Consider a uniform mesh with interval [0,1] in which $0 = x_0 < x_1 < x_2 < ... < x_N = 1$ where $h = \frac{1}{N}$ and $x_i = ih, i = 0, 1, 2, ...N$.
For each segment $[x_i, x_{i+1}]$ the non-polynomial cubic spline $S_\Delta(x)$ has the following form

$$\begin{aligned} S_\Delta(x) = a_i \sin w(x-x_i) + b_i \cos w(x-x_i) \\ +c_i(e^{w(x-x_i)} - e^{-w(x-x_i)}) \\ +d_i(e^{w(x-x_i)} + e^{-w(x-x_i)}), \end{aligned} \tag{2.5}$$

where, a_i, b_i, c_i and d_i are unknown coefficients, $w \neq 0$ parameter used to increase the accuracy of the method.
To determine the unknown coefficients in Eq. (2.5) in terms of y_i, y_{i+1}, M_i and M_{i+1} first we define:

$$S_\Delta(x_i) = y_i, S_\Delta(x_{i+1}) = y_{i+1}, S'_\Delta(x_i) = M_i, S''_\Delta(x_{i+1}) = M_{i+1}. \tag{2.6}$$

After some algebraic manipulation, the coefficients in Eq. (2.5) becomes

$$\begin{cases} a_i = \frac{w^2 y_{i+1} - M_{i+1} + (M_i - y_i w^2)\cos(\theta)}{2w^2 \sin(\theta)}, \\ b_i = \dfrac{y_i w^2 - M_i}{2w^2}, \\ c_i = \frac{2(y_{i+1}w^2 + M_{i+1}) - (y_i w^2 + M_i)(e^\theta + e^{-\theta})}{4w^2(e^\theta - e^{-\theta})}, \\ d_i = \frac{y_i w^2 + M_i}{4w^2}. \end{cases} \tag{2.7}$$

Using the continuity condition of the first derivative at x_i, $S'_{\Delta-1}(x_i) = S'_\Delta(x_i)$, we have

$$a_{i-1}\cos(\theta) + b_{i-1}\sin(\theta) + c_{i-1}(e^\theta - e^{-\theta}) + d_{i-1}(e^\theta + e^{-\theta}) = a_i + 2c_i. \tag{2.8}$$

Reducing indices of Eqs. (2.7) by one and substituting into Eq. (2.8), we obtain

$$\begin{aligned} (\frac{-1}{2\sin(\theta)} - \frac{1}{e^\theta - e^{-\theta}})y_{i-1} + (\frac{\cos(\theta)}{\sin(\theta)} + \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}})y_i \\ +(\frac{-1}{2\sin(\theta)} - \frac{1}{e^\theta - e^{-\theta}})y_{i+1} = (\frac{-h^2}{2\theta^2 \sin(\theta)} - \frac{h^2}{\theta^2(e^\theta - e^{-\theta})})M_{i-1} \\ +(\frac{h^2\cos(\theta)}{\theta^2\sin(\theta)} + \frac{h^2(e^\theta + e^{-\theta})}{\theta^2(e^\theta - e^{-\theta})})M_i + (\frac{-h^2}{2\theta^2\sin(\theta)} - \frac{h^2}{\theta^2(e^\theta - e^{-\theta})})M_{i+1}. \end{aligned} \tag{2.9}$$

Multiplying both side of Eq. (2.9) by $\frac{-2\sin(\theta)(e^\theta - e^{-\theta})}{e^\theta - e^{-\theta} + 2\sin(\theta)}$, we get

$$y_{i-1} + \rho y_i + y_{i+1} = h^2(\alpha M_{i-1} + \beta M_i + \alpha M_{i+1}), \tag{2.10}$$

where

$$\begin{cases} \rho = -2(\frac{e^\theta(\cos\theta + \sin\theta) + e^{-\theta}(\sin\theta - \cos\theta)}{e^\theta - e^{-\theta} + 2\sin\theta}), \\ \alpha = \frac{e^\theta - e^{-\theta} - 2\sin\theta}{\theta^2(e^\theta - e^{-\theta} + 2\sin\theta)}, \\ \beta = -2(\frac{e^\theta(\cos\theta - \sin\theta) - e^{-\theta}(\sin\theta + \cos\theta)}{\theta^2(e^\theta - e^{-\theta} + 2\sin\theta)}). \end{cases}$$

