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Common Fixed Point Theorems for Expansive Type Mappings in Cone Heptagonal Metric Spaces

Gezachew Haile[1], Kidane Koyas[1] and Aynalem Girma[2,*]

[1] *Department of Mathematics, College of Natural Sciences*
Jimma University, Jimma, Ethiopia
e-mail : gizah21@gmail.com(Gizachew Haile); kidanekoyas@yahoo.com(Kidane Koyas)

[2] *Department of Mathematics, College of Natural Sciences*
Arba Minch University, Arba Minch, Ethiopia
e-mail : aynalemgirma46@yahoo.com(Aynalem Girma)

Abstract In this paper, we establish common fixed point theorems for a pair of self-mappings involving expansive type conditions in cone heptagonal metric spaces. The proved results generalize, improve and extend some known results in the literature. We also provide an example in support of our main result.

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1. Introduction

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis. In this area, the first important and significant result was proved by Banach in 1922. Due to its wide applications; the study of existence and uniqueness of fixed point of a mapping has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces. The notion of a generalized metric was introduced by Branciari [1] during deriving fixed point theorems for metric-like spaces. Some generalizations of a metric space was introduced by several authors and they presented an application for their obtained results.

Recently, in [2] Abdeljawad, Karapnar, Panda and Mlaiki considered a new distance structure, extended Branciari b-distance, to combine and unify several distance notions and obtained fixed point results that cover several existing ones in the corresponding literature. As an application of their obtained result, they presented a solution for a fourth order differential equation boundary value problem.

More recently, Panda, Abdeljawad and Swamy in [3] introduced the notion of $(\omega - F)$-contraction and presented fixed point results for such contractions. Thereafter, by using

*Corresponding author.

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the technique of fixed point, they proposed a simple solution for a nonlinear integral equation. In the same year Panda, Abdeljawad and Ravichandran in [4] introduced an extended F-metric and proved related fixed point results. Subsequently, they mainly focused on solution for the Atangana-Baleanu fractional integral and L^P-type solution $(1 < p < 1)$ for the linear Fredholm integral equation of the second kind.

In 1984, Wang, Li, Gao and Iseki [5] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces which correspond to some contractive mappings. In 2007, Huang and Zhang [6] introduced the concept of cone metric spaces which generalized the concept of the metric spaces, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mapping satisfying different contractive conditions. Many authors, [7],[8],[9],[10] have obtained coincidence point and fixed point results for mappings satisfying expansive type conditions in cone metric spaces. Azam, Arshad and Beg [11] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting. Further Kannan's fixed point theorem, Reich type contraction and more results were proved in [12],[13],[14] and [15] for these spaces. Patil and Salunke [16] proved some fixed point theorems for mappings satisfying expansive conditions for these spaces. Garg and Agarwal [17] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting. Further Auwalu and Hincal [18] proved some fixed point theorems for mappings satisfying expansive conditions in non-normal cone pentagonal metric spaces. Recently, Garg and Agarwal [19] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting. Very recently, Ampadu [20] introduced the notion of cone heptagonal metric space and proved Chatterjea contraction mapping principle in a normal cone heptagonal metric space setting. In the same year Auwalu and Denker [21] proved Banach contraction principle in cone heptagonal metric space setting. Motivated and inspired by the results of [16], [18], [20], it is our purpose in this paper to continue the study of fixed point of mappings in cone heptagonal metric space setting. Our results extend, improve and generalize some known results in the literature.

2. Preliminaries

We give some facts and definitions required to concepts concerning cones and cone metric spaces.

Definition 2.1. [6] Let E be a real Banach space with the zero vector θ. A subset P of E is called a cone if the following conditions are satisfied:

(i) P is closed, non-empty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P \ \ and \ -x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$ we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$ and we write $x < y$ if $x \leq y$ and $x \neq y$. Likewise, we shall write x $\ll$ y if $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 2.2. [6] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \leq d(x, y), \ \ for \ \ all \ \ x, y \in X \ \ and \ \ d(x, y) = \theta \ \ if \ \ and \ \ only \ \ if \ \ x = y,$

(ii) $d(x, y) = d(y, x) \ \ for \ \ all \ \ x, y \in X,$

(*iii*) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x,y,z \in X$. Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.3. [11] (Azam, et al., 2009) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(i) $\theta \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(iii) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$ for all $x,y,z,w \in X$, for all distinct $w,z \in X - \{x,y\}$. (rectangular property)
Then d is called a cone rectangular metric on X and (X,d) is called a cone rectangular metric space.

