Thai Journal of Mathematics
Volume 20 Number 2 (2022)
Pages 629–637

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Asymptotically Ideal Invariant Equivalence of Double Sequences

Erdinç Dündar[1,*], Uğur Ulusu[2] and Fatih Nuray[1]

[1] *Department of Mathematics, Afyon Kocatepe University, 03200, Afyonkarahisar, Turkey*
e-mail : edundar@aku.edu.tr (Erdinç Dündar); fnuray@aku.edu.tr (Fatih Nuray)

[2] *Sivas Cumhuriyet University, 58140, Sivas, Turkey*
e-mail : ugurulusu@cumhuriyet.edu.tr (Uğur Ulusu)

Abstract In this study, the concepts of asymptotically $\mathcal{I}_2^\sigma$-equivalence, asymptotically invariant equivalence, strongly asymptotically invariant equivalence and p-strongly asymptotically invariant equivalence for double sequences are defined. Also, we investigate the relationships between these new type equivalence concepts.

MSC: 34C41; 40A35
Keywords: Asymptotically equivalence; invariant convergence; $\mathcal{I}$-convergence; double sequences; statistical convergence

Submission date: 09.05.2018 / Acceptance date: 09.01.2020

1. Introduction

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^k(n) \neq n$ for all positive integers n and k, where $\sigma^k(n)$ denotes the k th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are

*Corresponding author.

Published by The Mathematical Association of Thailand.

equal, is the set of almost convergent sequences $\hat{c}$. It can be shown that

$$V_\sigma = \left\{ x = (x_k) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \quad \text{uniformly in } n \right\}.$$

Several authors have studied on invariant convergent sequences (see, [16–18, 20, 21, 26–28, 30, 31, 33]). The concept of strongly σ-convergence was defined by Mursaleen in [17]:

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.

In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly σ-convergence was generalized by Savaş [27] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$.

If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

The idea of statistical convergence was introduced by Fast [11] and then studied by many authors. The concept of σ-statistically convergent sequence was presented by Savaş and Nuray in [30]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [13] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Similar concepts can be seen in [12, 20].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Recently, the concepts of σ-uniform density of subset A of the set $\mathbb{N}$ and corresponding $\mathcal{I}_\sigma$-convergence for real sequences was introduced by Nuray et al. [20].

Marouf [15] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many researchers (see, [22–24, 29, 33, 34, 36]).

Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$. It is denoted by $x \sim y$.

Convergence and $\mathcal{I}$-convergence of double sequences in a metric space and some properties of this convergence, and similar concepts which are noted following can be seen in [1–4, 6–8, 14, 19, 25, 32, 35].

A double sequence $x = (x_{mn})$ is said to be bounded if $\sup_{m,n} x_{mn} < \infty$. The set of all bounded double sequences will be denoted by ℓ_∞^2.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in\mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let (X,ρ) be a metric space and $\mathcal{I}_2$ be a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. A double sequence $x=(x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L\in X$ if for every $\varepsilon>0$

$$A(\varepsilon)=\big\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},L)\geq\varepsilon\big\}\in\mathcal{I}_2.$$

It is denoted by $\mathcal{I}_2-\lim\limits_{m,n\to\infty}x_{mn}=L$.

Let $A\subseteq\mathbb{N}\times\mathbb{N}$ and

$$s_{mn}:=\min_{s,t}\big|A\cap\big\{\big(\sigma(s),\sigma(t)\big),\big(\sigma^2(s),\sigma^2(t)\big),...,\big(\sigma^m(s),\sigma^n(t)\big)\big\}\big|$$

and

$$S_{mn}:=\max_{s,t}\big|A\cap\big\{\big(\sigma(s),\sigma(t)\big),\big(\sigma^2(s),\sigma^2(t)\big),...,\big(\sigma^m(s),\sigma^n(t)\big)\big\}\big|.$$

If the following

$$\underline{V_2}(A):=\lim_{m,n\to\infty}\frac{s_{mn}}{mn}\quad\text{and}\quad\overline{V_2}(A):=\lim_{m,n\to\infty}\frac{S_{mn}}{mn}$$

exist, then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A)=\overline{V_2}(A)$, then $V_2(A)=\underline{V_2}(A)=\overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A\subseteq\mathbb{N}\times\mathbb{N}$ with $V_2(A)=0$.

Throughout the paper, we let $\mathcal{I}_2^\sigma\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

Dündar et al. [5] studied on the concepts of invariant convergence, strongly invariant convergence, p-strongly invariant convergence and ideal invariant convergence of double sequences.