If $h \to 0$, then $\theta = hk \to 0$. Thus, using L'Hopitals rule $(\alpha, \beta, \rho) \to (\frac{1}{6}, \frac{4}{6}, -2)$. Substituting $S''_{\Delta}(x_i) = y''_i = M_i$ into Eq. (2.4), we get

$$\begin{cases} M_i = r_i - p_i y'_i - q_i y_i, \\ M_{i-1} = r_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}, \\ M_{i+1} = r_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}. \end{cases} \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.10), we obtain

$$\begin{aligned} \frac{1}{h^2}(y_{i-1} + \rho y_i + y_{i+1}) &= \alpha(r_{i-1} - p_{i-1} y'_{i-1} - q_{i-1} y_{i-1}) \\ &+\beta(r_i - p_i y'_i - q_i y_i) + \alpha(r_{i+1} - p_{i+1} y'_{i+1} - q_{i+1} y_{i+1}). \end{aligned} \tag{2.12}$$

From the approximations of the first derivatives of y, according to [1], we have:

$$\begin{aligned} y'_{i-1} &\approx \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}, \\ y'_{i+1} &\approx \frac{3y_{i+1} - 4y_i + 3y_{i-1}}{2h}, \\ y'_i \approx \left(\frac{1 - 2h^2\omega q_{i+1} - h\omega(3p_{i+1} + p_{i-1})}{2h}\right) &y_{i+1} \\ +2\Psi(p_{i+1} + p_{i-1}) &y_i \\ + \left(\frac{1 - 2h^2\omega q_{i-1} - h\omega(p_{i+1} + 3p_{i-1})}{2h}\right) &y_{i-1} \\ +h\omega(r_{i+1} - r_{i-1}&), \end{aligned} \tag{2.13}$$

where ω is parameter used to raise the accuracy of the method.
Substituting Eq.(2.13) into Eq.(2.12) and rearranging, we get

$$\begin{aligned} &(1 - \frac{h}{2}u_i + \alpha h^2 q_{i-1})y_{i-1} - (-\rho + \frac{h}{2}v_i - \beta h^2 q_i)y_i \\ &\qquad + (1 - \frac{h}{2}w_i + \alpha h^2 q_{i+1})y_{i+1} \\ &= h^2((\alpha + \beta p_i \omega h)r_{i-1} + \beta r_i + (\alpha - \beta p_i \omega h)r_{i+1}), \end{aligned} \tag{2.14}$$

where

$$\begin{cases} u_i = 3\alpha p_{i-1} + \beta p_i h\omega(p_{i+1} + 3p_{i-1}) - \beta p_i h^2 \omega q_{i-1} - \alpha p_{i+1} + \beta p_i, \\ v_i = 4\alpha p_{i-1} - 4\beta p_i h\omega(p_{i+1} + p_{i-1}) - \alpha p_{i+1}, \\ w_i = \alpha p_{i-1} + \beta p_i h\omega(3p_{i+1} + p_{i-1}) - 3\alpha p_{i+1} - \beta p_i + \beta p_i h^2 \omega q_{i+1}. \end{cases}$$

Finally, we get the three term recurrence relation of the form

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 1, 2, ..., n-1, \tag{2.15}$$

where

$$\begin{cases} E_i = 1 - \frac{h}{2}u_i + \alpha h^2 q_{i-1}, \\ F_i = -\rho + \frac{h}{2}v_i - \beta h^2 q_i, \\ G_i = 1 - \frac{h}{2}w_i + \alpha h^2 q_{i+1}, \\ H_i = h^2((\alpha + \beta p_i \omega h)r_{i-1} + \beta r_i + (\alpha - \beta p_i \omega h)r_{i+1}). \end{cases}$$

3. Truncation Error

From Eqs. (2.13), we have

$$\begin{cases} e'_{i+1} = Y'(x_{i+1}) - y'_{i+1} = \frac{h^2 y_i^{(3)}}{3} + \frac{h^3 y_i^{(4)}}{12} + \frac{h^4 y_i^{(5)}}{30}, \\ e'_{i-1} = Y'(x_{i-1}) - y'_{i-1} = \frac{h^2 y_i^{(3)}}{3} + \frac{h^3 y_i^{(4)}}{12} + \frac{h^4 y_i^{(5)}}{30}, \\ e'_i = Y'(x_i) - y'_i = -h^2(\frac{1}{6} + 2\omega) y_i^{(3)} - \frac{\omega h^2 y_i^{(5)}}{3} + \frac{h^4 y_i^{(5)}}{120}. \end{cases} \tag{3.1}$$

From Eq.(2.11) and Eq. (2.10),

$$\begin{aligned} &y_{i-1} + \rho y_i + y_{i+1} = h^2(\alpha(r_{i-1} - p_{i-1}y'_{i-1} - q_{i-1}y_{i-1}) \\ &+\beta(r_i - p_i y'_i - q_i y_i) + \alpha(r_{i+1} - p_{i+1}y'_{i+1} - q_{i+1}y_{i+1})). \end{aligned} \tag{3.2}$$