Remark 2.4. It is clear that any cone metric space is a cone rectangular metric space but the converse is not true in general.

Definition 2.5. [17] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(i) $\theta \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,y)$ for all $x,y,z,u,w \in X$, for all distinct $u,w,z \in X - \{x,y\}$. [Pentagonal property]
Then d is called a cone pentagonal metric on X and the pair (X,d) is called a cone pentagonal metric space.

Remark 2.6. Every cone rectangular metric space and so cone metric space is a cone pentagonal metric space. The converse is not necessarily true.

Definition 2.7. [19] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(i) $\theta \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,v) + d(v,y)$ for all $x,y,z,u,v,w \in X$, for all distinct $u,v,w,z \in X - \{x,y\}$. [Hexagonal property]
Then d is called a cone hexagonal metric on X and (X,d) is called a cone hexagonal metric space.

Definition 2.8. [20] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:
(i) $\theta \leq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(iii) $d(x,y) \leq s[d(x,z)+d(z,w)+d(w,u)+d(u,v)+d(v,r)+d(r,y)]$ for all $x,y,z,u,v,w,r \in X$, for all distinct $u,v,w,z,r \in X - \{x,y\}$. [heptagonal property]
Then d is called a cone heptagonal metric on X and (X,d) is called a cone heptagonal metric space.

Remark 2.9. Every cone hexagonal metric space, cone pentagonal metric space and so cone rectangular metric space is cone heptagonal metric space. The converse is not true.

Definition 2.10. [20] Let (X, d) be a cone heptagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_\theta$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.11. [20] Let (X, d) be a cone heptagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in X.

Definition 2.12. [20] Let (X, d) be a cone heptagonal metric space. If every Cauchy sequence is convergent in (X, d) then X is called a complete cone heptagonal metric space.

Definition 2.13. [22] Let f and g be two self maps of a nonempty set X. If $fx = gx = y$ for some $x \in X$, then x is called the coincidence point of f and g and y is called the point of coincidence of f and g.

Definition 2.14. [22] Two self mappings f and g are said to be weakly compatible if they commute at their coincidence points,that is $fx = gx$ implies that $fgx = gfx$.

Remark 2.15. [23] Let P be a cone in a real Banach space E and let $a, b, c \in P$, then,
(a) If $a \le b$ and $b \ll c$, then $a \ll c$.
(b) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(c) If $\theta \le u \ll c$, for each $c \in P^0$, then $u = \theta$.
(d) If $c \in P^0$, and $a_n \to \theta$, then there exists, $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n \ll c$.
(e) If $\le a_n \le bn$, for each n and $a_n \to a; bn \to b$, then $a \le b$.
(f) If $a \le \lambda a$ where $0 < \lambda < 1$, then $a = \theta$.

Lemma 2.16. *[22] Let T and S be weakly compatible self-mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

The following definition is about expansive mapping in cone heptagonal metric space.

Definition 2.17. Let (X, d) be a cone heptagonal metric space. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > 1$ such that $d(Tx, Ty) \ge kd(x, y)$ for all $x, y \in X$.

3. Main Results

In this section, we prove common fixed point theorems for a pair of self mappings in cone heptagonal metric spaces.

Theorem 3.1. *Let (X, d) be a cone heptagonal metric space. Suppose the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Sy) \ge kd(Tx, Ty) \tag{3.1}$$

for all $x, y \in X$, where $k > 1$. Suppose that $T(X) \subseteq S(X)$ and either of $S(X)$ or $T(X)$ is complete, then the mappings S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.

Proof. Let x_0 be any arbitrary point in X. Since $T(X) \subseteq S(X)$, we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_0$ and also we can choose $x_2 \in X$ such that $Sx_2 = Tx_1$. Continuing this process in the same way, we construct a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = Tx_n = Sx_{n+1}$ for all $n = 0, 1, 2, \cdots$.
If for some $n \in \mathbb{N}$, $y_n = y_{n-1}$ then

$$Tx_n = Sx_{n+1} = y_n = y_{n-1} = Tx_{n-1} = Sx_n$$

this implies that $Tx_n = Sx_n$ and x_n is a coincidence point of S and T.
Now, we assume that $y_n \neq y_{n-1}$ for all $n \in \mathbb{N}$. It follows from (3.1) that

$$\begin{aligned} d(y_n, y_{n-1}) &= d(Sx_{n+1}, Sx_n) \\ &\geq kd(Tx_{n+1}, Tx_n) \\ &= kd(y_{n+1}, y_n). \end{aligned}$$