A double sequence $x=(x_{kl})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to L if for every $\varepsilon>0$, $A(\varepsilon)=\big\{(k,l):|x_{kl}-L|\geq\varepsilon\big\}\in\mathcal{I}_2^\sigma$; that is, $V_2\big(A(\varepsilon)\big)=0$. It is denoted by $\mathcal{I}_2^\sigma-\lim x=L$ or $x_{kl}\to L(\mathcal{I}_2^\sigma)$.

The set of all $\mathcal{I}_2$-invariant convergent double sequences will be denoted by $\mathfrak{I}_2^\sigma$.

A double sequence $x=(x_{kl})$ is said to be strongly invariant convergent to L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\big|x_{\sigma^k(s),\sigma^l(t)}-L\big|=0,$$

uniformly in s,t. It is denoted by $x_{kl}\to L\big([V_\sigma^2]\big)$.

A double sequence $x=(x_{kl})$ is said to be p-strongly invariant convergent to L, if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\big|x_{\sigma^k(s),\sigma^l(t)}-L\big|^p=0,$$

uniformly in s,t, where $0<p<\infty$. It is denoted by $x_{kl}\to L\big([V_\sigma^2]_p\big)$.

The set of all p-strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.

Hazarika [9] introduced the concept of asymptotically $\mathcal{I}$-equivalence and investigated it's some properties. Definitions of P-asymptotically equivalence, asymptotically statistical equivalence and asymptotically $\mathcal{I}_2$-equivalence for double sequences were presented by Hazarika and Kumar [10] as following:

Two non-negative double sequences $x = (x_{kl})$ and $x = (y_{kl})$ are said to be P-asymptotically equivalent if $P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = 1$. It is denoted by $x \sim^{P} y$.

Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left| \left\{ k \leq m, l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \right\} \right| = 0.$$

It is denoted by $x \sim^{\mathcal{S}^L} y$ and simply called asymptotically statistical equivalent if $L = 1$.

Two non-negative double sequences $x = (x_{kl})$ and $x = (y_{kl})$ are said to be asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$

$$\left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

It is denoted by $x \sim^{\mathcal{I}_2^L} y$ and simply called asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

2. Asymptotically $\mathcal{I}_2^\sigma$-Equivalence

In this section, the concepts of asymptotically $\mathcal{I}_2^\sigma$-equivalence, asymptotically σ_2-equivalence, strongly asymptotically σ_2-equivalence and p-strongly asymptotically σ_2-equivalence for double sequences are defined. Also, we investigate the relationships between these new type equivalence concepts.

Definition 2.1. Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically invariant equivalent or asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} = L,$$

uniformly in s, t. In this case, we write $x \overset{V_{2(L)}^\sigma}{\sim} y$ and simply called asymptotically σ_2-equivalent if $L = 1$.

Definition 2.2. Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L if for every $\varepsilon > 0$,

$$A_\varepsilon := \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2^\sigma,$$

i.e., $V_2(A_\varepsilon) = 0$. In this case, we write $x \overset{\mathcal{I}_{2(L)}^\sigma}{\sim} y$ and simply called asymptotically $\mathcal{I}_2^\sigma$-equivalent if $L = 1$.

The set of all asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L sequences will be denoted by $\mathfrak{I}_{2(L)}^\sigma$.

Theorem 2.3. *Suppose that $x = (x_{kl})$ and $y = (y_{kl})$ are bounded double sequences. If x and y are asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L, then these sequences are σ_2-asymptotically equivalent of multiple L.*

Proof. Let $m, n, s, t \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Now, we calculate

$$u(m,n,s,t) := \left| \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|.$$

We have

$$u(m,n,s,t) \leq u^{(1)}(m,n,s,t) + u^{(2)}(m,n,s,t),$$

where

$$u^{(1)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \geq \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|$$

and

$$u^{(2)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| < \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|.$$

We have $u^{(2)}(m,n,s,t) < \varepsilon$, for every $s, t = 1, 2, \ldots$. The boundedness of $x = (x_{kl})$ and $y = (y_{kl})$ implies that there exists an $M > 0$ such that

$$\left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \leq M,$$

for $k, l = 1, 2, \ldots,\ s, t = 1, 2, \ldots$. Then, this implies that

$$\begin{aligned} u^{(1)}(m,n,s,t) &\leq \frac{M}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \\ &\leq M \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} \\ &= M \frac{S_{mn}}{mn}, \end{aligned}$$

hence x and y are σ_2-asymptotically equivalent to multiple L. ■

The converse of Theorem 2.3 does not hold. For example, $x = (x_{kl})$ and $y = (y_{kl})$ are the double sequences defined by following;

$$x_{kl} := \begin{cases} 2 \ , & \text{if } k+l \text{ is an even integer,} \\ 0 \ , & \text{if } k+l \text{ is an odd integer.} \end{cases}$$

$$y_{kl} := 1$$

When $\sigma(s) = s+1$ and $\sigma(t) = t+1$, these sequences are asymptotically σ_2-equivalent but they are not asymptotically $\mathcal{I}_2^{\sigma}$-equivalent.