Putting the exact solution in Eq. (3.2), we have

$$\begin{aligned} &Y(x_{i-1}) + \rho Y(x_i) + Y(x_{i+1}) \\ &= h^2(\alpha(r_{i-1} - p_{i-1}Y'(x_{i-1}) - q_{i-1}Y(x_{i-1})) \\ &+\beta(r_i - p_i Y'(x_i) - q_i Y(x_i)) \\ &+\alpha(r_{i+1} - p_{i+1}Y'(x_{i+1}) - q_{i+1}Y(x_{i+1}))) + T_0(h), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} T_0(h) = -(2+\rho)y_i + (2\alpha+\beta-1)h^2 y''_i + (12\alpha - 1)\frac{h^4 y_i^{(4)}}{12} \\ +(2\alpha+\beta-1)h^6 y_i^6. \end{aligned} \tag{3.4}$$

Subtract Eq. (3.2) from Eq. (3.3) and putting $e_i = Y(x_j) - y_j, j = i \pm 1$, we obtain

$$\begin{aligned} &(1 + h^2\alpha q_{i-1})e_{i-1} + (\rho + \beta h^2 q_i)e_i + (1 + h^2\alpha q_{i+1})e_{i+1} \\ &= -h^2(\alpha p_{i-1}e'_{i-1} + \beta p_i e'_i + \alpha p_{i+1}e'_{i+1}) + T_0(h). \end{aligned} \tag{3.5}$$

Using Eq. (3.1) and (3.5) and by letting

$$\begin{cases} p_{i+1} = p_i + hp'_i + \frac{h^2 p''_i}{2} + O(h^3), \\ p_{i-1} = p_i - hp'_i + \frac{h^2 p''_i}{2} + O(h^3), \end{cases} \tag{3.6}$$

we have

$$(1 + h^2\alpha q_{i-1})e_{i-1} + (\rho + \beta h^2 q_i)e_i + (1 + h^2\alpha q_{i+1})e_{i+1} = T_i(h), \tag{3.7}$$

where

$$\begin{aligned} &T_i(h) = -(2+\varrho)y_i + (2\alpha+\beta-1)h^2 y''_i \\ &+(\frac{-2\alpha}{3} + \beta(\frac{1}{6} + 2\omega)h^2 p_i)y_i^{(3)} + (12\alpha - 1)\frac{h^4 y_i^{(4)}}{12} + O(h^6), \end{aligned} \tag{3.8}$$

where $T_i(h)$ is a local truncation error associated with the scheme developed in Eq. (2.15). Thus,for different values of α, β, ω in the scheme Eq.(2.15), the following different orders are indicated.

i. For $\rho = -2, 2\alpha + \beta = 1$ and for any value of ω, the scheme of Eq. (2.15) gives the second order method.

ii. For $\rho = -2, \alpha = \frac{1}{12}, \beta = \frac{10}{12}, \omega = \frac{-1}{20}$, from Eq.(2.15) the fourth order method is derived.

4. Stability and Convergence Analysis

Definition 4.1. A matrices A is said to be L-matrices if and only if $a_{ii} > 0, i = 1, 2, ..., n-1$ and $a_{ij} \leq 0, i \neq j, i = 1, 2, ..., n-1$.

Theorem 4.2. *For any partition $J \bigcup K$ of the index set $1, 2, ..., n$ of an of matrix A, if there exists $j \in J$ and $k \in K$ such that $a_{j,k} \neq 0$, then A is an irreducible matrix, [20].*

Theorem 4.3. *If A is an L-matrix which is symmetric, irreducible and weak diagonal dominance, then A is a monotone matrix, [21].*

Writing the tri-diagonal system in Eq. (2.15) in matrix vector form, we have:

$$A\overline{Y} = C, \tag{4.1}$$

where $A = (B_0 + B_1 + h^2 B_2 Q)$ is a tri diagonal matrix of order $n-1$. Multiplying both sides of Eq. (2.14) by (-1), we get

$$\begin{cases} B_0 = \begin{pmatrix} -\rho & -1 & & & \\ -1 & -\rho & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -\rho & -1 \\ & & & -1 & -\rho \end{pmatrix}, B_1 = \begin{pmatrix} \frac{hv_1}{2} & \frac{hw_1}{2} & & & \\ \frac{hu_2}{2} & \frac{hv_2}{2} & \frac{hw_2}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{hu_{n-2}}{2} & \frac{hv_{n-2}}{2} & \frac{hw_{n-2}}{2} \\ & & & \frac{hu_{n-1}}{2} & \frac{hv_{n-1}}{2} \end{pmatrix}, \\ B_2 = \begin{pmatrix} -\beta & -\alpha & & & \\ -\alpha & -\beta & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & -\beta & -\alpha \\ & & & -\alpha & -\beta \end{pmatrix}, Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n-2} \\ q_{n-1} \end{pmatrix}. \end{cases} \tag{4.2}$$

