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Generalized Ćirić's Pairs of Maps and Some Systems of Nonlinear Integral Equations

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Abstract In this article, we establish common fixed point results for a class of pair of Ćirić's type mappings assuming weakly compatibility, which is a minimal noncommuting notion for contractive pair of mappings. Additionally, we prove existence and uniqueness common of fixed points for this pair of mappings satisfying alternative properties as the so-called property E.A., reciprocal continuity and CLR_S-property. Those results allow us to analyze the existence of solutions for some system of nonlinear integral equations on (not necessarily complete) metric spaces.

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1. Introduction and Preliminary Notions

In 1976, G. Jungck [14] began the study of common fixed points for a pair of mappings, by generalizing the Banach-Caccioppoli contraction principle through the consideration of two commuting mappings, proving a common fixed point theorem for these mappings. Afterwards, the commutative property of the mappings assumed by Jungck has been relaxed by introducing "weak" alternative notions as *weak commutating*, (*non-*) *compatibility*, *R-weak commutativity* and *weak compatibility* among others, which allowed to extend several well-known common fixed point results for Lipschitz type of mapping pairs.

Here, firstly, we are going to establish existence and uniqueness results of common fixed point for a pair of contractive-type of mappings, whose contractive parameters are non-constants and its contractive inequality is controlled by a positive function satisfying a stability condition at 0 (see (2.2)). As a particular case, our results are valid if we control the mentioned inequality by using the well-known *altering distance functions*, which have been used to solve several problems in the metric fixed point theory (see, e.g., [10, 11, 13, 22, 23, 25, 26, 32]). Then, with those results and relation (4.2), we can analyze

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the existence of solutions for a system of nonlinear integral equations defined on a not necessarily complete metric space.

To attain our goals, we will assume that the mappings under consideration are *weakly compatible*, which is a minimal noncommuting notion for contractive type of mapping pairs (see, Section (5)). Also, alternatively, we are going to assume that the pair of mappings satisfy some conditions like *property E. A.* introduced in 2002 by Aamri and Moutawaki [1], *reciprocal continuous* introduced by R. Pant [29] in 1998 and the so-called CLR_S-*property* given by Sintunavarat and Kumam in 2011 [33].

1.1. Preliminary notions and results

A point $x \in M$ is called a *coincidence point* (CP) of S and T if $Sx = Tx$. The set of coincidence points of S and T will be denoted by C(S,T). If $x \in C(S,T)$, then $w = Sx = Tx$ is called a *point of coincidence* (POC) of S and T. In order to establish our results the following notions will be needed: A pair of mappings (S,T) on a metric space (M,d) is said to be

Non-trivially weakly compatible (WC), [16]: if they commute at their coincidence points, that is, $STu = TSu$ whenever $Su = Tu$, for some $u \in M$.

Occasionally weakly compatible (OWC), [3]: if there exists some $x \in C(S,T)$ such that $STx = TSx$.

Compatible, [15]: if and only if $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$, for whenever sequence $(x_n)_n \subset M$ is such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in M$.

Noncompatible, [1]: if there exits at least one sequence $(x_n)_n \subset M$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in M$, but

$$\lim_{n\to\infty} d(STx_n, TSx_n)$$

is either nonzero or non-existent.

For more details on comparisons of the early notions, we refer to [17, 24].

The following notion is a generalization of noncompatibility: A pair of self mappings (S,T) is said to satisfy the *property E.A.*, [1], if there exists a sequence $(x_n)_n \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in M$. If two maps are noncompatible, they satisfy property E.A., the converse however is not true [4]. It is important to point out that property E.A. requires either completeness of the whole space or any of the range spaces or continuity of the maps. But, on the contrary, the following notion does not need such conditions. A pair of self-mappings (S,T) is said to satisfy the *common limit in the range property with respect to* S (in short CLR_S) [33], if there exists a sequence $(x_n)_n \subset M$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = St,$$

for some $t \in M$. Notice that a pair (S,T) satisfying property E.A. along with closedness of the subspace $S(M)$ always enjoy CLR_S-property, (see [33]). It may be observed that the CLR_S-property avoids the requirement of the condition of closedness of the ranges of the involved mappings.

Two maps S and T are called *reciprocal continuous* (RC) if $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$, whenever $(x_n)_n$ is a sequence such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in M$. If S and T are both continuous, then they are reciprocal continuous, but the converse is not true [29]. As an application of this concept, in 1998, R. Pant [29] obtained the first result that establish a situation in which a collection of mappings has a fixed point which is a point of discontinuity for all the mappings. In 2011, Pant et al [31] generalized the notion of reciprocal continuity by introducing a new concept, the *weak reciprocal continuity*, as follows: Two self mappings S and T are weak reciprocal continuous (WRC) if $\lim_{n\to\infty} STx_n = Sz$ or $\lim_{n\to\infty} TSx_n = Tz$, whenever $(x_n)_n$ is a sequence such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in M$. If S and T are reciprocally continuous, they are weakly reciprocally continuous but the converse is not true [31].

On the other hand, the following results in will be useful in the sequel.

Lemma 1.1 ([5]). *Let (M, d) be a metric space. Let $(x_n)_n$ be a sequence in M such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $(x_n)_n$ is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $(m(k))_k$ and $(n(k))_k$ with

$$m(k) > n(k) > k$$

such that,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0, \quad d(x_{m(k)-1}, x_{n(k)}) < \varepsilon_0$$

and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon_0,$

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0,$

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon_0.$

Lemma 1.2 ([2]). *Let S and T be self mappings of a metric space (M, d). If the pair (S, T) is WC and has a unique POC, then it has a unique common fixed point.*

2. The Generalized Ćirić-type Class of Mappings

In order to introduce the class of mappings which will be the focus of study in this paper, as in [20], we are going to use the functions $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ which satisfy that $\alpha(t) + \beta(t) < 1$, for all $t \in \mathbb{R}_+$, and

$$\limsup_{s\to 0^+} \alpha(s) < 1$$

$$\limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1, \quad \forall t > 0. \tag{2.1}$$

In the sequel, by Ψ we mean the class of all functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ continuous satisfying that

$$\psi(t) \to 0 \text{ implies that } t_n \to 0. \tag{2.2}$$

Remark 2.1. Since every non decreasing map ψ satisfies (2.2) (but the converse is not true), then all the results that we are going to prove here are valid, in particular, if we replace functions satisfying the condition (2.2) for functions belonging to the well-known class of altering distance functions [18, 34].