Definition 2.4. Two non-negative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be strongly asymptotically invariant equivalent or strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| = 0,$$

uniformly in s, t. In this case, we write $x \overset{[V^{\sigma}_{2(L)}]}{\sim} y$ and simply called strongly asymptotically σ_2-equivalent if $L = 1$.

Definition 2.5. Let $0 < p < \infty$. Two non-negative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be p-strongly asymptotically invariant equivalent or p-strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p = 0,$$

uniformly in s, t. In this case, we write $x \overset{[V^{\sigma}_{2(L)}]_p}{\sim} y$ and simply called p-strongly asymptotically σ_2-equivalent if $L = 1$.

The set of all p-strongly asymptotically σ_2-equivalent of multiple L sequences will be denoted by $[\mathcal{V}^{\sigma}_{2(L)}]_p$.

Theorem 2.6. *Let* $0 < p < \infty$. *Then,* $x \overset{[\mathcal{V}^{\sigma}_{2(L)}]_p}{\sim} y \Rightarrow x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$.

Proof. Assume that $x \overset{[\mathcal{V}^{\sigma}_{2(L)}]_p}{\sim} y$ and $\varepsilon > 0$ is given. For every $s, t \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{k,l=1,1}^{m,n} \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\ &\geq \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon}}^{m,n} \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\ &\geq \varepsilon^p \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^p \max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\ &\geq \varepsilon^p \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \tfrac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} \\ &= \varepsilon^p \frac{S_{mn}}{mn}. \end{aligned}$$

From by the assumption, this implies that $\lim_{m,n\to\infty} \frac{S_{mn}}{mn} = 0$ and so we get $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$. ∎

Theorem 2.7. *Let $0 < p < \infty$ and $x, y \in \ell^2_\infty$. Then, $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y \Rightarrow x \overset{[V^{\sigma}_{2(L)}]_p}{\sim} y$.*

Proof. Suppose that $x, y \in \ell^2_\infty$ and $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$. Let $\varepsilon > 0$. By assumption, we have $V_2(A_\varepsilon) = 0$. The boundedness of x and y implies that there exists an $M > 0$ such that

$$\left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \le M$$

for $k, l = 1, 2, ...,\ \ s, t = 1, 2, ...$. Observe that, for every $s, t \in \mathbb{N}$ we have

$$\begin{aligned}
&\frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\
&= \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \ge \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\
&\quad + \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| < \varepsilon}}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p \\
&\le M \frac{\max_{s,t} \left| \left\{ 1 \le k \le m, 1 \le l \le n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \ge \varepsilon \right\} \right|}{mn} + \varepsilon^p \\
&\le M \frac{S_{mn}}{mn} + \varepsilon^p.
\end{aligned}$$

Hence, the assumption, we obtain

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right|^p = 0,$$

uniformly in s, t. ∎

Theorem 2.8. *Let $0 < p < \infty$. Then, $\mathfrak{I}^{\sigma}_{2(L)} \cap \ell^2_\infty = [\mathcal{V}^{\sigma}_{2(L)}]_p \cap \ell^2_\infty$.*

Proof. This is an immediate consequence of Theorem 2.6 and Theorem 2.7. ∎

Now, we give definition of asymptotically S^{σ}_2-equivalence for double sequences and shall state a theorem that gives a relationship between asymptotically $\mathcal{I}^{\sigma}_2$-equivalence and asymptotically S^{σ}_2-equivalence of double sequences.

Definition 2.9. Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically S^{σ}_2-equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ k \le m, l \le n : \left| \frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in $s,t = 1,2,...$, (denoted by $x \overset{S^{\sigma}_{2(L)}}{\sim} y$) and simply called asymptotically S^{σ}_2-equivalent if $L = 1$.

Theorem 2.10. *The double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are asymptotically $\mathcal{I}^{\sigma}_2$-equivalent to multiple L if and only if they are asymptotically S^{σ}_2-equivalent of multiple L.*

Acknowledgements

This work is supported by the Scientific Research Project Fund of Afyon Kocatepe University under the project number 18.KARIYER.76.

References

[1] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Slovaca 58(5) (2008) 605–620.

[2] E. Dündar, B. Altay, Multipliers for bounded $\mathcal{I}_2$-convergent of double sequences, Math. Comput. Modelling 55(3-4) (2012) 1193–1198.

[3] E. Dündar, B. Altay, $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Acta Math. Sci. 34B(2) (2014) 343–353.

[4] E. Dündar, B. Altay, $\mathcal{I}_2$-uniform convergence of double sequences of functions, Filomat, 30(5) (2016) 1273–1281.

