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Quasi-Homeomorphisms Between Ordered Topological Spaces

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Abstract In this paper, we introduce the notion of quasi-homeomorphism in the category of ordered topological spaces. Some related results are given. We give also some relation of quasi-homeomorphisms with T_0-ordered topological spaces.

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1. Introduction

For topological spaces, Grothendieck introduced, for the first time, the notion of quasi-homeomorphism (see [1]). After that, Kai-Wing presented an other definition of this notion in [2] that is equivalent to the previous one given by Grothendieck.

L. Nachbin, introduced the definition of ordered topological spaces in [3]. Many recent references was investigated some properties of ordered topological spaces, we cite for examples [4–11] . In this paper we give a generalization of the two definitions of quasi-homeomorphism to the category of ordered topological spaces **OrdTop** which has continuous increasing maps as arrows. We study the equivalence between the two definitions in this category. The case of quasi-homeomorphism for topological spaces will be seen as a particular case from our work when we take the equality as an order.

The remainder of this paper is organized as follows: Section 2 is devoted to the presentation of quasi-homeomorphisms in **OrdTop**. In Section 3, we study some relations between quasi-homeomorphism and T_0-ordered topological spaces.

Now, we introduce some basic notions and definitions that will be needed throughout this paper.

Let $f : (X, \tau) \longrightarrow (Y, \gamma)$ be a continuous map between topological spaces. f is called a quasi-homeomorphism according the Grothendiek definition if the map $U \longmapsto f^{-1}(U)$ is a bijection between the set of all open sets of Y and the set of all open sets of X.

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f is said to be a quasi-homeomorphism according the Yip definition if it satisfies the following conditions:

- For any closed set A in X, $f^{-1}\left(\overline{f(A)}\right) = A$.
- For any closed set B in Y, $\overline{f(f^{-1}(B))} = B$.

The two previous definitions are equivalent by Proposition 2.2 in [12].

$(X, \tau, \leq)$ is an object of **OrdTop** if $(X, \leq)$ is a partially ordered set and (X, τ) is a topological space.

Let $(X, \tau, \leq)$ be an ordered topological space and A be a nonempty subset of X. We say that A is an increasing (resp. decreasing) set if, when A contain x and $x \leq y$ (resp. $y \leq x$), then A must contain y.

The closed increasing (resp. decreasing) hull of A is the smallest closed increasing (resp. decreasing) set containing A it will be denoted by $I(A)$ (resp. $D(A)$). We denote also by $C(A)$ the set $I(A) \cap D(A)$. You can find all this previous notations and definitions in [13]. If x is an element of $(X, \tau, \leq)$ we denote $I(x)$, $D(x)$ and $C(x)$ instead of $I(\{x\})$, $D(\{x\})$ and $C(\{x\})$

It is easy to prove that $I(x) = I(y)$ and $D(x) = D(y)$ is equivalent to $C(x) = C(y)$. These previous assertions define an equivalence relation in an ordered topological spaces needed in section 3.

2. Preliminary Results

To introduce a similar definition of Grothendieck's quasi-homeomorphism in **OrdTop** we will use the collection of all closed increasing sets of an ordered topological space $(X, \tau, \leq)$ which will be denoted by $FI(X)$.

According to the Grothendieck's definition we introduce the notion of quasi-homeomorphism in **OrdTop** as follows

Definition 2.1. Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an continuous increasing map between two ordered topological spaces. f is said to be a quasi-homeomorphism if the map

$$\begin{array}{rccc} \varphi_f : & FI(Y) & \longrightarrow & FI(X) \\ & A & \longmapsto & f^{-1}(A) \end{array}$$

is bijective.

Remark 2.2.

- It's clear that every isomorphism is a quasi-homeomorphism.
- If we replace FI by OD in the previous definition we get an equivalent definition of quasi-homeomorphism, when $OD(X)$ denote the collection of all open decreasing sets of $(X, \tau, \leq)$.

Proof. The first items is straightforward.

For the second one, we use the fact that the complement of an closed increasing set is an open decreasing set. ∎

When we take equality as orders in the previous definition we obtain the Grothendiek's definition of quasi-homeomorphism for topological spaces.

Lemma 2.3. *Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$, $g : (Y, \gamma, \sqsubseteq) \longrightarrow (Z, \delta, \preceq)$ and $h = g \circ f$. The set $\{f, g, h\}$ can not contain exactly two quasi-homeomorphisms.*

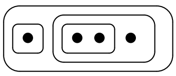
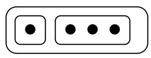
Proof. It is sufficient to see that if two of the three maps ψ_f, ψ_g, ψ_h are bijective, then so is the third one. ∎

Proposition 2.4. *let* $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ *be a continuous increasing map between two ordered topological spaces. Then,*

(1) f *may induce a quasi-homeomorphism in* ***Top*** *from* (X, τ) *to* (Y, γ) *but* f *is not a quasi-homeomorphism in* ***OrdTop****.*
(2) f *may be a quasi-homeomorphism in* ***OrdTop*** *but the induced map in* ***Top*** *may not be a quasi-homeomorphism.*

Proof.

