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The Characterization of Fuzzy and Anti Fuzzy Ideals in AG-groupoid

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Abstract Fuzzy and anti fuzzy ideals in ordered AG-groupoid are the modern tool for handling uncertainty in many decisions making problems. The purpose of this paper is to investigate, the characterizations of different classes of non-associative ordered semigroups by using anti fuzzy left (resp. right) interior, weakly regular, $(2, 2)$-regular ideals. The algebraic properties of ideals in AG-groupoid are examined for the newly introduced fuzzy algebra. The results presented are justified by providing examples of well-defined and well-established AG-groupoid.

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1. Introduction

Semigroup is an algebraic structure formulated by defining an associative binary operation of a non-empty set. If the binary operation is commutative, then the semigroup is called a commutative semigroup. So in this case for any three elements a, b and c we have $abc = cba$ known as ternary commutative law. In 1972, Kazim and Naseeruddin [8] introduced a generalization of commutative semigroup by introducing in braces on the left side of ternary law and explored a new pseudo associative law, that is $(ab)c = (cb)a$ This they called the left invertive law. A groupoid S is said to be left almost semigroup

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(abbreviated as LA-semigroup) if it satisfies the left invertive law $(ab)c = (cb)a$. Holgate [5] has called the same structure as left invertive groupoid. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid) by [20]. In fact an AG-groupoid is non-commutative and non-associative semigroup. It is a midwaystructure between a commutative semigroup and a groupoid. Ideals in AG-groupoids have been investigated by [19]. In [7] left(resp. right [3]) a groupoid is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$)). According to Kazim and Naseeruddin [8], an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. However, by [20], every AG-groupoid with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$. In [9], if $(S, \cdot, \leq)$ is an ordered semigroup and $A \subseteq S$, we denote by $(A]$, the subset of S defined as follows: $(A] = \{s \in S : s \leq a$ for some $a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$. The notions of ideals play a crucial role in the study of ring, semiring, near-ring, semigroup, ordered semigroup theory etc. A non-empty subset A of S is called a left (resp. right) ideal of S if following hold:

(1) $SA \subseteq A$ (resp. $AS \subseteq A$);
(2) if $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).
A non-empty subset A of S is called an interior ideal of S if

(1) $SAS \subseteq A$;
(2) if $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

An ordered semigroup S is said to be regular [11, 12] if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows:

(1) $A \subseteq (ASA]$ for every $A \subseteq S$.
(2) $a \in (aSa]$ for every $a \in S$.

An ordered semigroup S is said to be $(2, 2)$-regular if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2xa^2$. Equivalent definitions are as follows:

(1) $A \subseteq (A^2SA^2]$ for every $A \subseteq S$.
(2) $a \in (a^2Sa^2]$ for every $a \in S$.

An ordered semigroup S is said to be weakly regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq axay$. Equivalent definitions are as follows:

(1) $A \subseteq ((AS)^2]$ for every $A \subseteq S$.
(2) $a \in ((aS)^2]$ for every $a \in S$.

An ordered semigroup S is an intra-regular [10, 12] if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent definitions are as follows:

(1) $A \subseteq (SA^2S]$ for every $A \subseteq S$.
(2) $a \in (Sa^2S]$ for every $a \in S$.

The idea of ordering of AG-groupoids has initiated by Shah *et al.* [23], [24], have investigated the concept of m-(resp. $n-, i-$) systems in ordered AG-groupoids. We define anti fuzzy left (resp. right) interior ideals in ordered AG-groupoids, basically an ordered AG-groupoid is non-commutative and non-associative ordered semigroup.
In this present paper, we characterize regular (resp. right regular, left regular, $(2, 2)$-regular, weakly regular and intra-regular) ordered AG-groupoids in terms of anti fuzzy left (resp. right, interior) ideals. In this regard, we prove that in regular, right regular,

weakly regular ordered AG-groupoids, the concept of anti fuzzy interior, two-sided ideals coincide. The concept of anti fuzzy interior, two-sided ideals coincide in $(2,2)$ intra-regular ordered AG-groupoids with left identity.

2. Preliminaries

Shah *et al.* [24] introduced ordered AG-groupoid S as a partially ordered set, at the same time an AG-groupoid such that $a \leq b$, implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leq (cb)d$ for all $a, b, c, d \in S$. An ordered AG-groupoid is also called a po-AG-groupoid for short.

Example 2.1. Consider a set $S = \{e, f, a, b, c\}$ with the following multiplication " $\cdot$ " and order relation " $\leq$ ":

$\cdot$	e	f	a	b	c
e	e	f	a	b	c
f	f	f	f	b	c
a	a	f	c	b	c
b	c	c	c	f	b
c	b	b	b	c	f

$$\leq := \{(e,e),(e,a),(e,b),(e,c),(f,f),(f,b),(f,c),(a,a),(a,c),(b,b),(b,c),(c,c)\}.$$

Then $(S, \cdot, \leq)$ is an ordered AG-groupoid with left identity e.

Let S be an ordered AG-groupoid and $A \subseteq S$, we define a subset $(A] = \{s \in S : s \leq a$ for some $a \in A$ of S and obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, then $AB := \{ab \mid a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$, then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general. For $\emptyset \neq A \subseteq S$. Then A is called an AG-subgroupoid of S if $A^2 \subseteq A$. A is called a left (resp. right) ideal of S if the following hold:

(1) $SA \subseteq A$ (resp.$AS \subseteq A$).
(2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

Equivalent definition: $A$is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$). A is called an ideal of S if A is both a left and a right ideal of S. If A, B are ideals of S then $A \cup B$ and $A \cap B$ are also ideals of S.
A non-empty subset A of S is called an interior ideal of S if

(1) $(SA)S \subseteq A$.
(2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ (or$(A] \subseteq A$).

