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On the Stability of a Cubic Functional Equation

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Abstract : In this paper, we establish the general solution of the following cubic functional equation

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = 48f(y)$$

and examine its generalized Hyers-Ulam-Rassias stability problem on Banach spaces.

Keywords : Cubic functional equation, Hyers-Ulam-Rassias stability
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1 Introduction

In *A Collection of Mathematical Problems*[7], the author, S.M. Ulam, posed a famous question about the stability of a functional equation. The question is that "if we replace a given functional equation by functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the strict equation?" The problem has been considered for many different types of equations in different spaces by a number of writers. For the Cauchy equation on Banach spaces, D.H. Hyers obtained the first affirmative answer in 1941 [4]. Then in 1978, Th. M. Rassias gave a generalization of Hyers's theorem [6] in the way to weaken the condition of the Cauchy differerence. Their method became a powerful tool for studying the stability of several functional equations nowadays, and have been called Hyers-Ulam-Rassias stability.

The Hyers-Ulam-Rassias stability problem of functional equation of cubic type has appeared in a paper written by K.-W. Jun and H.-M. Kim in 2002 [5]. In fact, they investigated that kind of stability for the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.1}$$

in real vector spaces.

In 2005, Chang, Jun and Jung [1] introduced a cubic type functional equation different from (1.1) as follows:

$$f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) + 7f(x) + f(-x)$$

$$= 2f(x+y) + 4f(x+z) + f(x-z) + f(y+z) + f(y-z),$$

and investigated the Hyers-Ulam-Rassias stability for this equation. Recently, Chang and Jung [2] extended there old result to the n-dimensional equation.

Consider the following functional equation:

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = 48f(y). \tag{1.2}$$

Since the cubic function $f(x) = cx^3$ satisfies this equation, it is said to be another cubic equation and every solution will be called a cubic function. In this paper, we establish the general solution of equation (1.2) and examine its generalized Hyers-Ulam-Rassias stability problem on Banach spaces. We also obtain 2 consequent corollaries in sense of Hyers-Ulam stability and Hyers-Ulam-Rassias stability, respectively.

2 The General Solution

Let E_1 and E_2 be real Banach spaces.

Theorem 2.1. *A function $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if there exists a tri-additive symmetric function $A_3 : E_1^3 \to E_2$ such that $f(x) = A_3(x, x, x)$ for all $x \in E_1$.*

Proof. (Necessity). From (1.2), putting $x = y = 0$ yields $f(0) = 0$. Note that the left hand side of (1.2) changes sign when y is replaced by $-y$. Thus f is odd.

One can verify that $f(2x) = 8f(x)$ and $f(3x) = 27f(x)$. By induction, we infer that $f(kx) = k^3 f(x)$ for all positive integer k.

Replacing x by $x + 2y$ in (1.2), we have

$$f(x+5y) - 3f(x+3y) + 3f(x+y) - f(x-y) = 48f(y). \tag{2.1}$$

According to (1.2) and (2.1), we obtain

$$f(x+5y) - 4f(x+3y) + 6f(x+y) - 4f(x-y) + f(x-3y) = 0. \tag{2.2}$$

The substitution $x = x + 3y/2$ and $y = y/2$ in (2.2) gives the relation

$$f(x+4y) - 4f(x+3y) + 6f(x+2y) - 4f(x+y) + f(x) = 0. \tag{2.3}$$

Thus f also satisfies the difference functional equation

$$\Delta_y^4 f(x) = 0. \tag{2.4}$$

Here Δ_y is the forward difference operator with the span y.

For each $n = 0, 1, 2, \ldots$, let $A_n : E_1^n \to E_2$ be a symmetric n-additive function and let $A^n : E_1 \to E_2$ be its diagonalization. Let $s \in \mathbb{N}$, it is known that [3] the general solution of the difference functional equation

$$\Delta_y^{s+1} f(x) = 0.$$

is given by a generalized polynomial of degree at most s such that

$$f(x) = \sum_{n=0}^{s} A^n(x).$$

Hence the general solution of (2.4) is in the form

$$f(x) = A^0(x) + A^1(x) + A^2(x) + A^3(x), \tag{2.5}$$

where $A^0(x) \equiv A^0$ is taken to be a constant.