This implies that,

$$d(y_{n+1}, y_n) \leq \lambda d(y_n, y_{n-1}), \ where\ \lambda = \frac{1}{k} \in (0, 1). \tag{3.2}$$

By (3.2) , we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}).$$

Similarly,

$$d(y_{n+1}, y_n) \leq \lambda^2 d(y_{n-1}, y_{n-2}).$$

Continuing the same procedure, we get

$$d(y_{n+1}, y_n) \leq \lambda^n d(y_1, y_0)\ for\ all\ n \geq 0. \tag{3.3}$$

Now we consider,

$$\begin{aligned} d(y_{n+1}, y_{n-1}) &= d(Sx_{n+2}, Sx_n) \\ &\geq kd(Tx_{n+2}, Tx_n) \\ &= kd(y_{n+2}, y_n). \end{aligned}$$

This implies that,

$$d(y_{n+2}, y_n) \leq \lambda d(y_{n+1}, y_{n-1}), \ where\ \lambda = \frac{1}{k} \in (0, 1). \tag{3.4}$$

By (3.4), we have

$$d(y_{n+1}, y_{n-1}) \leq \lambda d(y_n, y_{n-2}).$$

Similarly,

$$d(y_{n+2}, y_n) \leq \lambda^2 d(y_n, y_{n-2}).$$

Continuing the same procedure, we get

$$d(y_{n+2}, y_n) \leq \lambda^n d(y_2, y_0)\ for\ all\ n \geq 0. \tag{3.5}$$

For the sequence $\{y_n\}$, we consider $d(y_n, y_{n+p})$ in two cases, when p is even and it is odd.
Case (i): Suppose p is even, let $p = 2m, m \geq 2$ then by (3.3), (3.5) and the heptagonal

property we have,

$$\begin{aligned} d(y_n, y_{n+p}) &= d(y_n, y_{n+2m}) \\ &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) \\ &\quad + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+2m}) \\ &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) \\ &\quad + d(y_{n+5}, y_{n+6}) + ... + d(y_{n+2m-1}, y_{n+2m}) \\ &\leq \lambda^n d(y_0, y_2) + \lambda^{n+2} d(y_0, y_1) + \lambda^{n+3} d(y_0, y_1) + \lambda^{n+4} d(y_0, y_1) \\ &\quad + \lambda^{n+5} d(y_0, y_1) + ... + \lambda^{n+2m-1} d(y_0, y_1) \\ &= \lambda^n d(y_0, y_2) + \lambda^n [\lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 ... + \lambda^{2m-1}] d(y_1, y_0) \\ &\leq \lambda^n d(y_0, y_2) + (\frac{\lambda^{n+2}}{1-\lambda}) d(y_1, y_0). \end{aligned}$$

Case (2): Suppose p is odd, let $p = 2m+1, m \geq 1$ then by (3.3) and the heptagonal property, we have,

$$\begin{aligned} d(y_n, y_{n+p}) &= d(y_n, y_{n+2m+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) \\ &\quad + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+2m+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) \\ &\quad + d(y_{n+4}, y_{n+5}) + ... + d(y_{n+2m}, y_{n+2m+1}) \\ &\leq \lambda^n d(y_1, y_0) + \lambda^{n+1} d(y_1, y_0) + \lambda^{n+2} d(y_1, y_0) + \lambda^{n+3} d(y_1, y_0) \\ &\quad + \lambda^{n+4} d(y_1, y_0) + ... + \lambda^{n+2m} d(y_1, y_0) \\ &= \lambda^n [1 + \lambda + \lambda^2 + \lambda^3 + ... + \lambda^{2m}] d(y_1, y_0) \\ &\leq (\frac{\lambda^n}{1-\lambda}) d(y_1, y_0). \end{aligned}$$