Now, we introduce the following class of pair of contraction-type of mappings which generalizes several classes of mappings, by considering α, β constants functions, as well as ψ, T be the identity map.

Definition 2.2. Let (M, d) be a metric space and let $S, T : M \longrightarrow M$ be mappings. Then, the pair (S, T) es called $\psi - (\alpha, \beta)$-Ćirić generalized contraction pair ($\psi - (\alpha, \beta)$-Cgc) if for all $x, y \in M$

$$\psi(d(Sx, Sy)) \leq \alpha\left(d(Tx, Ty)\right)\psi\left(d(Tx, Ty)\right) + \beta\left(d(Tx, Ty)\right)\psi\left(M(x, y)\right) \tag{2.3}$$

where $\psi \in \Psi$, α, β satisfy (2.1) and

$$M(x, y) = \max\left\{d(Tx, Ty), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}. \tag{2.4}$$

The next result shows that a $\psi - (\alpha, \beta)$-Cgc map can has, at most, one POC.

Proposition 2.3. *Let S and T two self mappings of a metric space (M, d). Assume that the pair (S, T) is a $\psi - (\alpha, \beta)$-Cgc pair. If S and T have a POC in M, then it is unique.*

Proof. Let $\omega \in M$ be a POC of S and T. Then there exists $x \in M$ such that $Sx = Tx = \omega$. Suppose that for some $y \in M$, $Sy = Ty = \upsilon$ and $\omega \neq \upsilon$. From (2.3) we have that

$$\begin{aligned}\psi(d(\omega, \upsilon)) =& \psi\left(d(Sx, Sy)\right) \leq \alpha\left(d(Tx, Ty)\right)\psi\left(d(Tx, Ty)\right) \\ &+ \beta\left(d(Tx, Ty)\right)\psi\left(M(x, y)\right).\end{aligned} \tag{2.5}$$

It follows that,

$$\psi(d(\omega, \upsilon)) = \alpha\left(d(\omega, \upsilon)\right)\psi\left(d(\omega, \upsilon)\right) + \beta\left(d(\omega, \upsilon)\right)\psi\left(M(\omega, \upsilon)\right).$$

Using (2.4) we obtain $M(\omega, \upsilon) = d(\omega, \upsilon)$ and substituting it into (2.5) we obtain

$$\psi(d(\omega, \upsilon)) \leq [\alpha\left(d(\omega, \upsilon)\right) + \beta\left(d(\omega, \upsilon)\right)]\psi\left(d(\omega, \upsilon)\right) < \psi\left(d(\omega, \upsilon)\right).$$

Since $\psi \in \Psi$ we conclude that $d(\omega, \upsilon) < d(\omega, \upsilon)$ which is a contradiction. Thus, $\omega = \upsilon$. ∎

In the next section, we will use the following proposition to prove the existence of a unique common fixed point of a $\psi - (\alpha, \beta)$-Cgc pair.

Proposition 2.4. *Let (M, d) be a metric space and let S and T be self mappings of M with $SM \subset TM$. If the pair (S, T) is a $\psi - (\alpha, \beta)$-Cgc pair, then for any $x_0 \in M$ the sequence defined by*

$$y_n = Sx_n = Tx_{n+1}, \qquad n = 0, 1, \ldots$$

satisfies

(i) $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
(ii) *$(y_n)_n$ is a Cauchy sequence in M.*

Proof.

(i) Let x_0 be an arbitrary point. Since $SM \subset TM$, there exits an $x_1 \in M$ such that $Sx_0 = Tx_1$. In this way we construct a sequence $y_n \in M$ by the formula $y_n = Sx_n = Tx_{n+1}$, $n = 0, \dots$. From (2.3) we have that

$$\begin{aligned}&\psi\left(d(Tx_{n+1}, Tx_{n+2})\right) = \\ &\psi\left(d(Sx_n, Tx_{n+1})\right) \le \alpha\left(d(Tx_n, Tx_{n+1})\right)\psi\left(d(Tx_n, Tx_{n+1})\right) \\ &+ \beta\left(d(Tx_n, Tx_{n+1})\right)\psi\left(M(x_n, x_{n+1})\right).\end{aligned}$$

Using (2.4) we obtain

$$\begin{aligned}&M(x_n, x_{n+1}) \\ &= \max\left\{d(Tx_n, Tx_{n+1}), d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2}), \frac{1}{2}d(Tx_n, Tx_{n+2})\right\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}d(Tx_n, Tx_{n+1}) \le& \frac{1}{2}[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x_{n+2})] \\ \le& \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\}.\end{aligned}$$

Therefore,

$$M(x_n, x_{n+1}) = \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\}.$$

If $M(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1})$, then

$$\begin{aligned}&\psi\left(d(Tx_{n+1}, Tx_{n+2})\right) \\ &\le [\alpha\left(d(Tx_n, Tx_{n+1})\right) + \beta\left(d(Tx_n, Tx_{n+1})\right)]\psi\left(d(Tx_n, Tx_{n+1})\right)]\end{aligned} \tag{2.6}$$

and when $M(x_n, x_{n+1}) = d(Tx_{n+1}, Tx_{n+2})$, we have

$$\psi\left(d(Tx_{n+1}, Tx_{n+2})\right) \le \frac{\alpha\left(d(Tx_n, Tx_{n+1})\right)}{1 - \beta\left(d(Tx_n, Tx_{n+1})\right)}\psi\left(d(Tx_n, Tx_{n+1})\right). \tag{2.7}$$