An ordered AG-groupoid S is said to be regular if for every $a \in S$ there exists $x \in S$ such that $a \leq (ax)a$. Equivalent definitions are as follows:

(1) $A \subseteq ((AS)A]$ for every $A \subseteq S$.
(2) $a \in ((aS)a]$ for every $a \in S$.

An ordered AG-groupoid S is left (resp. right) regular if for every $a \in S$, there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$). Equivalent definitions are as follows:

(1) $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for every $A \subseteq S$.
(2) $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for every $a \in S$.

An ordered AG-groupoid S is said to be completely regular if it is regular, left regular, right regular. An ordered AG-groupoid S is said to be strongly regular if for every $a \in S$, there exists $x \in S$ such that $a \leq (ax)a$ and $ax = xa$. Every strongly regular ordered AG-groupoid is right regular ordered AG-groupoid. An ordered AG-groupoid S is said to be weakly regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq (ax)(ay)$ Equivalent definitions are as follows:

(1) $A \subseteq ((AS)^2]$ for every $A \subseteq S$.
(2) $a \in ((aS)^2]$ for every $a \in S$.

An ordered AG-groupoid S is called intra-regular if for every $a \in S$, here exist $x, y \in S$ such that $a \leq (xa^2)y$ Equivalent definitions are as follows:

(1) $A \subseteq ((SA^2)S]$ for every $A \subseteq S$.
(2) $a \in ((Sa^2)S]$ for every $a \in S$.

We denote by $L(a), R(a), I(a)$ the left ideal, the right ideal and the ideal of S, respectively generated by a. We have $L(a) = \{s \in S : s \leq a$ or $s \leq xa$ for some $x \in S\} = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

Example 2.2. Let $S = \{a, b, c, d, e\}$. Define multiplication " $\cdot$ " in S as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

and $\leq: \; = \{(a,a),(b,b),(c,c),(d,d),(e,e)\}$. Then S is an ordered AG-groupoid. $A = \{c, d, e\}$ is an AG-subgroupoid of S and $I = \{a, c, d, e\}$ is an ideal of S.

Remark 2.3. Every ideal whether right, left or two-sided is an AG-subgroupoid but the converse is not true in general.

An ordered AG-groupoid S is said to be locally associative if for every $a \in S$, $(a.a).a = a.(a.a)$.

Example 2.4. Let $S = \{a, b, c\}$. Define multiplication " $\cdot$ " in S as follows:

$\cdot$	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

and $\leq: \; = \{(a,a),(b,b),(c,c)\}$. Then $(S, \cdot, \leq)$ is a locally associative ordered AG-groupoid.

In a locally associative ordered AG-groupoids S, we define powers of an element as follow : $a^1 = a,\ a^{n+1} = a^n a$. If S has a left identity e, we define $a^0 = e$, as left identity is unique in an ordered AG-groupoid. A locally associative ordered AG-groupoid S with left identity e has associative powers.

3. Fuzzy interior ideals in ordered AG-groupoids

A fuzzy set μ of a given set X is described as an arbitrary function $\mu : X \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers. The fundamental concept of a fuzzy set,

introduced by Zadeh [25] in 1965, which gives a natural frame work for the generalizations of some basic notions of algebra, for example set (resp. group, semigroup, ring, near-ring, semiring) theory, groupoids, real analysis, topology, differential equations and so forth. Rosenfeld [21], introduced the concept of fuzzy set in groups. The study of fuzzy set in semigroups investigated by Kuroki [15–17]. He studied fuzzy interior, bi-, quasi-, semiprime quasi ideals in semigroups. Dib and Galham in [4], examined the definition of fuzzy groupoid (resp. semigroup). They studied fuzzy ideals and fuzzy bi-ideals of fuzzy semigroups. A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [18], where one can find theoretical results on fuzzy semigroups and their use in fuzzy finite state machines and fuzzy languages. Fuzzy sets in ordered semigroups/ordered groupoids established by Kehayopulu and Tsingelis [13]. They also studied fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups [13, 14]. Biswas [2], introduced the concept of anti fuzzy subgroups of groups and studied the basic properties of groups in terms of anti fuzzy subgroups. Hong and Jun [6] modified the Biswas idea and applied it into BCK-algebra. Akram and Dar defined anti fuzzy left h-ideals of hemiring and discussed the basic properties of hemiring [1].
By a fuzzy set μ of an ordered AG-groupoid S, we mean a function $\mu : S \to [0,1]$ and the complement of μ is denoted by μ' is a fuzzy set in S given by $\mu'(x) = 1 - \mu(x)$ for all $x \in S$.
A fuzzy set μ of S is called an anti fuzzy AG-subgroupoid of S if $\mu(xy) \leq max\{\mu(x), \mu(y)\}$ for all $x, y \in S$. μ is called an anti fuzzy left (resp. right) ideal of S if

(1) $\mu(xy) \leq \mu(y)$ (resp. $\mu(xy) \leq \mu(x)$).
(2) $x \leq y$ implies $\mu(x) \leq \mu(y)$ for all $x, y \in S$.