Therefore, combining Case (1) and Case (2), we obtain

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \lambda^n d(y_0, y_2) + (\frac{\lambda^{n+2}}{1-\lambda}) d(y_1, y_0) + (\frac{\lambda^n}{1-\lambda}) d(y_1, y_0) \\ &= \lambda^n d(y_0, y_2) + \lambda^n (\frac{\lambda^2+1}{1-\lambda}) d(y_1, y_0). \end{aligned}$$

for all $n, p \in \mathbb{N}$.
Since, $\lambda = \frac{1}{k} \in (0,1)$, then $\lambda^n d(y_0, y_2) \to \theta$ and $\lambda^n (\frac{\lambda^2+1}{1-\lambda}) d(y_1, y_0) \to \theta$ as $n \to \infty$. So by (a) and (d) of Remark 2.15, for every $c \in E$ with $\theta \ll c$, there exits $n_0 \in \mathbb{N}$ such that $d(y_n, y_{n+p}) \ll c$ for all $n > n_0$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Suppose $T(X)$ is complete, then there exists $q \in T(X) \subseteq S(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = q.$$

Hence, $q \in S(X)$.
Let $u \in X$, be such that $Su = q$. For $\theta \ll c$ we can choose a natural number $n_0 \in \mathbb{N}$, such that $d(q, y_{n-1}) \ll \frac{c}{6}$, $d(y_n, Su) \ll \frac{kc}{6}$ and $d(y_{n-1}, y_n) = d(y_n, y_{n+2}) = d(y_{n+2}, y_{n+3}) = d(y_{n+1}, y_{n+3}) \ll \frac{c}{6}$ for all $n \geq n_0$.

By (3.1) we have

$$\begin{aligned} d(y_n, Su) &= d(Sx_{n+1}, Su) \\ &\geq kd(Tx_{n+1}, Tu) \\ &= kd(y_{n+1}, Tu). \end{aligned}$$

Thus,

$$d(y_{n+1}, Tu) \leq \frac{1}{k} d(y_n, Su)$$

for all $n \geq n_0$.
Now by heptagonal property we have:

$$\begin{aligned} d(q, Tu) &\leq d(q, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+1}, y_{n+3}) \\ &\quad + d(y_{n+1}, Tu) \\ &\leq d(q, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+1}, y_{n+3}) \\ &\quad + \frac{1}{k} d(y_n, Su). \end{aligned}$$

Thus,

$$d(q, Tu) \ll \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} = c$$

for all $n \geq n_0$.
Thus $Tu = q$ hence $Tu = Su = q$, which means that q is a point of coincidence of T and S. Next, we show that q is unique. Suppose, there exists another point of coincidence q^* such that $Tu^* = Su^* = u^*$ for some $u^* \in X$. Then, from (3.1), we have

$$\begin{aligned} d(q, q^*) &= d(Su, Su^*) \\ &\geq kd(Tu, Tu^*) \\ &= kd(q, q^*). \end{aligned}$$

Since $k > 1$, we have $d(q, q^*) = \theta$. This implies that $q = q^*$. Therefore S and T have a unique point of coincidence in X. Since S and T are weakly compatible, then by Lemma 2.16, S and T have a unique common fixed point in X. ∎

Theorem 3.2. *Let (X, d) be a cone heptagonal metric space. Suppose the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Sy) \geq k_1 d(Tx, Sx) + k_2 d(Ty, Sy) + k_3 d(Tx, Ty) + k_4 d(Sx, Ty) \tag{3.6}$$

for all $x, y \in X$, $x \neq y$ where $k_1, k_2, k_4 \geq 0$ with $k_1, k_2, k_4 < 1$ and $k_3 > 1$. Suppose that $T(X) \subseteq S(X)$ and either of $S(X)$ or $T(X)$ is complete, then the mappings S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.

Proof. Let x_0 be any arbitrary point in X. Since $T(X) \subseteq S(X)$, we can choose a point $x_1 \in X$ such that $Sx_1 = Tx_0$ and also we can choose $x_2 \in X$ such that $Sx_2 = Tx_1$. Continuing this process in the same way, we construct a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = Tx_n = Sx_{n+1}$ for all $n = 0, 1, 2, \cdots$.
If for some $n \in \mathbb{N}$, $y_n = y_{n-1}$ then

$$Tx_n = Sx_{n+1} = y_n = y_{n-1} = Tx_{n-1} = Sx_n$$

this implies that $Tx_n = Sx_n$ and x_n is a coincidence point of S and T.
Now, we assume that $y_n \neq y_{n-1}$ for all $n \in \mathbb{N}$. It follows from (3.6) that

$$\begin{aligned} d(y_n, y_{n-1}) &= d(Sx_{n+1}, Sx_n) \\ &\geq k_1 d(Tx_{n+1}, Sx_{n+1}) + k_2 d(Tx_n, Sx_n) + k_3 d(Tx_{n+1}, Tx_n) + k_4 d(Sx_{n+1}, Tx_n) \\ &= k_1 d(y_{n+1}, y_n) + k_2 d(y_n, y_{n-1}) + k_3 d(y_{n+1}, y_n) + k_4 d(y_n, y_n) \\ &= k_1 d(y_{n+1}, y_n) + k_2 d(y_n, y_{n-1}) + k_3 d(y_{n+1}, y_n). \end{aligned}$$