Making use of the oddness of f, we have $A^0 = 0$ and $A^2(x) = 0$, for all $x \in E_1$. Thus (2.5) is reduced to

$$f(x) = A^1(x) + A^3(x). \tag{2.6}$$

Substitute (2.6) into (1.2) to obtain

$$\begin{aligned} & A_1(x+3y) + A^3(x+3y) - 3A_1(x+y) - 3A^3(x+y) \\ & +3A_1(x-y) + 3A^3(x-y) - A_1(x-3y) - A^3(x-3y) \\ = \ & 48A_1(y) + 48A^3(y). \end{aligned}$$

On account of the additivity of A_1, it follows that

$$A^3(x+3y) - 3A^3(x+y) + 3A^3(x-y) - A^3(x-3y) = 48A_1(y) + 48A^3(y).$$

Finally, replacing y by x in the above equation, we arrive at

$$A^3(4x) - 3A^3(2x) - A^3(-2x) = 48A_1(x) + 48A^3(x).$$

Observing that $A^3(nx) = n^3A^3(x)$ for all $n \in \mathbb{Z}$ and all $x \in E_1$, the above equation turns into

$$48A^3(x) = 48A_1(x) + 48A^3(x).$$

Now we can conclude that $A_1(x) \equiv 0$ for all $x \in E_1$. That is $f(x) = A^3(x) = A_3(x,x,x)$ for all $x \in E_1$.

(Sufficiency). Assume that there exists a tri-additive symmetric function $A_3 : E_1^3 \to E_2$ such that $f(x) = A_3(x,x,x)$ for all $x \in E_1$. Making use of the fact that

$$\Delta_y^3 A^3(x) = 3!A(y)$$

[3], we have

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = \Delta_{2y}^3 f(x-3y) = 3!f(2y) = 48f(y).$$

That is f satisfies the equation (1.2). This completes the proof of the theorem. □

3 The Generalized Hyers-Ulam-Rassias Stability

Theorem 3.1. *Let an even function $\phi : E_1 \times E_1 \to [0, \infty)$ satisfy the following conditions*

$$\sum_{k=0}^{\infty} \frac{\phi(0, 3^k y)}{27^k} < \infty \quad and \quad \lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{27^n} = 0, \tag{3.1}$$

or

$$\sum_{k=1}^{\infty} 27^k \phi(0, \frac{y}{3^k}) < \infty \quad and \quad \lim_{n\to\infty} 27^n \phi(\frac{x}{3^n}, \frac{y}{3^n}) = 0, \tag{3.2}$$

for all $x, y \in E_1$. Suppose that a function $f : E_1 \to E_2$ satisfies

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\| \le \phi(x, y) \tag{3.3}$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \to E_2$ which satisfies the inequality

$$\|f(y) - C(y)\| \le \begin{cases} \dfrac{1}{48} \displaystyle\sum_{k=0}^{\infty} \frac{1}{27^k} \left(\phi(0, 3^k y) + \frac{\phi(0, 3^{k+1} y)}{27} \right) & if\ (3.1)\ holds, \\ \dfrac{1}{48} \displaystyle\sum_{k=1}^{n} 27^k \left(\phi(0, \frac{y}{3^k}) + \frac{1}{27}\phi(0, \frac{y}{3^{k-1}}) \right) & if\ (3.2)\ holds, \end{cases} \tag{3.4}$$

for all $y \in E_1$.

Proof. Replacing y by $-y$ in (3.3) and adding the result to (3.3) yield

$$\|f(y) + f(-y)\| \le \phi(x, y)/24 \tag{3.5}$$

for all $x \in E_1$. Since (3.3) and (3.5) hold for any x, let us fix $x = 0$ for convenience. Thus, from the two inequalities, we have that

$$\begin{aligned} \|2f(3y) - 54f(y)\| &\le \phi(0, y) + \|f(3y) + f(-3y) - 3(f(y) + f(-y))\| \\ &\le \phi(0, y) + \|f(3y) + f(-3y)\| + 3\|f(y) + f(-y)\| \\ &\le \frac{27}{24}\phi(0, y) + \frac{1}{24}\phi(0, 3y) \end{aligned} \tag{3.6}$$

for all $y \in E_1$. We will first consider for the case when the condition (3.1) holds. Dividing both sides by 54, lead us to

$$\|\frac{f(3y)}{27} - f(y)\| \le \frac{1}{48}\left(\phi(0, y) + \frac{\phi(0, 3y)}{27} \right) \tag{3.7}$$