In both cases, from inequalities (2.6) and (2.7) we conclude that

$$\psi\left(d(Tx_{n+1}, Tx_{n+2})\right) < \psi\left(d(Tx_n, Tx_{n+1})\right),$$

since $\psi \in \Psi$ it follows that $(d(Tx_n, Tx_{n+1}))_n$ is a monotone sequence of positive real numbers and consequently there exits $L \ge 0$ such that $\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = L$. Notice that

$$\begin{aligned}0 \le& \psi(L) = \lim_{n\to\infty} \psi\left(d(Tx_{n+1}, Tx_{n+2})\right) \\ \le& \limsup_{n\to\infty} \alpha\left(d(Tx_n, Tx_{n+1})\right)\psi(L) + \limsup_{n\to\infty} \beta\left(d(Tx_n, Tx_{n+1})\right)\psi(L).\end{aligned}$$

From here, L should be identically equal to 0.

(ii) We shall prove that $(Sx_n)_n = (Tx_{n+1})_n$ is a Cauchy sequence in $T(M)$. Suppose that it is not true. Then there exists an $\varepsilon > 0$ and there exist sequences $(m(k))_k$ and $(n(k))_k$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon, \qquad d(x_{m(k)-1}, x_{n(k)}) < \varepsilon.$$

From Lemma 1.1 we have

$$0 < \psi(\varepsilon) = \limsup_{k\to\infty} \psi\left(d(Tx_{m(k)}, Tx_{n(k)})\right) = \limsup_{k\to\infty} \psi\left(d(Sx_{m(k)-1}, Tx_{n(k)-1})\right)$$

$$\leq \limsup_{k\to\infty} \alpha\left(d(Tx_{m(k)-1}, Tx_{n(k)-1})\right)\psi\left(d(Tx_{m(k)-1}, Tx_{n(k)-1})\right)$$
$$+ \limsup_{k\to\infty} \beta\left(d(Tx_{m(k)-1}, Tx_{n(k)-1})\right)\psi\left(M(x_{m(k)-1}, x_{n(k)-1})\right)$$

where,

$$\begin{aligned}
&M(x_{m(k)-1}, x_{n(k)-1}) \\
&= \max\Big\{d(Tx_{m(k)-1}, Tx_{n(k)-1}), d(Sx_{m(k)-1}, Sx_{n(k)-1}), \\
&\quad d(Sx_{m(k)-1}, Tx_{n(k)-1}), \frac{1}{2}\left[d(Sx_{m(k)-1}, Tx_{n(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)-1})\right]\Big\} \\
&= \max\Big\{d(Tx_{m(k)-1}, Tx_{n(k)-1}), d(Tx_{m(k)}, Sx_{n(k)-1}), \\
&\quad d(Tx_{m(k)}, Tx_{n(k)-1}), \frac{1}{2}\left[d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)-1})\right]\Big\}.
\end{aligned}$$

It follows that $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}) = \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon$. Therefore,

$$\begin{aligned}
0 < \psi(\varepsilon) &= \limsup_{k\to\infty}\left(\alpha\left(d(Tx_{m(k)-1}, Tx_{n(k)-1})\right) + \beta\left(d(Tx_{m(k)-1}, Tx_{n(k)-1})\right)\right)\psi(\varepsilon) \\
&< \psi(\varepsilon),
\end{aligned}$$

Which is a contradiction. Hence $(T(x_n))_n$ is a Cauchy sequence in TM. ∎

Now, we prove that the (S,T)-Picard iteration scheme given in Proposition 2.4, converges to the common fixed point of the pair (S,T).

Proposition 2.5. *Let (M,d) be a metric space and let S and T be self mappings of M with $SM \subset TM$ and TM closed. If the pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair with a common fixed point, then for any $x_0 \in M$ the sequence defined by*

$$y_n = Sx_n = Tx_{n+1}, \qquad n = 0, 1, \ldots$$

converges to the common fixed point of (S,T).

Proof. Let x_0 be an arbitrary point. Since $SM \subset TM$, there exits an $x_1 \in M$ such that $Sx_0 = Tx_1$. In this way we construct a sequence $y_n \in TM$ by the formula $y_n = Sx_n = Tx_{n+1}$, $n = 0, \ldots$. Since $(y_n)_n$ is a Cauchy sequence and TM is closed, $y_n \to y \in TM$ as $n \to \infty$. Let $p \in M$ be a common fixed point of (S,T) such that $p \neq y$, from (2.3) we have that

$$\begin{aligned}
&\psi\left(d(Sx_n, Sp)\right) \leq \alpha\left(d(Tx_n, Tp)\right)\psi\left(d(Tx_n, Tp)\right) \\
&+ \beta\left(d(Tx_n, Tp)\right)\psi\left(M(x_n, p)\right).
\end{aligned}$$

Using (2.4) we obtain

$$\begin{aligned}
&M(x_n, p) \\
&= \max\left\{d(Tx_n, Tp), d(Sx_n, Tx_n), d(Sp, Tp), \frac{1}{2}(d(Sx_n, Tp) + d(Sp, Tx_n))\right\}.
\end{aligned}$$

Notice that $\lim_{n\to\infty} \psi\left(M(x_n, p)\right) = \psi\left(d(y, p)\right)$. Therefore,

$$\begin{aligned}
\psi(d(y,p)) = \lim_{n\to\infty} \psi\left(d(Sx_n, Sp)\right) &\leq \limsup_{n\to\infty}[\alpha\left(d(Tx_n, Tp)\right)\psi\left(d(Tx_n, Tp)\right) \\
&\quad + \beta\left(d(Tx_n, Tp)\right)\psi\left(M(x_n, p)\right)] \\
&= \limsup_{n\to\infty}[\alpha\left(d(Tx_n, Tp)\right)
\end{aligned}$$

$$+\beta\left(d(Tx_n,Tp)\right)]\psi\left(d(y,p)\right)$$
$$<\psi\left(d(y,p)\right).$$

Which is a contradiction, thus

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = y = p = Sp = Tp.$$

That is, the (S,T)-Picard iteration process converges to the common fixed point of (S,T). ■

3. Common Fixed Points for $\psi-(\alpha,\beta)$-Cgc Pair of Mappings

In this section we prove our main results concerning to the existence and uniqueness of common fixed points for a $\psi-(\alpha,\beta)$-Cgc pair of mappings. We are going to give common fixed point results assuming the completeness only on a specific subspace of the space M and the maps S and T are not necessarily continuous.