μ is an anti fuzzy ideal of S if μ is both an anti fuzzy left and an anti fuzzy right ideal of S. Equivalently, μ is called an anti fuzzy ideal of S if

(1) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$.
(2) $x \leq y$, implies $\mu(x) \leq \mu(y)$ for all $x, y \in S$.

Every anti fuzzy ideal (whether right, left, two-sided) is an anti fuzzy AG-subgroupoid but the converse is not true in general.
A fuzzy set μ of S is called an anti fuzzy interior ideal of S if

(1) $\mu((xa)y) \leq \mu(a)$.
(2) $x \leq y$, implies $\mu(x) \leq \mu(y)$ for all $x, a, y \in S$.

We denote by $F(S)$, the set of all fuzzy subsets of S. We define an order relation " $\subseteq$ " on $F(S)$ such that $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$. Then $(F(S), \circ, \subseteq)$ is an ordered AG-groupoid. By the symbols $f \wedge g$ and $f \vee g$, we mean the following fuzzy subsets:

$$(\forall x \in S)\, (f \wedge g : S \to [0,1], x \longmapsto (f \wedge g)(x) = min\{f(x), g(x)\})\,;$$
$$(\forall x \in S)\, (f \vee g : S \to [0,1], x \longmapsto (f \vee g)(x) = max\{f(x), g(x)\})\,.$$

Let $a \in S$, we define a set $A_a = \{(y, z) \in S \times S \mid a \leq yz\}$. Let f and g be fuzzy subsets of S, the product $f \circ g$ of f and g is defined by:

$$f \circ g : S \to [0,1], a \longmapsto f \circ g(a) = \begin{cases} \vee_{(y,z) \in A_a} min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset \end{cases}$$

For a non-empty family of fuzzy subsets $\{f_i\}_{i\in I}$ of S, the fuzzy subsets $\vee_{i\in I} f_i$ and $\wedge_{i\in I} f_i$ of S are defined as follows:

$$\vee_{i\in I} f_i \quad : \quad S \to [0,1], a \mapsto (\vee_{i\in I} f_i)(a) := \sup_{i\in I}\{f_i(a)\}$$
$$\text{and } \wedge_{i\in I} f_i \quad : \quad S \to [0,1], a \mapsto (\wedge_{i\in I} f_i)(a) := \inf_{i\in I}\{f_i(a)\}.$$

If I is a finite set, say $I = \{1, 2, ...n\}$, then clearly,

$$\vee_{i\in I} f_i(a) = max\{f_1(a), f_2(a), ..., f_n(a)\}$$
$$\text{and } \wedge_{i\in I} f_i(a) = min\{f_1(a), f_2(a), ..., f_n(a)\}.$$

For S, the fuzzy subsets "0" and "1" are defined as follows:

$$0 \quad : \quad S \to [0,1], x \mapsto 0(x) := 0.$$
$$1 \quad : \quad S \to [0,1], x \mapsto 1(x) := 1.$$

Clearly, the fuzzy subset "0" (resp. "1") of S is the least (resp. the greatest) element of the ordered set $(F(S), \leq)$. The fuzzy subset "0" is the zero element of $(F(S), \circ, \leq)$, that is, $f \circ 0 = 0 \circ f = 0$ and $0 \leq f$ for every $f \in F(S)$). For $\emptyset \neq A \subseteq S$,the anti characteristic function of A is denoted by χ_A^C and defined by:

$$\chi_A^C(a) = \begin{cases} 0 \text{ if } a \in A \\ 1 \text{ if } a \notin A \end{cases}$$

An ordered AG-groupoid S can be considered a fuzzy subset of itself and we write $S = \chi_S^C$, that is, $S(x) = \chi_S^C(x) = 0$ for all $x \in S$. This imply that $S(x) = 1$ for all $x \in S$. For $A, B \subseteq S$, then $A \subseteq B$ if and only if $\chi_A^C \geq \chi_B^C$, $\chi_A^C \cap \chi_B^C = \chi_{A\cap B}^C$ and $\chi_A^C \circ \chi_B^C = \chi_{(AB]}^C$. Let μ be a fuzzy subset of S, then for all $t \in (0,1]$, we define a set $L(\mu; t) = \{x \in S \mid \mu(x) \leq t\}$, which is called lower t-level cut of μ and can be used to the characterization of μ.

Example 3.1. Let $S = \{a, b, c, d\}$. Define multiplication $\cdot$ in S as follows:

$\cdot$	a	b	c	d
a	c	d	a	b
b	b	c	d	a
c	a	b	c	d
d	d	a	b	c

and $\leq := \{(a,a), (b,b), (c,c), (d,d)\}$. Then S is an ordered AG-groupoid. Let μ be a fuzzy subset of S. We define $\mu(a) = \mu(c) = 0.7$, $\mu(b) = \mu(d) = 0$. Hence μ is an anti fuzzy AG-subgroupoid of S.

Example 3.2. Let $S = \{a, b, c, d\}$. Define multiplication $\cdot$ in S as follows:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	d	a
d	a	a	c	d

and $\leq := \{(a,a), (b,b), (c,c), (d,d)\}$. Then S is an ordered AG-groupoid. Let μ be a fuzzy subset of S. We define $\mu(a) = \mu(c) = \mu(d) = 0$, $\mu(b) = 0.7$. Hence μ is an anti fuzzy right ideal of S.

Remark 3.3. Example 3.1 and 3.2 show that, every anti fuzzy ideal (whether right, left, two-sided) is an anti fuzzy AG-subgroupoid, but the converse is not true.