This implies that,

$$d(y_{n+1}, y_n) \leq \lambda d(y_n, y_{n-1}), \ where \ \lambda = \frac{1-k_2}{k_1+k_3} \in (0,1). \tag{3.7}$$

By (3.7), we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}).$$

Similarly,

$$d(y_{n+1}, y_n) \leq \lambda^2 d(y_{n-1}, y_{n-2}).$$

Continuing the same procedure, we get

$$d(y_{n+1}, y_n) \leq \lambda^n d(y_1, y_0) \ for \ all \ n \geq 0. \tag{3.8}$$

Now we consider,

$$\begin{aligned} d(y_{n+1}, y_{n-1}) &= d(Sx_{n+2}, Sx_n) \\ &\geq k_1 d(Tx_{n+2}, Sx_{n+2}) + k_2 d(Tx_n, Sx_n) + k_3 d(Tx_{n+2}, Tx_n) + k_4 d(Sx_{n+2}, Tx_n) \\ &= k_1 d(y_{n+2}, y_{n+2}) + k_2 d(y_n, y_{n-1}) + k_3 d(y_{n+1}, y_n) + k_4 d(y_n, y_n) \\ &= k_1 d(y_{n+2}, y_{n+1}) + k_2 d(y_n, y_{n-1}) + k_3 d(y_{n+2}, y_n) + k_4 d(y_{n+1}, y_n). \end{aligned}$$

This implies that,

$$d(y_{n+2}, y_n) \leq \frac{1}{k_3} d(y_{n+1}, y_{n-1}) - \frac{k_1}{k_3} d(y_{n+2}, y_{n+1}) - \frac{k_2}{k_3} d(y_n, y_{n-1}) - \frac{k_4}{k_3} d(y_{n+1}, y_n). \tag{3.9}$$

By (3.9) and heptagonal property,

$$\begin{aligned} d(y_{n+2}, y_n) &\leq \frac{1}{k_3}[d(y_{n+1}, y_{n-1}) - k_1 d(y_{n+2}, y_{n+1}) - k_2 d(y_n, y_{n-1}) - k_4 d(y_{n+1}, y_n)] \\ &\leq \frac{1}{k_3}[d(y_{n-1}, y_{n+5}) + d(y_{n+5}, y_{n+4}) + d(y_{n+4}, y_{n+3}) + d(y_{n+3}, y_{n+2})] \\ &\quad + \frac{1}{k_3}[d(y_{n+2}, y_n) + d(y_n, y_{n+1})] - \frac{k_1}{k_3} d(y_{n+2}, y_{n+1}) - \frac{k_2}{k_3} d(y_n, y_{n-1}) \\ &\quad - \frac{k_4}{k_3} d(y_{n+1}, y_n). \end{aligned} \tag{3.10}$$

Again by heptagonal property,

$$\begin{aligned} d(y_{n-1}, y_{n+5}) &\leq d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) \\ &\quad + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}). \end{aligned} \tag{3.11}$$

By (3.10) and (3.11) we have

$$\begin{aligned} d(y_{n+2},y_n) &\leq \frac{1-k_2}{k_3-1}d(y_{n-1},y_n)+\frac{2-k_4}{k_3-1}d(y_n,y_{n+1})+\frac{1-k_1}{k_3-1}d(y_{n+1},y_{n+2}) \\ &\quad +\frac{2}{k_3-1}d(y_{n+2},y_{n+3})+\frac{2}{k_3-1}d(y_{n+3},y_{n+4}) \\ &\quad +\frac{2}{k_3-1}d(y_{n+4},y_{n+5}) \\ &= \alpha d(y_{n-1},y_n)+\beta d(y_n,y_{n+1})+\delta d(y_{n+1},y_{n+2}) \\ &\quad +\eta d(y_{n+2},y_{n+3})+\gamma d(y_{n+3},y_{n+4}) \\ &\quad +\phi d(y_{n+4},y_{n+5}). \end{aligned} \tag{3.12}$$