Theorem 3.1. *Let S and T be WC self mappings of a metric space (M,d) such that*

(i) $SM \subset TM$.
(ii) $TM \subset TM$ *is a complete subspace of* M.
(iii) *The pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair.*

Then, S and T have a unique common fixed point.

Proof. Let $x_0 \in M$ be an arbitrary point. Since $SM \subset TM$, as before we construct the sequence $y_n = Sx_n = Tx_{n+1}$, $n = 0,\ldots$. By Proposition 2.4 we have that $(y_n)_n$ is a Cauchy sequence in TM which is complete, so there exists $z \in M$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = z$ and therefore we can find an $u \in M$ such that $Tu = z$. Now, we shall prove that $Su = Tu$. Suppose that $Su \neq Tu$. Using (iii) we obtain

$$\begin{aligned}\psi\left(d(Sx_{n+1},Su)\right) \leq &\alpha\left(d(Tx_{n+1},Tu)\right)\psi\left(d(Tx_{n+1},Tu)\right)\\ &+\beta\left(d(Tx_{n+1},Tu)\right)\psi\left(M(x_{n+1},u)\right)\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}M(x_{n+1},u) = \max\Big\{&d(Tx_{n+1},Tu), d(Sx_{n+1},Tx_{n+1}), d(Su,Tu),\\ &\frac{1}{2}[d(Su,Tx_{n+1}+d(Sx_{n+1},Tu))]\Big\}.\end{aligned} \tag{3.2}$$

Letting $n\to\infty$ in (3.1) and (3.2) we obtain

$$\begin{aligned}\psi\left(d(z,Su)\right) \leq &\limsup_{n\to\infty}\alpha\left(d(Tx_{n+1},Tu)\right)\psi\left(d(z,Tu)\right)\\ &+\limsup_{n\to\infty}\beta\left(d(Tx_{n+1},Tu)\right)\psi\left(d(Su,Tu)\right) < \psi\left(d(z,Su)\right)\end{aligned}$$

which is a contradiction, thus $Su = Tu = z$. Therefore z is a POC of S and T. By Proposition 2.3 z is the unique POC. Now, due to the fact that (S,T) is WC, from Lemma 1.2, z is the unique common fixed point of S and T. ■

Example 3.2. Let $M = [1,\infty) \subset \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$. Define $S,T : M \longrightarrow M$ by $Sx = 2x-1$ and $Tx = x^2$, $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\psi(t) = \frac{1}{2}t^2$, $t \in \mathbb{R}_+$ and $\alpha(t) = \beta(t) = \frac{t}{3}\chi_{[0,1]}(t)$, where $\chi_{[0,1]}(t)$ is the characteristic function of the interval $[0,1]$ of the interval $[0,1]$. Then:

(i) $\alpha(t)+\beta(t) < 1$.

(ii) $Sx = Tx$, iff $x = 1$.
(iii) S and T are WC since they commute at the coincidence point $x = 1$.
(iv) The pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair.

Theorem 3.1 guarantees that $x = 1$ is the unique common fixed point of S and T. □

We can relax the condition "$TM \subset M$ be a complete subspace" for the condition "$TM \subset M$ is closed", as follows.

Theorem 3.3. *Let (M,d) be a metric space and let $S,T : M \longrightarrow M$ be WC self mappings satisfying property E.A. Let us assume that the pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair. If $TM \subset M$ is closed, then S and T have a unique common fixed point.*

Proof. Since the pair (S,T) satisfies property E.A., there exists a sequence $(x_n)_n \subset M$ such that $\lim_{n\to\infty} Sx_n = Tx_{n+1} = z$ for some $z \in M$. Since $TM \subset M$ is closed, then $z \in TM$ and $z = Tu$ for some $u \in M$. As in the proof of Theorem 3.1 we can prove that $z = Tu = Su$ and z is the unique POC of S and T. Finally, since the pair (S,T) is WC, then z is the unique fixed point of S and T. ■

Now, since two noncompatible self mappings of a metric space (M,d) satisfy property E.A., we obtain the following result:

Corollary 3.4. *Let S and T be two noncompatible WC mappings of a metric space (M,d) satisfying*

(i) *The pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair.*
(ii) *If $TM \subset M$ is closed.*

Then, S and T have a unique common fixed point.

Example 3.5. Let $M = [0,\infty) \subset \mathbb{R}$. Define $S,T : M \longrightarrow M$ by $Sx = \frac{1}{4}x$ and $Tx = \frac{3}{4}x$ for all $x \in M$. Using the sequence $x_n = \frac{1}{n}$ we can prove that S and T satisfy property E.A. $TM = [0,\infty)$ is closed in $\mathbb{R}$ and $C(S,T) = \{0\}$, thus S and T are WC mappings. Moreover, the pair (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair. The unique common fixed point is $x = 0$.

The next results prove the existence of a unique common fixed point for $\psi-(\alpha,\beta)$-Cgc pairs by using alternative notions to the noncompatibility.