(1) **Counter example 1**
The identity $Id : (\mathbb{R}, \tau_u, =) \to \mathbb{R}, \tau_u, \leq)$ induce a quasi-homeomorphism in **Top** but it is not a quasi-homeomorphism in **OrdTop**.
(2) **Counter example 2**
Let $X = \{1, 2, 3, 4\}$ equipped with the two topologies τ and γ defined by its closed sets represented by the following graphs and the natural order.

$\tau =$  $\gamma =$ 

Identity map from X to itself is a quasi-homeomorphism in **OrdTop** but not in **Top**.

∎

We present, now, the definition of quasi-homeomorphism in **OrdTop** according to Yip's definition in **Top**.

Definition 2.5. Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be an continuous increasing map between two ordered topological spaces. f is said to be a quasi-homeomorphism if it satisfies the following conditions:

(1) $\forall A \in FI(X)\ A = f^{-1}(I(f(A)))$.
(2) $\forall B \in FI(Y)\ B = I\left(f\left(f^{-1}(B)\right)\right)$.

Theorem 2.6. *Definition 2.1 is equivalent to Definition 2.5.*

Proof. Clearly Yip's Definition implies Grothendiek's Definition.

Conversely, suppose f satisfies the Grothendiek's Definition of quasi-homeomorphism.

Let $A \in FI(X)$. On one a hand, we have $A \subseteq f^{-1}(I(f(A)))$. On an other hand, let $D \in FI(Y)$ such that $A = f^{-1}(D)$ then $f(A) \subseteq D$.

- If $x \in f^{-1}(I(f(A)))$ then $f(x) \in I(f(A))$ or $I(f(A)) \subseteq D$ which implies that $x \in q^{-1}(D) = A$ so that $f^{-1}(I(f(A))) \subseteq A$.
- Let $B \in FI(Y)$. $f\left(f^{-1}(B)\right) \subseteq B$ then $I\left(f\left(f^{-1}(B)\right)\right) \subseteq B$ so $f^{-1}\left(I\left(f\left(f^{-1}(B)\right)\right)\right) \subseteq f^{-1}(B)$. Conversely, let $x \in f^{-1}(B)$ then $f(x) \in I\left(f\left(f^{-1}(B)\right)\right)$so that $x \in f^{-1}\left(I\left(f\left(f^{-1}(B)\right)\right)\right)$ and then $f^{-1}(B) \subseteq f^{-1}\left(I\left(f\left(f^{-1}(B)\right)\right)\right)$.We can see that $\varphi_f(B) = \varphi_f\left(I\left(f\left(f^{-1}(B)\right)\right)\right)$, since φ_f is bijective then $B = I\left(f\left(f^{-1}(B)\right)\right)$.

∎

Corollary 2.7. *let* $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ *be a continuous increasing map. Then,*

- *If f is one to one and satisfies the second condition of the Definition 2.5, then f is a quasi-homeomorphism.*
- *If f is onto and satisfies the first condition of the Definition 2.5, then f is a quasi-homeomorphism.*

Corollary 2.8. *let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be a continuous increasing map.f induce a lattice-isomorphism φ_f if and only if f is a quasi-homeomorphism.*

In **Top**, a bijective quasi-homeomorphism is a homeomorphism but in **OrdTop**, the bijectivity of a quasi-homeomorphism is not sufficient to be a homeomorphism since it miss the condition "bi-increasing."

Example 2.9. Counter example
It is sufficient to take the identity map from $X = \{0, 1\}$, with the discreet topology and the equality, to the same set with the same topology and the natural order.

Corollary 2.10. *A bijective bi-increasing quasi-homeomorphism is an homeomorphism in **OrdTop**.*

3. Quasi-homeomorphism and T_0-ordered Topological Space

Let $(X, \tau, \leq)$ be an ordered topological space. $(X, \tau, \leq)$ is said to be a T_0-ordered topological space if for all $x \neq y$ there exists a monotone (increasing or decreasing) open set which contain one of the point and does not contain the other point. This definition is equivalent to: $I(x) \cap D(x) = I(y) \cap D(y)$ implies $x = y$. We can find this definition in [14].

Let $(X, \tau, \leq)$ be an ordered topological space. The relation defined on X by $x \sim y$ if and only if $I(x) = I(y)$ and $D(x) = D(y)$ is an equivalence relation. Now we take the finite step order $\leq^0$ in the quotient set $X/\sim$ which is defined by:

$\bar{x} \leq^0 \bar{y}$ if and only if there exist $z_i, z_i^*, z_i^{'}$ $(i = 0, ..., n)$ such that $\overline{z_i} = \overline{z_i^*} = \overline{z_i^{'}}$ and $z_i^* \leq z_{i+1}^{'}$.