Where,

$$\alpha=\frac{1-k_2}{k_3-1},\beta=\frac{2-k_4}{k_3-1},\delta=\frac{1-k_1}{k_3-1},\eta=\frac{2}{k_3-1},$$

$$\gamma=\frac{2}{k_3-1}\quad\text{and}\quad\phi=\frac{2}{k_3-1}.$$

For the sequence $\{y_n\}$, we consider $d(y_n,y_{n+p})$ in two cases, when p is even and it is odd.
Case (i): Suppose p is even, let $p=2m, m\geq 2$ then by (3.8), (3.12) and the heptagonal property we have,

$$\begin{aligned} d(y_n,y_{n+p}) &= d(y_n,y_{n+2m}) \\ &\leq d(y_n,y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) \\ &\quad +d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+2m}) \\ &\leq d(y_n,y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) \\ &\quad +d(y_{n+5},y_{n+6})+...+d(y_{n+2m-1},y_{n+2m}) \\ &\leq \alpha d(y_{n-1},y_n)+\beta d(y_n,y_{n+1})+\delta d(y_{n+1},y_{n+2})+\eta d(y_{n+2},y_{n+3}) \\ &\quad +\gamma d(y_{n+3},y_{n+4})+\phi d(y_{n+4},y_{n+5})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4}) \\ &\quad +d(y_{n+4},y_{n+5})+d(y_{n+5},y_{n+6})+...+d(y_{n+2m-1},y_{n+2m}) \\ &\leq \alpha\lambda^{n-1}d(y_1,y_0)+\beta\lambda^n d(y_1,y_0)+\delta\lambda^{n+1}d(y_1,y_0)+\eta\lambda^{n+2}d(y_1,y_0) \\ &\quad +\gamma\lambda^{n+3}d(y_1,y_0)+\phi\lambda^{n+4}d(y_1,y_0)+\lambda^{n+2}d(y_1,y_0)+\lambda^{n+3}d(y_1,y_0) \\ &\quad +\lambda^{n+4}d(y_1,y_0)+\lambda^{n+5}d(y_1,y_0)+...+\lambda^{n+2m-1}d(y_1,y_0) \\ &= \alpha\lambda^{n-1}d(y_1,y_0)+\beta\lambda^n d(y_1,y_0)+\delta\lambda^{n+1}d(y_1,y_0)+(1+\eta)\lambda^{n+2}d(y_1,y_0) \\ &\quad +(1+\gamma)\lambda^{n+3}d(y_1,y_0)+(1+\phi)\lambda^{n+4}d(y_1,y_0) \\ &\quad +\lambda^{n+5}[1+\lambda+\lambda^2+\lambda^3+...+\lambda^{n+2m-2}]d(y_1,y_0) \\ &\leq \alpha\lambda^{n-1}d(y_1,y_0)+\beta\lambda^n d(y_1,y_0)+\delta\lambda^{n+1}d(y_1,y_0)+(1+\eta)\lambda^{n+2}d(y_1,y_0) \\ &\quad +(1+\gamma)\lambda^{n+3}d(y_1,y_0)+(1+\phi)\lambda^{n+4}d(y_1,y_0) \\ &\quad +(\frac{\lambda^{n+5}}{1-\lambda})d(y_1,y_0). \end{aligned}$$

Case (2): Suppose p is odd, let $p = 2m+1, m \geq 1$ then by (3.8) and the heptagonal property we have,

$$\begin{aligned} d(y_n, y_{n+p}) &= d(y_n, y_{n+2m+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) \\ &\quad + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+2m+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) \\ &\quad + d(y_{n+4}, y_{n+5}) + ... + d(y_{n+2m}, y_{n+2m+1}) \\ &\leq \lambda^n d(y_1, y_0) + \lambda^{n+1} d(y_1, y_0) + \lambda^{n+2} d(y_1, y_0) + \lambda^{n+3} d(y_1, y_0) \\ &\quad + \lambda^{n+4} d(y_1, y_0) + ... + \lambda^{n+2m} d(y_1, y_0) \\ &= \lambda^n [1 + \lambda + \lambda^2 + \lambda^3 + ... + \lambda^{2m}] d(y_1, y_0) \\ &\leq (\frac{\lambda^n}{1-\lambda}) d(y_1, y_0). \end{aligned}$$

