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Existence of Solutions of Set-valued Strong Vector Equilibrium Problems

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Abstract In this paper, we considered set-valued strong vector equilibrium problems and obtained some existence results with and without compactness assumptions in Hausdorff topological vector spaces ordered by a cone. Further, we established some existence results by making use of self-segment dense set, a special type of dense set. Our results in this paper are new which can be considered as a generalization of many known results in the literature.

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1. Introduction

Let X and Y be two topological vector spaces and K be a nonempty convex subset of X. Let $f : K \times K \to Y$ be a vector-valued mapping and C be a convex cone with nonempty interior in Y. Then consider the problem of finding $x_0 \in K$ such that

$$f(x_0, y) \notin -intC, \forall y \in K,$$

is known as weak vector equilibrium problem. This problem is the most interesting and intensively studied classes of problems which include many fundamental mathematical problems, like vector optimization problems, vector variational inequality problems, Nash equilibrium problems for vector-valued maps, fixed point problems, see for example, [1, 3–7, 12, 13] and the references therein.

The concept of vector variational inequality (in short, VVI) was first introduced by Giannessi [6] for finite dimensional Euclidean space, which is a generalization of a scalar variational inequality to the vector case by virtue of multi-criteria consideration. Since, then the VVI has been intensively studied in a general setting by many authors. Recently, Chen and Hou [1] reviewed and summarized representative existence results of solutions for vector variational inequalities, and pointed out that most of the results in this area touched upon a weak version of VVI and its generalization. Fang and Huang [3] obtained

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some existence results of solutions for a class of strong vector variational inequalities and partially answered the open problems proposed by Chen and Hou [1].

Recently, Kazmi and Khan [7] introduced a kind of equilibrium problems (EP) called generalized system (GS), which extends the strong vector variational inequality (SVVI) studied in Fang and Huang [3] in real Banach spaces, and obtained existence results of generalized system with and without monotonicity assumptions. However, they dealt with single-valued case of the bi-operator. Kum and Wong [8] considered a multi-valued version of generalized system (GS), called the multi-valued generalized system (MGS) and obtained existence results with and without monotonicity assumptions by using Brouwer and Fan-Browder Fixed point theorems. Recently, Sitthithakerngkiet and Plubtieng [13] established some existence results for the solutions of the generalized strong vector quasi-equilibrium problems with and without monotonicity by using the generalization of Fan-Browder fixed point theorem. Motivated and inspired by the work of Kum and Wong [8] and Sitthithakerngkiet and Plubtieng [13]. In this paper, we first obtained the existence results for set-valued strong vector equilibrium problem by making use of KKM-mapping and Ky-Fan lemma in Hausdorff topological vector spaces, and then established existence results for the same problem by making use of special property of self-segment dense set characterized by Lemma 4.1. The results presented in this paper give a positive answer to the open problem posed by Chen and Hou [1] in topological vector spaces.

We organize this work as follows:

In section 2, we introduced set-valued strong vector equilibrium problems, and listed some definitions and results which are needed in the sequel. The existence results for the set-valued strong vector equilibrium problems by using KKM-mapping and Ky-Fan lemma in Hausdorff topological vector spaces have been established in section 3. In section 4, some existence results for the set-valued strong vector equilibrium problem by using the concept of self-segment-dense set, a special type of dense set, have been discussed.

2. Preliminaries

Throughout this paper, unless otherwise specified, we assume that X and Y be the Hausdorff topological vector spaces and $K \subset X$ be a nonempty closed convex set.

Let $C : K \to 2^Y$ be a set-valued mapping, where 2^Y denotes the set of all nonempty subsets of Y, and C be a pointed closed convex cone with apex at origin with $intC \neq \phi$. For a given set-valued map $F : K \times K \to 2^Y$ with $F(x,x) = \{0\}$, for all $x \in K$. Then, the set-valued strong vector equilibrium problems is to find $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K \tag{2.1}$$

and

$$F(y, x_0) \subseteq -C, \forall y \in K. \tag{2.2}$$

The main aim of this paper is to establish some existence results for set-valued strong vector equilibrium problems (2.1).

Let us recall some definitions and lemmas which are needed in the main results of this paper.

Definition 2.1. Let Y be a topological vector space, then $C \subset Y$ is said to be *closed convex pointed cone* with apex at origin if and only if C is closed, and satisfying the following conditions:

(i) $\lambda C \subset C, \ \forall \ \lambda > 0$,
(ii) $C + C$=C,
(iii) $C \cap (-C) = \{0\}$.