Lemma 3.4. *Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then the anti characteristic function $\chi^C_{(A]}$ of $(A]$ is a fuzzy subset of S satisfying the condition $x \leq y \Rightarrow \chi^C_{(A]}(x) \leq \chi^C_{(A]}(y)$ for all $x, y \in S$.*

Proof. By the definition, $\chi^C_{(A]}$ is a mapping of S into $\{0,1\} \subseteq [0,1]$. Let $x \leq y$, $x, y \in S$. If $y \notin (A]$, by definition $\chi^C_{(A]}(y) = 1$, thus $\chi^C_{(A]}(x) \leq \chi^C_{(A]}(y)$. If $y \in (A]$, by definition $\chi^C_{(A]}(y) = 0$. Since $y \in (A]$, so there exists $z \in A$ such that $y \leq z$. Thus $x \leq z$, that is, $x \in (A]$ and $\chi^C_{(A]}(x) = 0$. Hence $\chi^C_{(A]}(x) \leq \chi^C_{(A]}(y)$. ∎

Proposition 3.5. *Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then $A = (A]$ if and only if fuzzy subset χ^C_A of S has the property $x \leq y \Rightarrow \chi^C_A(x) \leq \chi^C_A(y)$ for all $x, y \in S$.*

Proof. Suppose $A = (A]$, then the anti characteristic function χ^C_A of A is a fuzzy subset of S satisfying the condition $x \leq y \Rightarrow \chi^C_A(x) \leq \chi^C_A(y)$, by the Lemma 3.4.
Conversely, let $x \in (A]$, this imply that there exists $y \in A$ such that $x \leq y$. By the given condition, we have $\chi^C_A(x) \leq \chi^C_A(y)$. Since $y \in A$, we have $\chi^C_A(y) = 0$. Thus $\chi^C_A(x) = 0$, that is, $x \in A$. Hence $A = (A]$. ∎

Lemma 3.6. *Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is an AG-subgroupoid of S if and only if the anti characteristic function χ^C_A of A is an anti fuzzy AG-subgroupoid of S.*

Proof. Suppose A is an AG-subgroupoid of S and $x, y \in S$. If $x, y \notin A$, by definition $\chi^C_A(x) = 1 = \chi^C_A(y)$. Thus $\chi^C_A(xy) \leq \chi^C_A(x) \vee \chi^C_A(y)$. If $x, y \in A$, by definition $\chi^C_A(x) = 0 = \chi^C_A(y)$. $xy \in A$, A being an AG-subgroupoid of S this imply that $\chi^C_A(xy) = 0$. Thus $\chi^C_A(xy) \leq \chi^C_A(x) \vee \chi^C_A(y)$. Hence the anti characteristic function χ^C_A of A is an anti fuzzy AG-subgroupoid of S.
Conversely, let $xy \in A^2$, $x, y \in A$. By definition of anti characteristic function $\chi^C_A(x) = 0 = \chi^C_A(y)$. $\chi^C_A(xy) \leq \chi^C_A(x) \vee \chi^C_A(y) = 0$, χ^C_A being an anti fuzzy AG-subgroupoid of S. This imply that $\chi^C_A(xy) = 0$, that is, $xy \in A$. Hence A is an AG-subgroupoid of S. ∎

Lemma 3.7. *Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) ideal of S if and only if the anti characteristic function χ^C_A of A is an anti fuzzy left (resp. right) ideal of S.*

Proof. Suppose A is a left ideal of S and $x, y \in S$ such that $x \leq y$. This imply that $A = (A]$, A being a left ideal of S. Then $\chi^C_A(x) \leq \chi^C_A(y)$, by the Proposition 3.5. If $y \notin A$, by definition $\chi^C_A(y) = 1$. Thus $\chi^C_A(xy) \leq \chi^C_A(y)$. If $y \in A$, by definition $\chi^C_A(y) = 0$. $xy \in A$, A being a left ideal, so $\chi^C_A(xy) = 0$. Thus $\chi^C_A(xy) \leq \chi^C_A(y)$. Hence the anti characteristic function χ^C_A of A is an anti fuzzy left ideal of S.
Conversely, let $y \in A$ and $x \in S$ such that $x \leq y$. This imply that $\chi^C_A(x) \leq \chi^C_A(y)$, χ^C_A being an anti fuzzy left ideal of S. Then $A = (A]$, by the Proposition 3.5. Let $xy \in SA$, where $y \in A$, $x \in S$. By definition of anti characteristic function $\chi^C_A(y) = 0$. $\chi^C_A(xy) \leq \chi^C_A(y) = 0$, χ^C_A being an anti fuzzy left ideal of S. Thus $\chi^C_A(xy) = 0$, that is, $xy \in A$. Hence A is a left ideal of S. ∎

Proposition 3.8. *Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is an interior ideal of S if and only if the anti characteristic function χ_A^C of A is an anti fuzzy interior ideal of S.*