Theorem 3.6. *Let S and T be RC compatible self mappings of a metric space (M,d) such that the pair (S,T) satisfies property E.A. and moreover (S,T) is a $\psi-(\alpha,\beta)$-Cgc pair. If $TM \subset M$ is closed, then S and T have a unique common fixed point.*

Proof. Since T and S satisfy property E.A., there exists $(x_n)_n$ in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = t, \text{ for some } t \in M.$$

Since the pair (S,T) is compatible, then $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$. The reciprocal continuity of S and T implies that $STx_n \to Tt$. The last two limits together imply $St = Tt$. Thus t is a CP of S and T. The compatibility of the mappings S and T implies the commutativity at the coincidence points, that is $STt = TSt$. Therefore, the pair (S,T) is weakly compatible. Now, applying Theorem 3.1 we conclude that S and T have a unique common fixed point. ■

Theorem 3.7. *Let S and T be WRC compatible self mappings of a metric space (M,d) satisfying*

(i) $SM \subset TM$.
(ii) $TM \subset M$ *is a complete subspace of* M.
(iii) *The pair* (S,T) *is a* $\psi - (\alpha,\beta)$*-Cgc pair.*

Then, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in M$ be an arbitrary point. Since $SM \subset TM$, as before, we construct the Cauchy sequence in TM, $y_n = Sx_n = Tx_{n+1}$, $n = 0, 1, \ldots$. Due to the fact that TM is complete, there exits $t \in M$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = t.$$

Suppose that S and T are compatible mapping, from the fact that S and T are WRC, then $\lim_{n\to\infty} STx_n = Tt$ or $\lim_{n\to\infty} TSx_n = Tt$.

Let $\lim_{n\to\infty} TSx_n = Tt$. Then, the compatibility of S and T gives that

$$\lim_{n\to\infty} d(STx_n, Tt) = 0.$$

Hence $\lim_{n\to\infty} STx_n = Tt$. By construction we have

$$\lim_{n\to\infty} STx_{n+1} = \lim_{n\to\infty} SSx_n = Tt.$$

Using (ii) we get

$$\psi\left(d(St, SSx_n)\right) \le \alpha\left(d(Tt, TSx_n)\right)\psi\left(d(Tt, TSx_n)\right) + \beta\left(d(Tt, TSx_n)\right)\psi\left(M(t, x_n)\right)$$

where

$$M(t, Sx_n) = \max\left\{d(Tt, TSx_n), d(St, Tt), d(SSx_n, TSx_n), \frac{1}{2}[d(Tt, TSx_n) + d(SSx_n, Tt)]\right\}.$$

Letting $n \to \infty$, we obtain $M(t, Sx_n) = d(St, Tt)$ and

$$\psi\left(d(St, Tt)\right) \le \limsup_{n\to\infty} \alpha\left(d(Tt, TSx_n)\right)\psi\left(d(Tt, TSx_n)\right) + \limsup_{n\to\infty} \beta\left(d(Tt, TSx_n)\right)\psi\left(d(St, Tt)\right) < \psi\left(d(St, Tt)\right)$$

which is contradiction, so $St = Tt$. Again, the compatibility of S and T implies commutativity at a PC, hence that pair (S,T) is WC. By Theorem 3.1 S and T have a unique common fixed point. Now, suppose that $\lim_{n\to\infty} STx_n = St$, then from the fact $SM \subset TM$, there exists $u \in M$ such that $St = Tu$ and $\lim_{n\to\infty} STx_n = Tu$. The compatibility of S and T implies $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, hence $\lim_{n\to\infty} d(Tu, TSx_n) = 0$. Therefore, $\lim_{n\to\infty} TSx_n = Tu$, equivalently,

$$\lim_{n\to\infty} STx_{n+1} = \lim_{n\to\infty} SSx_n = Tu.$$

In a similar way, as above, using (ii) we get $St = Tt$ and the rest of the proof is similar. Consequently, S and T have a unique common fixed point. ∎

Due to the fact that if two mappings S and T are RC, then they are WRC we have the following result.

Corollary 3.8. *Let* S *and* T *be RC compatible self mappings of a metric space* (M,d) *satisfying*

(i) $SM \subset TM$.
(ii) $TM \subset M$ *is a complete subspace of* M.

(iii) *The pair* (S,T) *is a* $\psi-(\alpha,\beta)$*-Cgc pair.*

Then, S and T have a unique common fixed point.

We can drop the completeness of the subspace TM by using CLR$_S$-property or CLR$_T$-property.

Theorem 3.9. *Let S and T be two WC self mappings defined on a metric space (M,d) such that*

(i) $SM \subset TM$.
(ii) *The pair* (S,T) *satisfies the* CLR$_S$*-property or the* CLR$_T$*-property.*
(iii) *The pair* (S,T) *is a* $\psi-(\alpha,\beta)$*-Cgc pair.*

Then, S and T have a unique common fixed point.

Proof. First suppose that the pair (S,T) has the the CLR$_S$-property, so there exists $(x_n)_n$ in M such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in SM$. Since $SM \subset TM$ we have $t \in TM$, thus there exists $u \in M$ such that $t = Tu$, that is, $Sx = Tu$. We shall to show that $Su = Tu$. If it is false, then from hypothesis (iii) with $x = u$, $y = x_n$ we obtain

$$\psi\left(d(Su,Sx_n)\right) \leq \alpha\left(d(Tu,Tx_n)\right)\psi\left(d(Tu,Tx_n)\right) + \beta\left(d(Tu,Tx_n)\right)\psi\left(M(u,x_n)\right)$$

where

$$M(u,x_n) = \max\left\{d(Tu,Tx_n), d(Su,Tu), d(Sx_n,Tx_n), \frac{1}{2}[d(Su,Tx_n)+d(Sx_n,Tu)]\right\}.$$

Notice that $\lim_{n\to\infty} M(u,x_n) = d(Su,Tu)$, thus

$$\psi\left(d(Su,Tu)\right) \leq \limsup_{n\to\infty} \alpha\left(d(Tu,Tx_n)\right)\psi\left(d(Tu,Su)\right) + \limsup_{n\to\infty} \beta\left(d(Tu,Tx_n)\right)\psi\left(d(Su,Tu)\right) < \psi\left(d(Su,Tu)\right).$$

It follows that $\psi\left(d(Su,Tu)\right) < \psi\left(d(Su,Tu)\right)$, implying that $d(Su,Tu) < d(Su,Tu)$ which is a contradcition, so $Su = Tu = t$. t is the unique POC (see the proof of Theorem 3.1) of S and T. now, since (S,T) is a WC pair, Lemma 1.2 guarantees that t is the unique common fixed point of S and T. In the case when the pair (S,T) has the CLR$_T$-property its is similar. ∎

4. Existence of Solutions for a System of Nonlinear Integral Equation

We are going to study the existence of a unique solution for a system of nonlinear integral equations, posed on normed spaces of real-valued functions, through the results proved in the past section.