Therefore, combining Case (1) and Case (2), we obtain

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \alpha\lambda^{n-1} d(y_1, y_0) + \beta\lambda^n d(y_1, y_0) + \delta\lambda^{n+1} d(y_1, y_0) + (1+\eta)\lambda^{n+2} d(y_1, y_0) \\ &\quad + (1+\gamma)\lambda^{n+3} d(y_1, y_0) + (1+\phi)\lambda^{n+4} d(y_1, y_0) \\ &\quad + (\frac{\lambda^{n+5}}{1-\lambda}) d(y_1, y_0) + (\frac{\lambda^n}{1-\lambda}) d(y_1, y_0) \\ &= [\alpha\lambda^{n-1} + \beta\lambda^n + \delta\lambda^{n+1} + (1+\eta)\lambda^{n+2} + (1+\gamma)\lambda^{n+3} + (1+\phi)\lambda^{n+4}] d(y_1, y_0) \\ &\quad + [(\frac{\lambda^{n+5}}{1-\lambda}) + (\frac{\lambda^n}{1-\lambda})] d(y_1, y_0) \end{aligned}$$

for all $n, p \in \mathbb{N}$.
Since, $\alpha, \beta, \delta, \eta, \gamma, \phi \geq 0$ and $\lambda = \frac{1-k_2}{k_1+k_3} \in (0,1)$, then

$$[\alpha\lambda^{n-1} + \beta\lambda^n + \delta\lambda^{n+1} + (1+\eta)\lambda^{n+2} + (1+\gamma)\lambda^{n+3} + (1+\phi)\lambda^{n+4} + (\frac{\lambda^{n+5}}{1-\lambda}) + (\frac{\lambda^n}{1-\lambda})] d(y_1, y_0) \to \theta$$

as $n \to \infty$, so by (a) and (d) of Remark 2.15, for every $c \in E$ with $\theta \ll c$, there exits $n_0 \in \mathbb{N}$ such that $d(y_n, y_{n+p}) \ll c$ for all $n > n_0$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Suppose $T(X)$ is complete, then there exists $q \in T(X) \subseteq S(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = q.$$

Hence, $q \in S(X)$.
Let $u \in X$, be such that $Su = q$. For $\theta \ll c$ we can choose a natural number $n_0 \in \mathbb{N}$, such that $d(q, y_{n-1}) \ll \frac{c}{6}$, $d(y_n, Su) \ll \frac{k_3 c}{6}$ and $d(y_{n-1}, y_n) = d(y_n, y_{n+2}) = d(y_{n+2}, y_{n+3}) = d(y_{n+1}, y_{n+3}) \ll \frac{c}{6}$ for all $n \geq n_0$.
By (3.6) we have

$$\begin{aligned} d(y_n, Su) &= d(Sx_{n+1}, Su) \\ &\geq k_1 d(Tx_{n+1}, Sx_{n+1}) + k_2 d(Tu, Su) + k_3 d(Tx_{n+1}, Tu) + k_4 d(Sx_{n+1}, Tu) \\ &= k_1 d(y_{n+1}, y_n) + k_2 d(Tu, Su) + k_3 d(y_{n+1}, Tu) + k_4 d(y_n, Tu) \\ &\geq k_3 d(y_{n+1}, Tu). \end{aligned}$$

Thus,

$$d(y_{n+1}, Tu) \leq \frac{1}{k_3} d(y_n, Su)$$

for all $n \geq n_0$.
Now by heptagonal property we have:

$$\begin{aligned} d(q, Tu) &\leq d(q, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+1}, y_{n+3}) \\ &\quad + d(y_{n+1}, Tu) \\ &\leq d(q, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+1}, y_{n+3}) \\ &\quad + \frac{1}{k_3} d(y_n, Su). \end{aligned}$$

Thus,

$$d(q, Tu) \ll \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} = c$$

for all $n \geq n_0$.
Thus $Tu = q$ hence $Tu = Su = q$, which means that q is a point of coincidence of T and S. Next, we show that q is unique. Suppose, there exists another point of coincidence q^* such that $Tu^* = Su^* = u^*$ for some $u^* \in X$.
Then, from (3.6), we have

$$\begin{aligned} d(q, q^*) &= d(Su, Su^*) \\ &\geq k_1 d(Tu, Su) + k_2 d(Tu^*, Su^*) + k_3 d(Tu, Tu^*) + k_4 d(Su, Tu^*) \\ &= k_1 d(q, q) + k_2 d(q^*, q^*) + k_3 d(q, q^*) + k_4 d(q, q^*) \\ &= (k_3 + k_4) d(q, q^*). \end{aligned}$$