Proof. Suppose A is an interior ideal of S and $a, x, y \in S$ such that $x \leq y$. This imply that $A = (A]$, A being an interior-ideal. Then $\chi_A^C(x) \leq \chi_A^C(y)$, by the Proposition 3.5. If $a \notin A$, by definition $\chi_A^C(a) = 1$. Thus $\chi_A^C((xa)y) \leq \chi_A^C(a)$. If $a \in A$, by definition $\chi_A^C(a) = 0$. $(xa)y \in A$, A being an interior ideal, this imply that $\chi_A^C((xa)y) = 0$. Thus $\chi_A^C((xa)y) \leq \chi_A^C(a)$. Hence the anti characteristic function χ_A^C of A is an anti fuzzy interior ideal of S.
Conversely, let $y \in A$ and $x \in S$ such that $x \leq y$. This imply that $\chi_A^C(x) \leq \chi_A^C(y)$, χ_A^C being an anti fuzzy interior ideal of S. Then $A = (A]$,by the Proposition 3.5. Let $t \in (SA)S$, implies $t = (xa)y$, where $a \in A$ and $x, y \in S$. By definition of anti characteristic function $\chi_A^C(a) = 0$. $\chi_A^C((xa)y) \leq \chi_A^C(a) = 0$, χ_A^C being an anti fuzzy interior ideal of S. Thus $\chi_A^C((xa)y) = 0$, that is, $(xa)y \in A$. Hence A is an interior ideal of S. ■

Lemma 3.9. *Let μbe a fuzzy subset of an ordered AG-groupoid S. Then μ is an anti fuzzy AG-subgroupoid of S if and only if lower t-level $L(\mu; t)$ of μ is an AG-subgroupoid of S for all $t \in (0, 1]$.*

Proof. Suppose μ is an anti fuzzy AG-subgroupoid of S and $x, y \in L(\mu; t)$, this imply that $\mu(x), \mu(y) \leq t$. $\mu(xy) \leq \mu(x) \vee \mu(y) \leq t$, μ being an anti fuzzy AG-subgroupoid, that is, $xy \in L(\mu; t)$. Hence $L(\mu; t)$ is an AG-subgroupoid of S.
Conversely, we have to show that $\mu(xy) \leq \mu(x) \vee \mu(y)$, $x, y \in S$. We suppose a contradiction $\mu(xy) > \mu(x) \wedge \mu(y)$. Assume $\mu(x) = t = \mu(y)$, this imply that $\mu(x), \mu(y) \leq t$, that is, $x, y \in L(\mu; t)$. But $\mu(xy) > t$, that is, $xy \notin U(\mu; t)$ which is a contradiction. Hence $\mu(xy) \leq \mu(x) \vee \mu(y)$. ■

Lemma 3.10. *Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is an anti fuzzy left (resp. right) ideal of S if and only if lower t-level $L(\mu; t)$ of μ is a left (resp. right) ideal of S for all $t \in (0, 1]$.*

Proof. Suppose μ is an anti fuzzy left ideal of S. Let $y \in L(\mu; t)$ and $x \in S$ such that $x \leq y$, this imply that $\mu(y) \leq t$. $\mu(x) \leq \mu(y) \leq t$ and $\mu(xy) \leq \mu(y) \leq t$, μ being an anti fuzzy left ideal of S. Thus $x, xy \in L(\mu; t)$. Hence $L(\mu; t)$ is a left ideal of S.
Conversely, suppose $L(\mu; t)$ is a left ideal of S and $x, y \in S$ such that $x \leq y$. We have to show that $\mu(x) \leq \mu(y)$ and $\mu(xy) \leq \mu(y)$. We suppose a contradiction $\mu(x) > \mu(y)$ and $\mu(xy) > \mu(y)$. Let $\mu(y) = t$, this imply that $\mu(y) \leq t$, that is, $y \in L(\mu; t)$. But $\mu(x) > t$and $\mu(xy) > t$, that is, $x, xy \notin L(\mu; t)$, which is a contradiction. Hence $\mu(x) \leq \mu(y)$ and $\mu(xy) \leq \mu(y)$. ■

Proposition 3.11. *Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is an anti fuzzy interior ideal of S if and only if the lower t-level $L(\mu; t)$ of μ is an interior ideal of S for all $t \in (0, 1]$.*

Proof. Suppose μ is an anti fuzzy interior ideal of S. Let $y \in L(\mu; t)$ and $x \in S$ such that $x \leq y$, this imply that $\mu(y) \leq t$. $\mu(x) \leq \mu(y) \leq t$, μ being an anti fuzzy interior ideal of S. Thus $\mu(x) \leq t$, that is, $x \in L(\mu; t)$. Let $a \in L(\mu; t)$ and $x, y \in S$, by definition $\mu(a) \leq t$. $\mu((xa)y) \leq \mu(a) \leq t$, μ being an anti fuzzy interior ideal of S. Thus $\mu((xa)y) \leq t$, that is, $(xa)y \in L(\mu; t)$. Hence $L(\mu; t)$ is an interior ideal of S.

Conversely, suppose $L(\mu;t)$ is an interior ideal of S and $x, y, a \in S$ such that $x \leq y$. We have to show that $\mu(x) \leq \mu(y)$, we suppose a contradiction $\mu(x) > \mu(y)$. Let $\mu(y) = t$, this imply that $\mu(y) \leq t$, that is, $y \in L(\mu;t)$. But $\mu(x) > t,$, that is, $x \notin L(\mu;t)$, which is a contradiction. Hence $\mu(x) \leq \mu(y)$. We have to show that $\mu((xa)y) \leq \mu(a)$, we suppose a contradiction $\mu((xa)y) > \mu(a)$. Let $\mu(a) = t$, this imply that $\mu(a) \leq t$, that is, $a \in L(\mu;t)$. But $\mu((xa)y) > t$, that is, $(xa)y \notin L(\mu;t)$, which is a contradiction. Hence $\mu((xa)y) \leq \mu(a)$. ■

Lemma 3.12. *Every anti fuzzy right ideal of an ordered AG-groupoid S with left identity e, is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$. Now $\mu(xy) = \mu((ex)y) = \mu((yx)e) \leq \mu(yx) \leq \mu(y)$.. Hence μ is an anti fuzzy ideal of S. ■

Remark 3.13. The concept of anti fuzzy (right, two-sided) ideals coincide in ordered AG-groupoids S with left identity.