Let $(X,\|\cdot\|)$ be a normed space of functions $u : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ whose norm is ordering preserving; that is, $\|f\| \leq \|g\|$ whenever $|f(x)| \leq |g(x)|$ for all $x \in D$. We are interested in find solutions on X of the following system of equations.

$$\begin{cases} u(t) = & f_1(t) + \lambda_1 \int\limits_a^b g_1(s)h_1(t, u(s))ds =: Su(t) \\ u(t) = & f_2(t) + \lambda_2 \int\limits_a^b g_2(s)h_2(t, u(s))ds =: Tu(t) \end{cases} \tag{4.1}$$

where, $-\infty \leq a < b \leq \infty$, $\lambda_i \in \mathbb{R}$, $i = 1, 2$. In the operator theory framework, the problem is equivalent to find common fixed points for the pair (S, T). Equations as the given in the system (4.1) are widely studied since they appear in different problems and applications; for instance, in the analysis of the radial solutions of a nonlinear elliptic equation (see e.g., [19] and references therein). When the equations are posed on a concrete normed space $(X, \|\cdot\|)$, it is necessary to assume adequate growth conditions on the functions f_i, g_i or h_i, in order to guarantee the existence of at least one solution for the integral equations

$$u(t) = f_i(t) + \lambda_i \int\limits_a^b g_i(s)h_i(t, u(s))ds.$$

See, for instance, [7–9, 21, 28] and references in the topic. Here we will assume that the following relation between the equations holds:

$$\left|\lambda_1 \int\limits_a^b g_1(s)[h_1(t, u(s)) - h_1(t, v(s))]ds\right| \leq \left|\lambda_2 \int\limits_a^b g_2(s)[h_2(t, u(s)) - h_2(t, v(s))]ds\right| \tag{4.2}$$

for all $u, v \in X$. Notice that this condition is a comparative one, not a growth-type one, and it is independent of the normed space under consideration.

Now, are are going to use some functional associated with h-concave and quasilinear functions. This kind of functionals are introduced in [27]. Let C a convex cone in the linear space X over $\mathbb{R}$ and let L be a real number $L \neq 0$. A functional $F : C \longrightarrow \mathbb{R}$ is called *L-superadditive* in C if

$$f(x + y) \geq L(f(x) + f(y)), \text{ for any } x, y \in C.$$

Let K be a real non-negative function, a functional f satisfying

$$f(tx) = K(t)f(x)$$

for any $t \geq 0$ and $x \in C$, is called *K-positive homogeneous*. Notice that $K(1) = 1$. We are going to prove that condition (4.2) guarantees that the pair (S, T) is a $\psi - (\alpha, \beta)$-Cgc pair. To attain such a goal next lemma will be useful.

Lemma 4.1 ([27])**.** *Let $x, y \in C$ and $f : C \longrightarrow \mathbb{R}$ be a non-negative, L-superadditive and K-positive homogeneous functional on C. If $M \geq m > 0$ are such that $x - my$ and $My - x \in C$, then*

$$LK(m)f(y) \leq f(x) \leq \frac{1}{L}K(M)f(y).$$

Proposition 4.2. *Under condition* (4.2)*, the pair* (S,T) *given in* (4.1) *is a* $\psi-(\alpha,\beta)$*-Cgc pair.*

Proof. From condition (4.2) for all $u, v \in X$ we have that

$$|Su(t) - Sv(t)| \leq |Tu(t) - Tv(t)|, \quad \text{for all } t \in \mathbb{R}.$$

Then, from the preserving property of the norm we have that

$$\|Su - Sv\| \leq \|Tu - Tv\|.$$

Moreover, there exists $0 \geq m < 1$ depending of u and v such that

$$m\|Tu - Tv\| \leq \|Su - Sv\| \leq \|Tu - Tv\|. \tag{4.3}$$

Now, let ψ be a L-superadditive and K-positive homogeneous functional in Ψ, from (4.3), we are in the hypotheses of Lemma 4.1. Thus,

$$\psi\left(\|Su - Sv\|\right) \leq \frac{1}{L}\psi\left(\|Tu - Tv\|\right).$$

Let $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ satisfying (2.1) with $\frac{1}{L} \leq \alpha(t)$ for any $t \in \mathbb{R}_+$. Hence, we obtain

$$\begin{aligned}\psi\left(\|Su - Sv\|\right) \leq& \frac{1}{L}\psi\left(\|Tu - Tv\|\right)\\ \leq& \psi\left(\|Su - Sv\|\right)\psi\left(\|Tu - Tv\|\right) + \beta\left(\|Tu - Tv\|\right)\psi\left(M(u, v)\right).\end{aligned}$$

Therefore, (S, T) is a $\psi - (\alpha, \beta)$-Cgc pair. ∎

Remark 4.3. Notice that condition (4.2) is not sufficient to guarantee that the pair (S, T) below be a Ćirić pair, but for a preserving distance, L-superdditive and K-positive mapping as ψ below, the pair (S, T) is a ψ-Jungck contractive pair (ψ-JC pair), see Definition 2.1 in [22]. However, in order to guarantee the existence of common fixed points for ψ-JC pairs, is required that the underlying space be a complete metric space, hence, the results of this paper extend the given in [22]. In Example 4.7 we analyze a siystem of nonlinear integral equations on a normed (no Banach) space, thus this example cannot be analyzed within the results of [22].