Richmond and Kunzi proved that $\left(X/\sim, \tau/\sim, \leq^0\right)$ is the T_0-ordered reflection of $(X, \tau, \leq)$, you can see Theorem 3.1 in [14].

Theorem 3.1. *The canonical surjection $\mu_X : (X, \tau, \leq) \longrightarrow \left(X/\sim, \tau/\sim, \leq^0\right)$ is a quasi-homeomorphism*

Proof. Since μ_X is onto, then by Corollary 2.7 it is sufficient to prove the first condition in Definition 2.5.

We start by proving that $\forall A \in FI(X)$ we have $\mu_X^{-1}(\mu_X(A)) = A$. On one hand, the fact that $A \subseteq \mu_X^{-1}(\mu_X(A))$ is clear. On an other hand, if $x \in \mu_X^{-1}(\mu_X(A))$ then there exists $y \in A$ such that $y \sim x$ which implies that $x \in A$ and then $\mu_X^{-1}(\mu_X(A)) \subseteq A$.

Now, let us prove that $A \in FI(X)$ implies $\mu_X(A) \in FI(X/\sim)$.

Let $\bar{x} \in \mu_X(A)$ and $\bar{x} \leq^0 \bar{y}$. By the definition of $\leq^0$ there exists a set of elements $\left\{z_i, z_i^{'}, z_i^* \,;\, 0 \leq i \leq n\right\}$ such that $z_0^* \in A$ and $z_0^* \leq z_1^{'}$ which implies $z_1^{'} \in A$ and also $z_1 \in A$, by induction we can prove that $y \in A$ so that $\bar{y} \in p(A)$. Now, we can see that $p(A) \in FI(X/\sim)$.

As a conclusion for all $A \in FI(X)$ we have
$\mu_X^{-1}(I(\mu_X(A))) = \mu_X^{-1}(\mu_X(A)) = A$. This is sufficient to complete the proof. ∎

Theorem 3.2. *Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ is a quasi-homeomorphism such that X is a T_0-ordered topological space. Then,*

(1) *f is one to one.*
(2) *If f is onto then Y is a T_0-ordered topological space.*

Proof.

(1) Let $x \neq y \in X$. Since X is T_0 then there exists an increasing open set U of X which contain, for example, x and not contain y. Since f is a quasi-homeomorphism then there exists an open increasing set O of Y such that $f^{-1}(O) = A$. Finally O is a closed increasing set which contain $f(x)$ and not $f(y)$. So that $f(x) \neq f(y)$ and f is one to one.
(2) Let $f(x) \neq f(y) \in Y$. Since $x \neq y$ there exists an increasing open set U of X which contain, for example, x and not contain y. Since f is a quasi-homeomorphism then there exists an open increasing set O of Y such that $f^{-1}(O) = A$. Finally O is a closed increasing set which contain $f(x)$ and not $f(y)$. Which complete the proof.

■

Proposition 3.3. *Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be a continuous increasing map and $\tilde{f}$ be the map defined by the following diagram*

$$\begin{array}{ccc} (X, \tau, \preceq) & \xrightarrow{\ f\ } & (Y, \gamma, \sqsubseteq) \\ \downarrow{\scriptstyle \mu_X} & \circlearrowleft & \downarrow{\scriptstyle \mu_Y} \\ \left(X/\approx, \tau/\approx, \leq^0\right) & \xrightarrow[\tilde{f}]{} & \left(Y/\approx, \gamma/\approx, \sqsubseteq^0\right) \end{array}$$

when $\tilde{f}(\bar{x}) = \overline{f(x)}$. Then,
f is a quasi-homeomorphism if and only if $\tilde{f}$ is a quasi-homeomorphism.

Proof. It is sufficient to use the Lemma 2.3.

■

Theorem 3.4. *Let $f : (X, \tau, \leq) \longrightarrow (Y, \gamma, \sqsubseteq)$ be a quasi-homeomorphism. If Y is a T_0-ordered topological space. Then f is one to one if and only if X is a T_0-ordered topological space.*

Proof. Let $x \neq y \in X$. Since f is one to one then $f(x) \neq f(y)$. Now, Y is T_0 then there exists $A \in FI(Y)$ such that $Card(A \cap \{f(x), f(y)\}) = 1$. Then $f^{-1}(A) \in FI(X)$ and $Card\left(f^{-1}(A) \cap \{x, y\}\right) = 1$. Which is sufficient to prove that X is T_0. Conversely, it is the first result in Theorem 3.2.

■

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