Lemma 3.14. *Every anti fuzzy ideal of an ordered AG-groupoid S is an anti fuzzy interior ideal of S.*

Proof. Let μ be an anti fuzzy two-sided ideal of S and $x, a, y \in S$.. Now $\mu((xa)y) \leq \mu(xa) \leq \mu(a)$. Hence μ is an anti fuzzy interior ideal of S. ■

Proposition 3.15. *Let S be an ordered AG-groupoid with left identity e. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$. Now $\mu(xy) = \mu((ex)y) \leq \mu(x)$. Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.12. Converse is true by Lemma 3.14. ■

Lemma 3.16. *Every anti fuzzy right ideal of a regular ordered AG-groupoid S, is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq (xa)x$. Now $\mu(xy) \leq \mu(((xa)x)y) = \mu((yx)(xa)) \leq \mu(yx) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of S. ■

Remark 3.17. The concept of anti fuzzy (large right, two-sided) ideals coincide in regular ordered AG-groupoids S.

Proposition 3.18. *Let S be a regular ordered AG-groupoid. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq (xa)x$. Now $\mu(xy) \leq \mu(((xa)x)y) = \mu((yx)(xa)) \leq \mu(x)$. Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.16. Converse is true by Lemma 3.14. ■

Lemma 3.19. *Every anti fuzzy right (resp. left) ideal of $(2,2)$-regular ordered AG-groupoid S is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq (x^2a)x^2$. Now $\mu(xy) \leq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \leq \mu(yx^2) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of S. Let μ be an anti fuzzy left ideal of S. Now

$\mu(xy) \leq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a) \leq \mu((xx)a) = \mu((ax)x) \leq \mu(x)$. Hence μ is an anti fuzzy ideal of S. ■

Remark 3.20. The concept of anti fuzzy (right, left, two-sided) ideals coincide in $(2, 2)$ -regular ordered AG-groupoids S.

Proposition 3.21. *Let S be a $(2, 2)$-regular ordered AG-groupoid with left identity e. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq (x^2a)x^2$. Now

$$\begin{aligned} \mu(xy) &\leq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \leq \mu(x^2) \\ &= \mu(xx) = \mu((ex)x) \leq \mu(x). \end{aligned}$$

Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.19. Converse is true by Lemma 3.14. ■

Lemma 3.22. *Let S be a right regular ordered AG-groupoid. Then every anti fuzzy right (resp. left) ideal of S is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq x^2a$. Now

$$\begin{aligned} \mu(xy) &\leq \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \\ &= \mu((yx)(ax)) \leq \mu(yx) \leq \mu(y). \end{aligned}$$

Hence μ is an anti fuzzy ideal of S. Let μ be an anti fuzzy left ideal of S. Now

$$\begin{aligned} \mu(xy) &\leq \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \\ &= \mu((yx)(ax)) \leq \mu(ax) \leq \mu(x). \end{aligned}$$

Hence μ is an anti fuzzy ideal of S. ■

Remark 3.23. The concept of anti fuzzy (right, left, two-sided) ideals coincide in right regular ordered AG-groupoids S.

Proposition 3.24. *Let S be a right regular ordered AG-groupoid. Then μ is an anti fuzzy interior ideal if and only if $\mu$is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq x^2a$. Now $\mu(xy) \leq \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \leq \mu(x)$. Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.22. Converse is true by Lemma 3.14. ■

Lemma 3.25. *Let S be a left regular ordered AG-groupoid with left identity e. Then every anti fuzzy right (resp. left) ideal of S is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq ax^2$. Now

$$\begin{aligned} \mu(xy) &\leq \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \\ &= \mu((y(ax))x) \leq \mu(y(ax)) \leq \mu(y). \end{aligned}$$

Hence μ is an anti fuzzy ideal of S. Let μ be an anti fuzzy left ideal of S. Now

$$\begin{aligned}\mu(xy) &\leq \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y) \\ &= \mu((y(ax))x) \leq \mu((ax)x) \leq \mu(x).\end{aligned}$$

Hence μ is an anti fuzzy ideal of S. ∎

Remark 3.26. The concept of anti fuzzy (right, left, two-sided) ideals coincide in left regular ordered AG-groupoids with left identity.

Proposition 3.27. *Let S be a left regular ordered AG-groupoid with left identity e. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exists $a \in S$ such that $x \leq ax^2$. Now

$$\begin{aligned}\mu(xy) &\leq \mu((ax^2)y) = \mu((a(xx))y) \\ &= \mu((x(ax))y) = \mu(((ex)(ax))y) \\ &= \mu(((xx)(ae))y) = \mu((((ae)x)x)y) \leq \mu(x).\end{aligned}$$

Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.25. Converse is true by Lemma 3.14. ∎

Theorem 3.28. *Let S be a right regular locally associative ordered AG-groupoid with left identity e. Then for every anti fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{2n})$, where n is any positive integer, for all $a \in S$.*