$C = c_i$ is the column vector, where

$$\begin{cases} c_i = h^2((-\alpha - \beta p_1 \omega h)r_0 - \beta r_1 - (\alpha - \beta p_1 \omega h) r_2) + \varphi_0 - \frac{h\varphi_0}{2} u_1, i = 1, \\ c_i = h^2((-\alpha - \beta p_i \omega h) r_{i-1} - \beta r_i - (\alpha - \beta p_i \omega h) r_{i+1}), i = 2 \leq i \leq n-2, \\ c_i = h^2((-\alpha - \beta p_{n-1} \omega h) r_{n-2} - \beta r_{n-1} - (\alpha - \beta p_1 \omega h) r_n) + \alpha h^2 q_n) + \\ \varphi - \frac{h\varphi}{2} w_n, i = n-1, \end{cases} \tag{4.3}$$

and $\overline{Y} = [\overline{y_1}, \overline{y_2}, ..., \overline{y_{n-1}}]^t$ is is an approximation solution, $T(h) = O(h^6)$ for $\rho = -2, \alpha = \frac{1}{12}, \beta = \frac{10}{12}, \omega = \frac{-1}{12}$.

Now considering the above system with exact solution, we have

$$AY - T(h) = C. \tag{4.4}$$

From Eq. (4.1) and Eq. (4.4), we obtain

$$AE = T(h), \tag{4.5}$$

where $E = Y - \overline{Y} = (e_1, e_2, ..., e_{n-1})^t$.
Let

$$\begin{cases} |p_{i-1}| < Z_1, \\ |p_i| \leq Z_2, \\ |p_{i+1}| \leq Z_3, \\ |q_{i-1}| \leq K_1, \\ |q_i| \leq K_2, \\ |q_{i+1}| \leq K_3, \end{cases} \tag{4.6}$$

where Z_1, Z_2, Z_3, K_1, K_2 and K_3 are positive constants.
Let r_{ij} be the $(i,j)^{th}$ element of the matrix A^{-1}, then for $i = 1, 2, ..., N-1$,

$$|r_{i,j+1}| \leq \frac{h}{2}(\alpha Z_1 + \beta Z_2(1 + 2\omega h^2 K_3 + \omega h(3Z_3 + Z_1)) + 3\alpha Z_3 + 2\alpha h K_3),$$

$$|r_{i,j-1}| \leq \frac{h}{2}(3\alpha Z_1 + \beta Z_2(1 + 2\omega h^2 K_1 + \omega h(Z_3 + 3Z_1)) + \alpha Z_3 + 2\alpha h K_1).$$

Thus, for sufficiently small h, we have

$$\begin{cases} -1 + |r_{i,j+1}| \neq 0, i = 1, 2, ..., n-2, \\ -1 + |r_{i,j-1}| \neq 0, i = 2, ..., n-1. \end{cases}$$

Hence, by theorem (4.2) matrix A is irreducible. From Eq. (2.15), we have

$$\begin{aligned} E_i + G_i = -2 + \frac{h}{2}(4\alpha p_{i-1} - 4\beta p_i \omega h(p_{i+1} + p_{i-1}) - 4\alpha p_{i+1}) \\ -\alpha h^2(q_{i-1} + q_{i+1}) + 3\beta\omega p_i h^3(q_{i-1} + q_{i+1}), \end{aligned}$$

$$F_i = -2\varrho + \frac{h}{2}(-4\alpha p_{i-1} + 4\beta p_i \omega h(p_{i+1} + p_{i-1}) + 4\alpha p_{i+1}).$$

Thus, for sufficiently small h and $\rho = -2$ we get $|E_i + G_i| \leq |F_i|$. Hence, is diagonally dominant. Under this condition Thomas Algorithm, the method is stable ([8]).
Let S_i be the sum of the elements of the i^{th} row of the matrix A, then we have

$$\begin{cases} S_i = -\rho - 1 + \frac{h}{2}(-3\alpha p_{i-1} - \beta p_i + \alpha p_{i+1}) + \frac{h^2}{2}(\beta\omega p_i(p_{i+1} + 3p_{i-1}) \\ \qquad -2(\beta q_i + \alpha q_{i+1})) - h^3\beta\omega p_i q_{i+1}, i = 1, \\ S_i = -\rho - 2 + h^2(-\alpha q_{i-1} - \beta q_i - \alpha q_{i+1}) + O(h^3), 2 \leq i \leq n-2, \\ S_i = -\rho - 1 + \frac{h}{2}(-\alpha p_{i-1} + \beta p_i + 3\alpha p_{i+1}) + \frac{h^2}{2}(\beta\omega p_i(p_{i+1} + p_{i-1}) \\ \qquad -2(\beta q_i + \alpha q_{i-1})) + h^3\beta\omega p_i q_{i-1}, i = n-1. \end{cases}$$