for all $y \in E_1$. Making use of the triangle inequality, it follows that

$$\begin{aligned} \|\frac{f(3^n y)}{27^n} - f(y)\| &= \|\sum_{k=0}^{n-1} \left(\frac{f(3^{k+1}y)}{27^{k+1}} - \frac{f(3^k y)}{27^k} \right) \| \\ &\leq \sum_{k=0}^{n-1} \frac{1}{27^k} \|\frac{f(3^{k+1}y)}{27} - f(3^k y)\| \\ &\leq \frac{1}{48} \sum_{k=0}^{n-1} \frac{1}{27^k} \left(\phi(0, 3^k y) + \frac{\phi(0, 3^{k+1}y)}{27} \right) \end{aligned} \tag{3.8}$$

for all $y \in E_1$ and integer $n \geq 1$. For any positive integer m, we divide (3.11) by 27^m and replace y by $3^m y$ to obtain that

$$\|\frac{f(3^{n+m}y)}{27^{n+m}} - \frac{f(3^m y)}{27^m}\| \leq \frac{1}{48} \sum_{k=0}^{n-1} \frac{1}{27^{k+m}} \left(\phi(0, 3^{k+m}y) + \frac{\phi(0, 3^{k+m+1}y)}{27} \right) \tag{3.9}$$

for all $y \in E_1$. This shows that $\{f(3^n y)/27^n\}$ is a Cauchy sequence in E_1 because the right hand side of (3.9) converges to zero by the assumption of ϕ when $m \to \infty$. Since E_2 is a Banach space, it follows that the sequence $\{f(3^n y)/27^n\}$ converges. We define $C : E_1 \to E_2$ by

$$C(x) = \lim_{n \to \infty} \frac{f(3^n x)}{27^n}$$

for all $x \in E_1$. Letting $n \to \infty$ in (3.11), we get the inequality (3.4).

To show that C satisfies the equation (1.2) for all $x, y \in E_1$, we replace (x, y) by $(3^n x, 3^n y)$ in (3.3) and then divide by 27^n as the following:

$$\begin{aligned} &\frac{1}{27^n}\|f(3^n(x+3y)) - 3f(3^n(x+y)) + 3f(3^n(x-y)) - f(3^n(x-3y)) - 48f(3^n y)\| \\ &\qquad \leq \frac{1}{27^n}\phi(3^n\|x\|, 3^n\|y\|). \end{aligned}$$

Applying the condition (3.1), the right hand side of the above inequality becomes 0 as n approaches ∞. Now we can say that C is a cubic function.

Let $\tilde{C} : E_1 \to E_2$ be another cubic function satisfying (1.2) and (3.4). Hence it follows from (3.4) that

$$\begin{aligned} \|C(y) - \tilde{C}(y)\| &= 27^{-n}\|C(3^n y) - \tilde{C}(3^n y)\| \\ &\leq 27^{-n}(\|C(3^n y) - f(3^n y)\| + \|f(3^n y) - \tilde{C}(3^n y)\|) \\ &\leq \frac{1}{24} \sum_{k=0}^{\infty} \frac{1}{27^{n+k}} \left(\phi(0, 3^{k+n}y) + \frac{\phi(0, 3^{k+n+1}y)}{27} \right). \end{aligned}$$

for all $y \in E_1$. Taking the limit as $n \to \infty$, it is immediate that $C(x) = \tilde{C}(x)$ for all $x \in E_1$. This proves the uniqueness of C.

Next, for the case when the condition (3.2) holds, we can state the proof in the same pattern as we did in the first case. We start by replacing y by $y/3$ and dividing both sides by 2 in (3.6) to have that

$$\|f(y) - 27f(\frac{y}{3})\| \leq \frac{1}{48}\Big(27\phi(0, \frac{y}{3}) + \phi(0, y)\Big) \tag{3.10}$$

for all $y \in E_1$. Applying the triangle inequality, we can extend (3.10) to

$$\begin{aligned} \|27^n f(\frac{y}{3^n})\| &= \|\sum_{k=0}^{n-1}\Big(27^k f(\frac{y}{3^k}) - 27^{k+1} f(\frac{y}{3^{k+1}})\Big)\| \\ &\leq \sum_{k=0}^{n-1} 27^k \|f(\frac{y}{3^k}) - 27f(\frac{y}{3^{k+1}})\| \\ &\leq \frac{1}{48}\sum_{k=0}^{n-1} 27^k \Big(27\phi(0, \frac{y}{3^{k+1}}) + \phi(0, \frac{y}{3^k})\Big) \\ &= \frac{1}{48}\sum_{k=1}^{n} 27^k \Big(\phi(0, \frac{y}{3^k}) + \frac{1}{27}\phi(0, \frac{y}{3^{k-1}})\Big) \end{aligned} \tag{3.11}$$

for all $y \in E_1$ and a positive integer n.