Since $k_3 + k_4 > 1$, we have $d(q, q^*) = \theta$. This implies that $q = q^*$. Therefore S and T have a unique point of coincidence in X. Since S and T are weakly compatible, then by Lemma 2.16, S and T have a unique common fixed point in X. ■

The following is an example in support of Theorem 3.1

Example 3.3. Let $X = \{1, 2, 3, 4, 5, 6, 7\}, E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq \theta\}$, then P is a cone in E.
We define $d : X \times X \to E$ as follows:

$$\begin{aligned} d(x, x) &= \theta, \forall x \in X, \\ (14, 42) &= d(1, 2) = d(2, 1), \\ (1, 3) &= d(1, 3) = d(3, 1) = d(1, 4) = d(4, 1) = d(1, 5) = d(5, 1) = d(1, 6) = d(6, 1) \\ &= d(2, 3) = d(3, 2) = d(3, 5) = d(5, 3) = d(6, 3) = d(3, 6) = d(4, 5) = d(5, 4) \\ &= d(2, 5) = d(5, 2) = d(6, 5) = d(5, 6) = d(6, 2) = d(2, 6), \\ (5, 15) &= d(1, 7) = d(7, 1) = d(2, 7) = d(7, 2) = d(3, 7) = d(7, 3) = d(4, 7) = d(7, 4) \\ &= d(5, 7) = d(7, 5) = d(6, 7) = d(7, 6) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) \\ &= d(4, 6) = d(6, 4). \end{aligned}$$

Then (X, d) is a complete cone heptagonal metric space but not a cone hexagonal metric space because it lacks the hexagonal property as

$$\begin{aligned}(14,42) = d(1,2) &> d(1,3) + d(3,4) + d(4,5) + d(5,6) + d(6,2) \\ &= (1,3) + (5,15) + (1,3) + (1,3) + (1,3) \\ &= (9,27).\end{aligned}$$

Since,$(14, 42) - (9, 27) = (5, 15) \in P$.
Now define mappings $S, T : X \to X$ as follows:

$$T(x) = \begin{cases} 2 & \text{if } \ x = 4, \\ 6 & \text{if } \ x \neq 4. \end{cases}$$

$$S(x) = \begin{cases} 7 & \text{if } \ x = 1, \\ 4 & \text{if } \ x = 3, \\ 2 & \text{if } \ x \textit{ is either } 7 \textit{ or } 2, \\ 6 & \text{if } \ x \textit{ is either } 5 \textit{ or } 6, \\ 3 & \text{if } \ \textit{otherwise.} \end{cases}$$

Clearly $T(X) \subseteq S(X)$ and the mappings S and T are weakly compatible. Hence, all conditions of Theorem 3.1 hold for all $x, y \in X$, with $k \in (1, 5]$ and $6 \in X$ is the unique common fixed point of S and T.

Corollary 3.4. *Let (X, d) be a complete cone heptagonal metric space and let $S : X \to X$ be an onto mapping which satisfies:*

$$d(Sx, Sy) \geq k(d(x, y))$$

for all $x, y \in X$ where $k > 1$ is a constant. Then S has a unique fixed point in X.

Proof. The result follows by taking $T = I$ (Identity map on X) in Theorem 3.1 ■

Corollary 3.5. *Let (X, d) be a complete cone heptagonal metric space and let $S : X \to X$ be an onto mapping which satisfies:*

$$d(Sx, Sy) \geq k_1 d(x, Sx) + k_2 d(y, Sy) + k_3 d(x, y) + k_4 d(Sx, y) \tag{3.13}$$

for all $x, y \in X$, $x \neq y$ where $k_1, k_2, k_4 \geq 0$ with $k_1, k_2, k_4 < 1$ and $k_3 > 1$. Then S has a unique fixed point in X.

Proof. The result follows by taking $T = I$ (Identity map on X) in Theorem 3.2 ■

Conclusion:

We introduced and proved new theorems of common fixed point for expansive mappings in cone heptagonal metric spaces under a set of conditions. Our results generalize several well known comparable results in the literature. Also, we provided an example to support the validity of our main result.

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