Let $K = \min \| q_i| = |-q_i|$, for sufficiently small h by Theorem 4.2, A is monotone. Hence, A^{-1} exist and $A^{-1} \geq 0$.
From Eq. (4.5), we have

$$E = A^{-1}T(h) \Rightarrow \|E\| \leq \|A^{-1}\|\|T(h)\|. \tag{4.7}$$

For sufficiently small h, we have

$$\begin{cases} S_1 > h^2 K D_1, i = 1, \\ S_i > h^2 K D_2, 2 \leq i \leq n-2, \\ S_{n-1} > h^2 K D_1, i = n-1, \end{cases}$$

where $D = \min|-q_i| = \min|q_i|, D_1 = \alpha + \beta$ and $D_2 = 2\alpha + \beta$.
Let $A^{-1}_{i,k} \in A^{-1}$, we define

$$\|A^{-1}_{i,k}\| = \max_{1 \leq i \leq n-1} \sum_{k=1}^{n-1} |A^{-1}_{i,k}|.$$

Since, $A^{-1}_{i,k} \geq 0$, then from the theory of matrices, we have

$$\sum_{k=1}^{n-1} A^{-1}_{i,k} S_k = 1 for i = 1, 2, ..., n-1.$$

Hence,

$$\begin{cases} A^{-1}_{i,1} \leq \frac{1}{S_1} < \frac{1}{h^2DD_1}, k=1, \\ A^{-1}_{i,n-1} \leq \frac{1}{S_{n-1}} < \frac{1}{h^2DD_1}, k=n-1, \\ \sum_{k=2}^{n-2} A^{-1}_{i,k} \leq \frac{1}{\min S_k} < \frac{1}{h^2DD_2}, k=2,3,...,n-2. \end{cases} \tag{4.8}$$

From Eqs. (4.4-4.8) and (3.8), we get

$$\|E\| \leq (\frac{1}{h^2DD_1}, \frac{1}{h^2DD_1}, \frac{1}{h^2DD_2})O(h^6) = D*h^4,$$

where $D* = (\frac{1}{DD_1}, \frac{1}{DD_1}, \frac{1}{DD_2})$ which independent of mesh size h.
This establishes that the present method is fourth order convergent for$\rho = -2$, $\alpha = \frac{1}{12}, \beta = \frac{10}{12}, \omega = \frac{-1}{20}$.

5. Numerical Examples and Results

To demonstrate the applicability of the method, we implemented the method on four numerical examples, two with twin boundary layers and two with oscillatory behavior. Since those examples have no exact solution, the numerical solutions are computed using double mesh principle.

Example 5.1. Consider the singularly perturbed delay reaction-diffusion equation with layer behavior,

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) = 1,$$

subject to the interval and boundary conditions,

$$y(x) = 1, \delta \leq x \leq 0, y(1) = 0.$$

Example 5.2. Consider the singularly perturbed delay reaction-diffusion equation with layer behavior,

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) = 1,$$

subject to the interval and boundary conditions,

$$y(x) = 1, \delta \leq x \leq 0, y(1) = 0.$$

Example 5.3. Consider the singularly perturbed delay reaction-diffusion equation with oscilatory behavior,

$$\varepsilon y''(x) + 0.25y(x-\delta) + y(x) = 1,$$

subject to the interval and boundary conditions,

$$y(x) = 1, \delta \leq x \leq 0, y(1) = 0.$$

Example 5.4. Consider the singularly perturbed delay reaction-diffusion equation with oscilatory behavior,

$$\varepsilon y''(x) + y(x-\delta) + 2y(x) = 1,$$

subject to the interval and boundary conditions,

$$y(x) = 1, \delta \leq x \leq 0, y(1) = 0.$$

Table 1. The maximum absolute errors of Example 1, for different values of ε with $\delta = 0.5\varepsilon$.

$\delta \downarrow$	N=100	N=200	N=300	N=400	N=500
Present	meethod				
0.03	1.6500e-07	4.1449e-08	1.8438e-08	1.0375e-08	6.6408e-09
0.05	2.7015e-07	6.7734e-08	3.0122e-08	1.6947e-08	1.0847e-08
0.09	4.6696e-07	1.1693e-07	5.1987e-08	2.9246e-08	1.8718e-08
Swamy	et al.,	(2015)			
0.03	2.1999e-03	1.1041e-03	7.3705e-04	5.5315e-04	4.4269e-04
0.05	2.2012e-03	1.1049e-03	7.3749e-04	5.5345e-04	4.4293e-04
0.09	2.1999e-03	1.1038e-03	7.3676e-04	7.3676e-04	4.4247e-04

Table 2. The maximum absolute errors of Example 2, for different values of ε with $\delta = 0.5\varepsilon$.