The same idea was used to show that $\Big\{27^n f(\frac{y}{3^n})\Big\}$ is a Cauchy sequence. Thus a mapping

$$C(x) = \lim_{n\to\infty} 27^n f(\frac{x}{3^n})$$

from E_1 to E_2 is well-defined. Moreover, C satisfies (1.2) and (3.4). The proof for the uniqueness of C for this case proceeds similarly to that in the previous case, thus will be omitted. □

We also obtain the following corollaries concerning the stability of the equation (1.2) in the sense of Hyers-Ulam stability.

Corollary 3.2. *Suppose that a function $f : E_1 \to E_2$ satisfies*

$$\|f(x + 3y) - 3f(x + y) + 3f(x - y) - f(x - 3y) - 48f(y)\| \leq \varepsilon \tag{3.12}$$

for all $x, y \in E_1$. Then there exists a unique cubic function $C : E_1 \to E_2$ which satisfies the inequality

$$\|f(y) - C(y)\| \leq \frac{\delta}{26} \tag{3.13}$$

for all $y \in E_1$, where $\delta = \frac{7\varepsilon}{12}$.

Proof. According to Theorem3.1, if we select $\phi(x,y) = \varepsilon$ for all $x, y \in E_1$, the condition (3.1) is fulfilled. Consequently, there exists a unique cubic function $C : E_1 \to E_2$ such that

$$\begin{aligned} \|f(y) - C(y)\| &\leq \frac{1}{48}\sum_{k=0}^{\infty}\frac{1}{27^k}\left(\varepsilon + \frac{\varepsilon}{27}\right) \\ &= \frac{\delta}{26} \end{aligned} \tag{3.14}$$

for all $y \in E_1$ as desired. □

The following corollary is the Hyers-Ulam-Rassias stability of the equation (1.2) which also is an immediate consequence of Theorem(3.1).

Corollary 3.3. *Suppose that a function $f : E_1 \to E_2$ satisfies*

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{3.15}$$

for all $x, y \in E_1$, when $p \neq 3$. Then there exists a unique cubic function $C : E_1 \to E_2$ which satisfies the inequalities

$$\|f(y) - C(y)\| \leq \frac{\varepsilon(27 + 3^p)\|y\|^p}{48|27 - 3^p|} \tag{3.16}$$

for all $y \in E_1$.

Proof. This time we choose $\phi(x,y) = \varepsilon(\|x\|^p + \|y\|^p)$. After checking for the value of p to fulfill the 2 conditions in Theorem(3.1), we divide $p \neq 3$ into 2 intervals as follow.
If $0 < p < 3$, the condition (3.1) holds. Consequently, there exists a unique cubic function $C : E_1 \to E_2$ which satisfies the inequality

$$\begin{aligned} \|f(y) - C(y)\| &\leq \frac{1}{48}\sum_{k=0}^{\infty}\frac{1}{27^k}\left(\varepsilon\|3^k y\|^p + \frac{\varepsilon\|3^{k+1}y\|^p}{27}\right) \\ &= \frac{\varepsilon(27 + 3^p)\|y\|^p}{48 \cdot 27}\sum_{k=0}^{\infty}\left(\frac{3^p}{27}\right)^k \\ &= \frac{\varepsilon(27 + 3^p)\|y\|^p}{48(27 - 3^p)} \end{aligned} \tag{3.17}$$

for all $y \in E_1$.
If $p > 3$, the condition (3.2) holds Consequently, there exists a unique cubic

function $C : E_1 \to E_2$ which satisfies the inequality

$$\begin{aligned} \|f(y) - C(y)\| &\leq \frac{1}{48}\sum_{k=1}^{\infty} 27^k \left(\varepsilon \| \frac{y}{3^k} \|^p + \frac{\varepsilon \| \frac{y}{3^{k-1}} \|^p}{27} \right) \\ &= \frac{\varepsilon(27+3^p)\|y\|^p}{48 \cdot 27} \sum_{k=1}^{\infty} \left(\frac{27}{3^p} \right)^k \\ &= \frac{\varepsilon(27+3^p)\|y\|^p}{48(3^p-27)} \end{aligned} \tag{3.18}$$

for all $y \in E_1$. Both cases of consideration complete our proof. □

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