[5] E. Dündar, U. Ulusu, F. Nuray, On ideal invariant convergence of double sequences and some properties, Creat. Math. Inform. 27(2) (2018) 161–169.

[6] M. Gürdal, Some types of convergence, Doctoral Dissertation, Süleyman Demirel Univ. Isparta, 2004.

[7] M. Gürdal, A. Şahiner, Extremal $\mathcal{I}$-limit points of double sequences, Appl. Math. E-Notes, 8 (2008) 131–137.

[8] M. Gürdal, M.B. Huban, On $\mathcal{I}$-convergence of double sequences in the topology induced by random 2-norms, Mat. Vesnik. 66(1) (2014) 73–83.

[9] B. Hazarika, On asymptotically ideal equivalent sequences, J. Egyptian Math. Soc. 23 (2015) 67–72.

[10] B. Hazarika, V. Kumar, On asymptotically double lacunary statistical equivalent sequences in ideal context, J. Inequal. Appl. 2013:543 (2013) 1–15.

[11] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[12] P. Kostyrko, M. Macaj, T. Šalát, M. Sleziak, $\mathcal{I}$-convergence and extermal $\mathcal{I}$-limits points, Math. Slovaca, 55 (2005) 443–464.

[13] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-convergence, Real Anal. Exchange, 26(2) (2000) 669–686.

[14] V. Kumar, On $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Commun. 12 (2007) 171–181.

[15] M. Marouf, Asymptotic equivalence and summability, Int. J. Math. Math. Sci. 16(4) (1993) 755–762.

[16] M. Mursaleen, On finite matrices and invariant means, Indian J. Pure Appl. Math. 10 (1979) 457–460.

[17] M. Mursaleen, Matrix transformation between some new sequence spaces, Houston J. Math. 9 (1983) 505–509.

[18] M. Mursaleen, O.H.H. Edely, On the invariant mean and statistical convergence, Appl. Math. Lett. 22(11) (2009) 1700–1704.

[19] A. Nabiev, S. Pehlivan, M. Gürdal, On $\mathcal{I}$-Cauchy sequences, Taiwanese J. Math. 11(2) (2007) 569–576.

[20] F. Nuray, H. Gök, U. Ulusu, $\mathcal{I}_\sigma$-convergence, Math. Commun. 16 (2011) 531–538.

[21] F. Nuray, E. Savaş, Invariant statistical convergence and A-invariant statistical convergence, Indian J. Pure Appl. Math. 25(3) (1994) 267–274.

[22] N. Pancaroğlu Akın, E. Dündar, F. Nuray, Wijsman $\mathcal{I}$-invariant convergence of sequences of sets, Bull. Math. Anal. Appl. 11(1) (2019) 1–9.

[23] R.F. Patterson, On asymptotically statistically equivalent sequences, Demonstr. Math. 36(1) (2003) 149–153.

[24] R.F. Patterson, E. Savaş, On asymptotically lacunary statistically equivalent sequences, Thai J. Math. 4(2) (2006) 267–272.

[25] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (1900) 289–321.

[26] R.A. Raimi, Invariant means and invariant matrix methods of summability, Duke Math. J. 30(1) (1963) 81–94.

[27] E. Savaş, Some sequence spaces involving invariant means, Indian J. Math. 31 (1989) 1–8.

[28] E. Savaş, Strongly σ-convergent sequences, Bull. Calcutta Math. 81 (1989) 295–300.

[29] E. Savaş, On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences, Adv. Difference Equ. 2013 (2013) 111. doi:10.1186/1687-1847-2013-111

[30] E. Savaş, F. Nuray, On σ-statistically convergence and lacunary σ-statistically convergence, Math. Slovaca 43(3) (1993) 309–315.

[31] P. Schaefer, Infinite matrices and invariant means, Proc. Amer. Math. Soc. 36 (1972) 104–110.

[32] Ş. Tortop, E. Dündar, Wijsman $\mathcal{I}_2$-invariant convergence of double sequences of sets, J. Inequal. Spec. Funct. 9(4) (2018) 90–100.

[33] U. Ulusu, Asymptotically ideal invariant equivalence, Creat. Math. Inform. 27(2) (2018) 215–220.

[34] U. Ulusu, E. Dündar, Asymptotically lacunary $\mathcal{I}_2$-invariant equivalence, J. Intell. Fuzzy Systems 9(4) (2018) 90–100.

[35] U. Ulusu, E. Dündar, F. Nuray, Lacunary $\mathcal{I}_2$-invariant convergence and some properties, Int. J. Anal. Appl. 16(3) (2018) 317–327.

[36] U. Yamancı, M. Gürdal, On asymptotically generalized statistical equivalent double sequences via ideals, Electron. J. Math. Anal. Appl. 3(1) (2015) 89–96.