δ	N=100	N=200	N=300	N=400	N=500
Present	meethod				
0.03	2.6253e-06	6.5461e-07	2.9081e-07	1.6356e-07	1.0467e-07
0.05	4.5486e-06	1.1355e-06	5.0455e-07	2.8378e-07	1.8161e-07
0.09	8.7799e-06	2.1935e-06	9.7482e-07	5.4830e-07	3.5090e-07
Swamy	et al.,	(2015)			
0.03	3.1674e-03	1.6058e-03	1.0754e-03	8.0837e-04	6.4760e-04
0.05	3.1437e-03	1.5949e-03	1.0685e-03	8.0338e-04	6.4367e-04
0.09	3.0784e-03	1.5660e-03	1.0502e-03	7.9000e-04	6.3310e-04

Table 3. The maximum absolute errors of Example 3, for different values of $\varepsilon$with $\delta = 0.5\varepsilon$

δ	N=100	N=200	N=300	N=400	N=500
Present	meethod				
0.03	1.8840e-06	4.7776e-07	2.1290e-07	1.1987e-07	7.6746e-08
0.05	2.9946e-06	7.5533e-07	3.3625e-07	1.8925e-07	1.2115e-07
0.09	4.8487e-06	1.2185e-06	5.4207e-07	3.0501e-07	1.9524e-07
Swamy	et al.,	(2015)			
0.03	2.5991e-03	1.2872e-03	8.5528e-04	6.4039e-04	5.1179e-04
0.05	2.6270e-03	1.3013e-03	8.6474e-04	6.4750e-04	5.1749e-04
0.09	2.6813e-03	1.3289e-03	8.8320e-04	6.6139e-04	5.2863e-04

The Effect of Delay Term on the Solution Profile

To analyze the effect of the delay term on the solution profile of the problem, the numerical solution of the problem for different values of the delay parameters have been given by the following graphs.

Table 4. The maximum absolute errors of Example 4, for different values of $\varepsilon$with $\delta = 0.5\varepsilon$

δ	N=100	N=200	N=300	N=400	N=500
Present	meethod				
0.03	7.8747e-06	1.9980e-06	8.9036e-07	5.0129e-07	3.2097e-07
0.05	1.2983e-05	3.2763e-06	1.4587e-06	8.2100e-07	5.2558e-07
0.09	2.2358e-05	5.6247e-06	2.5027e-06	1.4084e-06	9.0152e-07
Swamy	et al.,	(2015)			
0.03	1.5929e-02	7.4850e-03	4.8816e-03	3.6202e-03	2.8764e-03
0.05	1.5470e-02	7.2782e-03	4.7473e-03	3.5209e-03	2.7975e-03
0.09	2.1396e-02	1.0097e-02	6.5922e-03	4.8916e-03	3.8879e-03

Table 5. The maximum absolute errors of Example 1, for different values of $\varepsilon$with $\delta = 0.5\varepsilon$

ε	N=16	N=32	N=64	N=128	N=256
Present method					
2^{-4}	1.2169e-05	3.8241e-06	1.0038e-06	2.5415e-07	6.3725e-08
2^{-5}	2.3766e-05	6.1005e-06	1.7097e-06	4.3897e-07	1.1046e-07
2^{-6}	5.9675e-05	8.3405e-06	2.5105e-06	6.7130e-07	1.7055e-07
2^{-7}	1.6709e-04	1.7610e-05	3.0449e-06	9.3490e-07	2.4448e-07
2^{-8}	8.1561e-04	4.6147e-05	5.3482e-06	1.1793e-06	3.3788e-07
2^{-9}	3.3844e-03	1.9563e-04	1.2967e-05	1.6731e-06	4.4173e-07
2^{-10}	1.1502e-02	8.6198e-04	4.6619e-05	3.7167e-06	5.3788e-07
Swamy et al., (2015)					
2^{-4}	1.8632e-02	9.6189e-03	4.8865e-03	2.4643e-03	1.2376e-03
2^{-5}	2.8161e-02	1.4818e-02	7.6255e-03	3.8713e-03	1.9509e-03
2^{-6}	3.7958e-02	2.0967e-02	1.0977e-02	5.6273e-03	2.8498e-03
2^{-7}	5.0640e-02	2.8316e-02	1.5267e-02	7.9105e-03	4.0287e-03
2^{-8}	6.3580e-02	3.7706e-02	2.0984e-02	1.1012e-02	5.6555e-03
2^{-9}	8.3843e-02	5.0477e-02	2.8297e-02	1.5261e-02	7.9111e-03
2^{-10}	9.9137e-02	6.3529e-02	3.7660e-02	2.0974e-02	1.1011e-02