We would like to recall that the weakly compatible property is a necessary property for the existence of common fixed point for $\psi - (\alpha, \beta)$-Cgc pairs, in Section 5 we discuss it with details. Therefore, we will require that the WC property holds for the mappings (S, T) given in (4.1).

On the other hand, we will assume that

$$\text{Rgo}(\lambda_1 g_1(\cdot)h_1(\cdot, \cdot)) \subset \text{Rgo}(\lambda_2 g_2(\cdot)h_2(\cdot, \cdot)). \tag{4.4}$$

Notice that this condition implies that $SX \subset TX$ and its is compatible with (4.2). Assuming that the functions $h_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and satisfy

$$\lim_{u\to\infty} h_i(t, u) = 0, \quad \text{for all } t \in \mathbb{R}, \ (i = 1, 2) \tag{4.5}$$

we obtain the following result.

Proposition 4.4. *Under condition* (4.5)*, the pair* (S, T) *given in* (4.1) *has the property E.A.*

Proof. Let $(u_n)_n \subset X$ such that $u_n \to \infty$ as $n \to \infty$. Then, from condition (4.5) we get that

$$\lim_{n\to\infty} h_i(t, u_n) = 0, \quad i = 1, 2.$$

Consequently, we obtain

$$\lim_{n\to\infty} \lambda_i \int_a^b g_i(s)h_i(t, u_n(s))ds = 0, \quad i = 1, 2.$$

Since $T0 = S0 = 0$, we conclude

$$\lim_{n\to\infty} TSu_n = \lim_{n\to\infty} STu_n = 0.$$

That is, the pair (S, T) has the property E.A. ■

Since all compatible pair satisfies the property E.A. we obtain the immediate consequence: *Under condition* (4.5)*, the pair* (S, T) *given in* (4.1) *is noncompatible.*

On the other hand, note that the functions

$$t \mapsto \lambda_i \int_a^b g_i(s)h_i(t, u(s))ds \quad i = 1, 2$$

are continuous if the following conditions are satisfy

$$\text{The functions } g_i(s)h_i(t, u) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \text{ are continuous.} \tag{4.6}$$

Due to the fact that a pair of continuous functions are reciprocal continuous, the following result holds.

Proposition 4.5. *Under condition* (4.6)*, the pair* (S, T) *given in* (4.1) *is RC.*

Also, we have the next immediate consequence: *Under conditions* (4.6)*, the pair* (S, T) *given in* (4.1) *is WRC.*

Now, by applying the results given in the previous sections we can establish conditions for the existence of solutions for system (4.1).

Theorem 4.6. *System* (4.1) *has a unique solution if conditions* (4.2) *and* (4.4) *are satisfy and one of the following conditions hold:*

(i) *The pair* (S, T) *is WC, and* $TX \subset X$ *is complete.*
(ii) (4.6) *and* $TX \subset X$ *is complete.*
(iii) (4.5) *and* $TX \subset X$ *is closed.*

Proof. The proof of (i) follows from Proposition 4.2 and Theorem 3.1. To prove (ii) we have from condition (4.2) and Proposition 4.2 that the pair (S, T) is a (α, β)-Cgc pair. Moreover, from (4.4) we have that $SX \subset TX$. Now, condition (4.6) assure the the pair (S, T) is RC or WRC (Proposition 4.5). Therefore the conclusion follows from Theorem 3.7 or Corollary 3.8. Finally we will prove (iii). The proof follows from Theorem 3.3 or Corollary 3.4 since (4.5) implies that (S, T) has the property E.A. or is noncompatible (Proposition 4.4). ■

Example 4.7. Let us consider the space $\mathcal{C}((0,1),\|\cdot\|)$ of all continuous functions $u:(0,1)\longrightarrow\mathbb{R}$ endowed with the maximum norm $\|u\|=\max|u(t)|$. At is known, this space is not a Banach space. We are going to analize the existence of a solution of the following system on integral equations

$$\begin{cases} u(t) = & \dfrac{1}{2}\displaystyle\int_0^1 t^2 s\sin(u(s))ds =: Su(t), \\ u(t) = & \dfrac{3}{4}\displaystyle\int_0^1 \sqrt{st}u(s)ds =: Tu(t). \end{cases}$$

First, we prove that the pair (S,T) satisfies (4.2). To attain such a goal we use the following estimate:

$$\begin{aligned} |\sin(u(t))-\sin(v(t))| &= \left|2\sin\left(\frac{u(t)-v(t)}{2}\right)\cos\left(\frac{u(t)+v(t)}{2}\right)\right| \\ &\le \left|2\sin\left(\frac{u(t)-v(t)}{2}\right)\right| \\ &\le |u(t)-v(t)|. \end{aligned} \tag{4.7}$$

Thus we obtain that

$$\begin{aligned} |Su(t)-Sv(t)| &= \frac{1}{2}\left|\int_0^1 t^2 s[\sin(u(s))-\sin(v(t))]ds\right| \\ &\le \frac{1}{2}\int_0^1 |t^2 s||\sin(u(s))-\sin(v(t))|ds \\ &\le \frac{1}{2}\int_0^1 t^2 s|u(s)-v(t)|ds, \quad s,t\in(0,1). \end{aligned}$$

On the other hand, for the operator T we have

$$|Tu(t)-Tv(t)| = \frac{3}{4}\left|\int_0^1 \sqrt{st}(u(s)-v(s))ds\right|.$$

Consequently, from the estimate (4.7) and the fact that $\sqrt{x}\le x^n$, for all $x\in(0,1)$ and all $n\in\mathbb{N}$, we conclude that

$$|Su(t)-Sv(t)|\le|Tu(t)-Tv(t)|, \quad \text{for all } u,v\in\mathcal{C}(0,1).$$

Therefore, the pair (S,T) satisfies (4.2). Thus, from Proposition 4.2, the pair (S,T) is a (α,β)-Cgc pair.