Proof. For $n = 1$, let $a \in S$, this imply that there exists $x \in S$ such that $a \leq a^2x$. Thus $\mu(a) \leq \mu(a^2x) = \mu((ea^2)x) \leq \mu(a^2) \leq max\{\mu(a), \mu(a)\} = \mu(a)$, ($\mu$ is an anti fuzzy ideal of S by Proposition 3.24). Hence $\mu(a) = \mu(a^2)$. Now $a^2 = aa \leq (a^2x)(a^2x) = a^4x^2$, then the result is true for $n = 2$. Suppose that result is true for $n = k$, that is, $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^ka \leq (a^{2k}x^k)(a^2x) = a^{2(k+1)}x^{(k+1)}$. Thus

$$\begin{aligned}\mu(a^{k+1}) &\leq \mu(a^{2(k+1)}x^{(k+1)}) = \mu((ea^{2(k+1)})x^{(k+1)}) \\ &\leq \mu(a^{2(k+1)}) = \mu(a^{2k+2}) = \mu(a^{k+1}a^{k+1}) \\ &\leq max\{\mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}).\end{aligned}$$

Therefore $\mu(a^{k+1}) = \mu(a^{2(k+1)})$. Hence by induction method, the result is true for all positive integers. ∎

Lemma 3.29. *Let S be a right regular locally associative ordered AG-groupoid with left identity e. Then for every anti fuzzy interior ideal μ of S, $\mu(ab) = \mu(ba)$ for all $a, b \in S$.*

Proof. Let $a, b \in S$. By using Theorem for $n = 1$. Now

$$\begin{aligned}\mu(ab) &= \mu((ab)^2) = \mu((ab)(ab)) \\ &= \mu((ba)(ba)) = \mu((ba)^2) = \mu(ba).\end{aligned}$$

∎

Theorem 3.30. *Let S be a regular and right regular locally associative ordered AG-groupoid with left identity e. Then for every anti fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{3n})$, where n is any positive integer, for all $a \in S$.*

Proof. For $n = 1$, let $a \in S$, this imply that there exists $x \in S$ such that $a \leq (ax)a$ and $a \leq a^2x$. Now $a \leq (ax)a \leq (ax)(a^2x) = a^3x^2$. Thus

$$\begin{aligned} \mu(a) &\leq \mu(a^3x^2) = \mu((ea^3)x^2) \leq \mu(a^3) \\ &= \mu(aa^2) \leq max\{\mu(a), \mu(a^2)\} \\ &\leq max\{\mu(a), \mu(a), \mu(a)\} = \mu(a). \end{aligned}$$

Hence $\mu(a) = \mu(a^3)$. Now $a^2 = aa \leq (a^3x^2)(a^3x^2) = a^6x^4$, then the result is true for $n = 2$. Suppose that result is true for $n = k$, that is, $\mu(a^k) = \mu(a^{3k})$. Now$a^{k+1} = a^ka \leq (a^{3k}x^{2k})(a^3x^2) = a^{3(k+1)}x^{2(k+1)}$. Thus

$$\begin{aligned} \mu(a^{k+1}) &\leq \mu(a^{3(k+1)}x^{2(k+1)}) = \mu((ea^{3(k+1)})x^{2(k+1)}) \leq \mu(a^{3(k+1)}) \\ &= \mu(a^{3k+3}) = \mu(a^{k+1}a^{2k+2}) \leq max\mu\left(a^{k+1}\right), \mu\left(a^{2k+2}\right)\} \\ &\leq max\{\mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right), \mu\left(a^{k+1}\right)\} = \mu\left(a^{k+1}\right). \end{aligned}$$

Therefore $\mu(a^{k+1}) = \mu(a^{3(k+1)})$. Hence by induction method, the result is true for all positive integers. ∎

Lemma 3.31. *Let S be a weakly regular ordered AG-groupoid. Then every anti fuzzy right (resp. left) ideal is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exist $a, b \in S$ such that $x \leq (xa)(xb)$. Now

$$\begin{aligned} \mu(xy) &\leq \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \\ &= \mu((((ab)x)x)y) = \mu((yx)((ab)x)) \\ &= \mu((yx)(nx)) \text{ say } ab = n \\ &\leq \mu(yx) \leq \mu(y). \end{aligned}$$

Hence μ is an anti fuzzy ideal of S. Let μ be an anti fuzzy left ideal of S. Now

$$\begin{aligned} \mu(xy) &\leq \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \\ &= \mu((((ab)x)x)y) = \mu((yx)((ab)x)) \\ &= \mu((yx)(nx)) \text{ say } ab = n \\ &\leq \mu(nx) \leq \mu(x). \end{aligned}$$

Hence μ is an anti fuzzy ideal of S. ∎

Remark 3.32. The concept of anti fuzzy (right, left, two-sided) ideals coincide in weakly regular ordered AG-groupoids S.