6. Discussion and Conclusion

The singularly perturbed boundary value problem for second-order singularly perturbed delay reaction-diffusion equations is considered. To obtain an approximation solution for such type of equation non-polynomial cubic spline method is presented. First, the second-order singularly perturbed delay reaction-diffusion equation transformed into an asymptotically equivalent singularly perturbed boundary value problem. Then, the non-polynomial cubic spline approximation is changed into a three-term recurrence relation, solved using Thomas Algorithm. The stability and convergence of the method have been established. Two model examples with twin layers behavior and two model examples of oscillatory layers have been considered and solved for different values of perturbation parameter ε delay parameter δ and mesh size h. The numerical solutions are tabulated (Tables 1 to 6) in terms of maximum absolute errors and observed that the present method improves the findings of [19]. Also, it is significant that all of the maximum absolute errors

TABLE 6. The maximum absolute errors of Example 2, for different values of $\varepsilon$with $\delta = 0.5\varepsilon$

ε	N=16	N=32	N=64	N=128	N=256
Present method					
2^{-4}	2.5892e-04	5.8139e-05	1.4129e-05	3.5115e-06	8.7625e-07
2^{-5}	4.3448e-04	8.6708e-05	2.0054e-05	4.9118e-06	1.2223e-06
2^{-6}	9.6797e-04	1.4201e-04	2.9130e-05	6.9208e-06	1.7050e-06
2^{-7}	2.6335e-03	2.7879e-04	4.6342e-05	9.9904e-06	2.4013e-06
2^{-8}	7.4070e-03	7.3858e-04	8.6268e-05	1.5323e-05	3.4509e-06
2^{-9}	1.8798e-02	2.3182e-03	2.0340e-04	2.7033e-05	5.1776e-06
2^{-10}	3.9544e-02	7.0349e-03	6.2763e-04	5.9354e-05	8.6381e-06
Swamy et al., (2015)					
2^{-4}	2.1118e-02	1.1692e-02	6.1941e-03	3.1887e-03	1.6178e-03
2^{-5}	2.7872e-02	1.6023e-02	8.6367e-03	4.4957e-03	2.2948e-03
2^{-6}	3.5711e-02	2.1293e-02	1.1869e-02	6.2731e-03	3.2240e-03
2^{-7}	4.6679e-02	2.8350e-02	1.6107e-02	8.6728e-03	4.5120e-03
2^{-8}	5.4895e-02	3.6018e-02	2.1373e-02	1.1929e-02	6.2847e-03
2^{-9}	5.7371e-02	4.7254e-02	2.8581e-02	1.6140e-02	8.6961e-03
2^{-10}	5.7878e-02	5.5695e-02	3.6153e-02	2.1406e-02	1.1956e-02

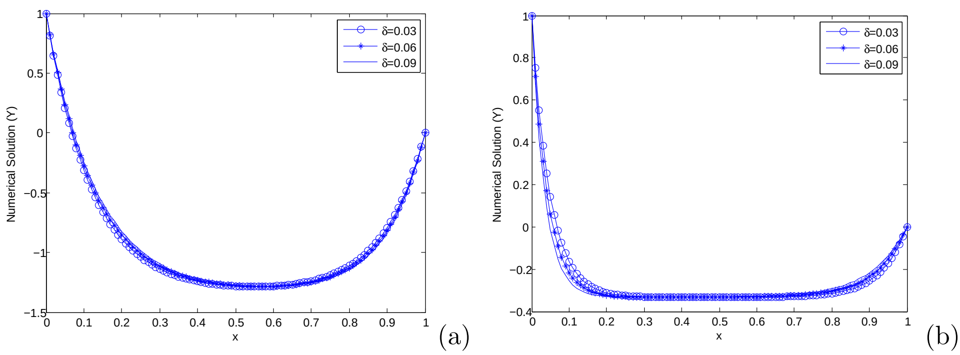

FIGURE 1. The numerical solution of Example 1 and Example 2 at $\varepsilon = 0.01$ and $N = 100$ respectively.

decrease rapidly as N increases. The stability and ε-uniform convergence of the method are investigated and established well. Further, numerical solutions have been presented using graphs to investigate the effect of delay on the problem's solution. Accordingly, when the delay term coefficient is $o(1)$, the delay affects the boundary layer solution but maintains the layer behavior (Figure 1 (b)). When the delay parameter is of $O(\varepsilon)$, the solution maintains layer behavior. However, the coefficient of the delay term in the equation of $O(1)$ and the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases (Figure 1(a)). For the oscillatory behavior case, one can conclude that the solution oscillates throughout the domain for different values of delay parameter δ (Figure 2 (a) and Figure 2 (b)). Concisely, the present method gives more accurate solution and is uniformly convergent for solving singularly perturbed