Now we are going to prove that $T\mathcal{C}(0,1)$ is a closed subspace of $\mathcal{C}(0,1)$. Let f be a continuous function in the closure $\overline{T\mathcal{C}(0,1)}$ of the set $T\mathcal{C}(0,1)$, and suppose that $f\notin T\mathcal{C}(0,1)$. Therefore, there is not exists a sequence u_n in $T\mathcal{C}(0,1)$ such that $u_n(t)\to f(t)$,

$t \in (0,1)$, as $n \to \infty$. However, notice that

$$f(t) = \lim_{n\to\infty} \frac{3}{4} \int_0^1 s^{-1/4} f(t) \left(1 + \frac{1}{n}\right) ds$$
$$= \lim_{n\to\infty} \frac{3}{4} \int_0^1 \sqrt{st} \frac{s^{-1/4}}{\sqrt{st}} f(t) \left(1 + \frac{1}{n}\right) ds.$$

Notice that the functions

$$u_n(t) = \frac{3}{4} \int_0^1 \sqrt{st} \frac{s^{-1/4}}{\sqrt{st}} f(t) \left(1 + \frac{1}{n}\right) ds, \quad n = 1, \dots$$

belong to $T\mathcal{C}(0,1)$, since the functions

$$\frac{s^{-1/4}}{\sqrt{st}} f(t) \left(1 + \frac{1}{n}\right)$$

are continuous for each $s \in (0,1)$ and for all $t \in (0,1)$, which contradicts the assumption $f \notin T\mathcal{C}(0,1)$, thus $\overline{T\mathcal{C}(0,1)} \subset T\mathcal{C}(0,1)$.

Finally, due to the fact that the pair (S,T) is RC, then from Theorem 4.6(iii) the system has a unique solution, namely $u(t) \equiv 0$.

5. Final Remarks

First, we would like to show that weak compatibility is a necessary, hence minimal, condition for the existence of common fixed points of contractive type mapping pairs. Suppose S and T be a contractive type pair of self-mappings of a metric space (M,d) having a common fixed point, say z then $z = Sz = Tz$ and $STz = TSz = Sz = Tz = z$. If it is possible, suppose that S and T are not weakly compatible. Then there exists a point w in M such that $Sw = Tw$ while $STw \neq TSw$, we thus have $Sw = Tw$ and $Sz = Tz$ with $Sw \neq Sz$. This is not possible in view of contractive conditions. For example, in this case

$$\psi\left(d(Sz,Sw)\right) \le \alpha\left(d(Tz,Tw)\right)\psi\left(d(Tz,Tw)\right) + \beta\left(d(Tz,Tw)\right)\psi\left(M(z,w)\right)$$
$$= \alpha\left(d(Sz,Sw)\right)\psi\left(d(Sz,Sw)\right) + \beta\left(d(Sz,Sw)\right)\psi\left(M(z,w)\right)$$

where

$$M(z,w) = \max\left\{d(Sz,Sz), d(Sz,Sz), d(Sw,Sw), \frac{1}{2}[d(Sz,Sw) + d(Sz,Sw)]\right\}$$
$$= d(Sz,Sw)$$

thus, $\psi\left(d(Sz,Sw)\right) < \psi\left(d(Sz,Sw)\right)$ which is a contradiction. This shows that weak compatibility is a necessary, hence minimal, condition for the existence of common fixed points of contractive type mappings pairs.

On the other hand, notice that in Theorem 3.1 we cannot replace non-trivially weakly compatible mappings by OWC. In fact, under the hypothesis (S,T) is a $\psi - (\alpha,\beta)$-Cgc pair in Theorem 3.1, assumption of OWC and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that (S,T) is a $\psi - (\alpha,\beta)$-Cgc pair. If S and T have a common fixed point, say z, then $z = Tz = Sz, TSz = STz = z$

and S and T are, therefore, OWC mappings. On the other hand, if S and T are OWC mappings such that $Tx = Sx$ and $TSx = STx = TTx = SSx$ for some x then, using the $\psi-(\alpha,\beta)$-Cgc pair inequality, we get

$$\psi\left(d(Sx,SSx)\right) \leq \alpha\left(d(Tx,TSx)\right)\psi\left(d(Tx,TSx)\right) + \beta\left(d(Tx,TSx)\right)\psi\left(M(x,Sx)\right),$$

where

$$M(x,Sx) = \max\left\{d(Tx,TSx), d(Sx,Tx), d(SSx,TSx), \frac{1}{2}[d(Sx,TSx)+d(SSx,Tx)]\right\}.$$

Then, $\psi\left(d(Sx,SSx)\right) < \psi\left(d(Sx,SSx)\right)$, so we conclude that $Sx = SSx$. Since (α,β)-Cgc pair condition exclude the existence of two coincidence points x, y for S and T, we get $Sx = SSx (= TSx)$. This means that $Sx = Tx$ is a common fixed point of S and T. Therefore, one should be really careful before using OWC under any contractive conditions (see also, [30]).

Finally, we would like to point out that due to the minor restrictions on the functions involved in the definition of the class of $\psi-(\alpha,\beta)$-Cgc pairs and the minimal commutative requirements of the mappings, our results extend several common fixed point theorems for classes of well-known contractive type of mappings, including various classes of contractive mappings with inequalities controlled by altering distance functions as well as contractive mappings of the integral type which were started by A. Branciari [6] in 2002, who extended the Banach contraction principle by using some Lebesgue integrable functions. However, we would like point out that in 2009, J. Jachymski [12] showed that some contractive conditions of integral type are consequences of the classical known ones (see [12] and references therein).

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