Proposition 3.33. *Let S be a weakly regular ordered AG-groupoid. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exist $a, b \in S$ such that $x \leq (xa)(xb)$. Now $\mu(xy) \leq \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \leq \mu(x)$. Thus μ is an anti fuzzy right ideal of S. Hence μ is an anti fuzzy ideal of S by Lemma 3.31. Converse is true by Lemma 3.14. ∎

Theorem 3.34. *Let S be an ordered AG-groupoid with left identity e. Then S is a weakly regular if and only if S is completely regular.*

Proof. Suppose $S$is a weakly regular ordered AG-groupoid. Let $a \in S$, . Then there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Now $a \leq (ax)(ay) = (aa)(xy) = a^2t$, for some $t \in S$, this imply that $a \leq a^2t$. Thus S is a right regular ordered AG-groupoid. Now $a \leq (ax)(ay) = (yx)(aa) = ta^2$, for some $t \in S$, this imply that $a \leq ta^2$. Thus S is a left regular ordered AG-groupoid. Now

$$\begin{aligned} a &\leq (ax)(ay) = (aa)(xy) = a^2t = (aa)t = (ta)a \\ &\leq (t(ta^2))a = (t(t(aa)))a = (t(a(ta)))a \\ &= (a(t(ta)))a = (as)a, \text{ say } t(ta) = s \end{aligned}$$

This imply that $a \leq (as)a$, for some $s \in S$. Thus S is a regular ordered AG-groupoid. Hence S is a completely regular ordered AG-groupoid.
Conversely, let Sbe a completely regular ordered AG-groupoid. Let $a \in S$, then there exists $x \in S$ such that $a \leq (ax)a$, $a \leq a^2x$ and $a \leq xa^2$. Now

$$\begin{aligned} a &\leq (ax)a \leq (ax)(xa^2) = (ax)(x(aa)) \\ &= (ax)(a(xa)) = (ax)(ay), \quad \text{say } xa = y \end{aligned}$$

This imply that $a \leq (ax)(ay)$, for some $x, y \in S$. Hence $S$is weakly regular ordered AG-groupoid. ∎

Lemma 3.35. *Every anti fuzzy right ideal of an intra-regular ordered AG-groupoid S is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy right ideal of S and $x, y \in S$, this imply that there exist $a, b \in S$ such that $x \leq (ax^2)b$. Now $\mu(xy) \leq \mu(((ax^2)b)y) = \mu((yb)(ax^2)) \leq \mu(yb) \leq \mu(y)$. Hence μ is an anti fuzzy ideal of S. ∎

Remark 3.36. The concept of anti fuzzy (right, two-sided) ideals coincide in intra-regular ordered AG-groupoids S.

Proposition 3.37. *Let S be an intra-regular ordered AG-groupoid with left identity e. Then μ is an anti fuzzy interior ideal if and only if μ is an anti fuzzy ideal of S.*

Proof. Let μ be an anti fuzzy interior ideal of S and $x, y \in S$, this imply that there exist $a, b \in S$ such that $x \leq (ax^2)b$. Now

$$\begin{aligned} xy &\leq ((ax^2)b)y = (yb)(ax^2) = n(a(xx)) = n(x(ax)), \text{ say } yb = n \\ &= (en)(x(ax)) = (ex)(n(ax)) = (ex)m, \text{ say } n(ax) = m \end{aligned}$$

Thus $\mu(xy) \leq \mu((ex)m) \leq \mu(x)$. Hence μ is an anti fuzzy ideal of S. Converse is true by Lemma 3.14. ∎

Theorem 3.38. *Let S be an intra-regular locally associative ordered AG-groupoid. Then for every anti fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{2n})$, where n is any positive integer, for all $a \in S$.*

Proof. For $n = 1$. Let $a \in S$, this imply that there exist $x, y \in S$ such that $a \leq (xa^2)y$. Thus $\mu(a) \leq \mu((xa^2)y) \leq \mu(a^2) = \mu(aa) \leq max\{\mu(a), \mu(a)\} = \mu(a)$, ($\mu$ is an anti fuzzy ideal of S by Proposition 3.37). Hence $\mu(a) = \mu(a^2)$. Now $a^2 = aa \leq ((xa^2)y)((xa^2)y) = ((xa^2)(xa^2))y^2 = (x^2a^4)y^2$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, that is, $\mu(a^k) = \mu(a^{2k})$. Now $a^{k+1} = a^k a \leq ((x^k a^{2k})y^k)((xa^2)y) = (x^{k+1}a^{2(k+1)})y^{k+1}$. Thus

$$\begin{aligned}\mu\left(a^{k+1}\right) &\leq \mu((x^{k+1}a^{2(k+1)})y^{k+1}) \leq \mu(a^{2(k+1)}) = \mu(a^{(k+1)}a^{(k+1)}) \\ &\leq max\{\mu\left(a^{(k+1)}\right), \mu\left(a^{(k+1)}\right)\} = \mu\left(a^{(k+1)}\right).\end{aligned}$$

Therefore $\mu(a^{k+1}) = \mu(a^{2(k+1)})$. Hence by induction method, the result is true for all positive integers. ■

Lemma 3.39. *Let S be an intra-regular locally associative ordered AG-groupoid with left identity e. Then for every anti fuzzy interior ideal μ of S, $\mu(ab) = \mu(ba)$ for all $a, b \in S$.*

Proof. Same as Lemma 3.29. ■

4. Conclusion

AG-groupoid is a widely studied algebraic structure as a generalization of both the group and semigroup. The ideals are defined similarly as in semigroups. Partial order is an important feature of several real-life problems subject to various contradicting criteria. The partially ordered algebraic structure have been a matter of great concern for researchers over the years. Fuzzy sets are defined to coop uncertainty and vagueness in a very better than the classical probability theory. In the preset study the authors examined fuzzy interior ideals in ordered AG-groupoids. The relation between AG-subgroupoid and anti fuzzy AG-subgroupoid is established by using anti characteristic function. It is proved that A is an interior ideal of S if and only if the anti characteristic functions are anti fuzzy interior ideal of S. Same properties are tested for regular, weakly regular and $(2, 2)$- regular ideals. The results discuss in this article provides an essential background for several allied areas including rings, and AG-groupoid rings.

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