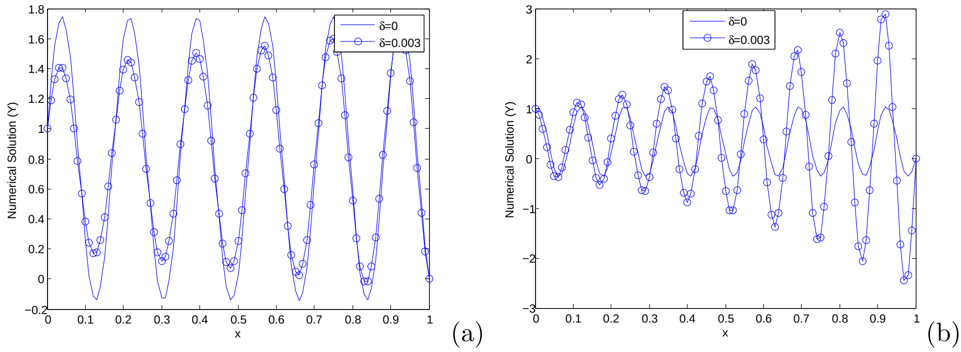

FIGURE 2. The numerical solution of Example 3 and Example 4 at $\varepsilon = 0.01$ and $N = 100$ respectively.

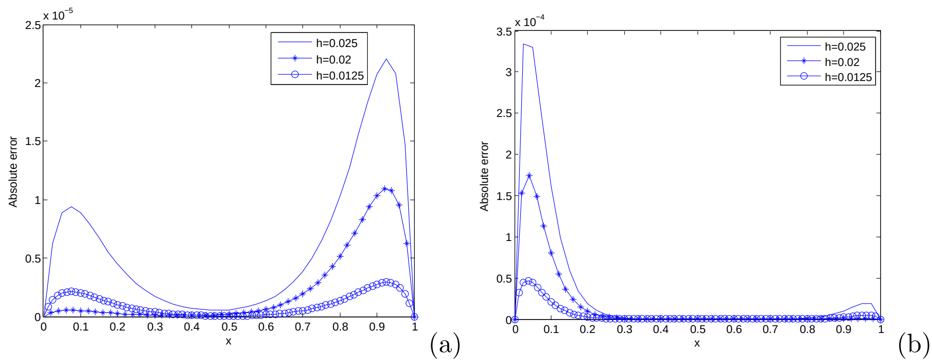

FIGURE 3. The point-wise absolute error of Example 1 and Example 2 for different values of mesh size h, $\varepsilon = 2^{-8}$ and $\delta = 0.05\varepsilon$ respectively.

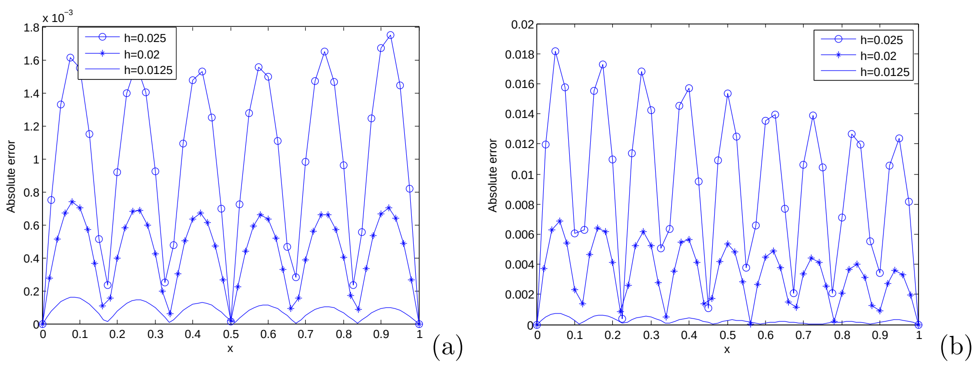

FIGURE 4. The point-wise absolute error of Example 3 and Example 4 for different values of mesh size h, $\varepsilon = 2^{-8}$ and $\delta = 0.05\varepsilon$ respectively.

delay reaction-diffusion equations with twin layer and oscillatory behavior. Also, as mesh size decreases, the absolute errors also decrease in both layer behavior (Figure 3 (a) and

Figure 3 (b)) and oscillatory behaver (Figure 4 (a) and Figure 4 (b)). The short coming of this method is when we extend to higher order the theoretical and experimental rate of convergence does not agree. The technique presented in this paper may also be done by higher order fitted mesh method.

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