Note that a closed convex pointed cone induces a partial ordering on Y. We define the following partial ordering and strict partial ordering relationships:

For $x, y \in Y$ and for any two subsets A, B of Y.

$$x \leq_C y \Leftrightarrow y - x \in C; \ x \nleq_C y \Leftrightarrow y - x \notin C,$$

$$x \leq_{C\setminus\{0\}} y \Leftrightarrow y - x \in C \setminus \{0\}; \ x \nleq_{C\setminus\{0\}} y \Leftrightarrow y - x \notin C \setminus \{0\},$$

$$A \leq_C B \Leftrightarrow B - A \subseteq C; \ A \nleq_C B \Leftrightarrow B - A \nsubseteq C,$$

$$A \leq_{C\setminus\{0\}} B \Leftrightarrow B - A \subseteq C \setminus \{0\}; \ A \nleq_{C\setminus\{0\}} B \Leftrightarrow B - A \nsubseteq C \setminus \{0\}.$$

It is easy to show that $C + C \setminus \{0\} = C \setminus \{0\}$ and $intC + C = intC$.

Definition 2.2. Let X and Y be two topological vector spaces, K be a nonempty convex subset X and C be a pointed closed convex cone in Y with $intC \neq \phi$. Then a mapping $F : K \times K \to 2^Y$ is said to be:

(i) *C-strongly pseudomonotone*, if it satisfies

$$F(x, y) \nsubseteq -C \setminus \{0\} \Rightarrow F(y, x) \subseteq -C, \forall x, y \in K.$$

(ii) *C-convex in the first argument*, if it satisfies

$$F(tx + (1 - t)y, z) \subseteq tF(x, z) + (1 - t)F(y, z) - C, \forall x, y, z \in K.$$

Definition 2.3. Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued map. Then a set-valued map $T : X \to 2^Y$ is said to be *upper semicontinuous* on X, if for each $x \in X$ and each open set V in Y containing $T(x)$, there exists an open neighborhood U of x in X such that $T(y) \subseteq V, \forall y \in U$, and a set-valued map $T : X \to 2^Y$ is said to be *lower semicontinuous* on X, if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \phi$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \phi$, for each $y \in U$. T is said to be *continuous*, if it is both *upper* and *lower semicontinuous*. It is also known that T is *lower semicontinuous* if and only if for each open set V in Y, the set $\{x \in X : T(x) \subset V\}$ is closed in X.

Definition 2.4. Let X, Y be the two topological spaces, and $C \subset Y$ be a closed convex cone with apex at origin, and $D \subset X$ be a nonempty subset. Let $T : D \to 2^Y$ be a set-valued map. Then, T is said to be:

(i) *C-lower semicontinuous at $x_0 \in D$* (in short, C-l.s.c), if for any open set V in Y with $T(x_0) \cap V \neq \phi$, there exists a neighborhood U of x_0 in D such that

$$T(x) \cap (V + C) \neq \phi, \text{ for each } x \in D \cap U.$$

(ii) *C-upper semicontinuous at $x_0 \in D$* (in short, C-u.s.c), if for any open set V in Y with $T(x_0) \subset V$, there exists a neighborhood U of x_0 in D such that

$$T(x) \subset V + C, \text{ for each } x \in D \cap U.$$

Lemma 2.5. *Let D, X, Y and C be as in the Definition 2.4 and $T : D \to 2^Y$ be a C-lower semicontinuous function. Then, the set $A = \{x \in D : T(x) \subseteq -C\}$ is closed in D.*

Proof. Let us assume that $intC \neq \phi$. Let $x \in D$ and a net $\{x_t\}$ in A such that $x_t \longrightarrow x$. To complete the proof, we need to show that $x \in A$. If $x \notin A$, then by definition of A, we have $T(x) \subset Y \setminus -C$. Hence, for $y_0 \in T(x)$ such that $Y \setminus -C$ is an open neighborhood of y_0. Since T is C-lower semicontinuous and $Y \setminus -C$ is an open neighborhood of y_0, then there exists a neighborhood U of x such that

$$T(x) \cap (Y \setminus -C + C) \neq \phi, \forall x \in U.$$

$$\Rightarrow T(x) \cap (Y \setminus -C) \neq \phi, \forall x \in U. \tag{2.3}$$

Since $x_t \longrightarrow x$, there exists t_0 such that $x_t \in U \cap D, \forall t \geq t_0$. Thus from (2.3), we have $T(x_t) \cap (Y \setminus -C) \neq \phi$, that is, $x_t \notin A$, a contradiction. ∎

Definition 2.6. (Knaster-Kuratowski-Mazurkiewicz) Let D be a nonempty convex subset of a vector space X. A set-valued mapping $F : D \to 2^X$ is called *KKM-mapping*, if for each finite subset $\{x_1, x_2....x_n\} \subset D$, we have

$$co\{x_1, x_2 \cdots x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $co(E)$ is a convex hull of a set D.

Lemma 2.7. [2] *(Fan-Lemma) Let X be a Hausdorff topological vector space, and let D be a non empty convex subset of X. Let $F : D \to 2^X$ be a KKM -mapping. If each $F(x)$ is closed and at least one $F(x)$ is compact, then $\bigcap_{x \in D} F(x) \neq \phi$.*

Self-segment Dense Set

Let X be a Hausdorff topological vector space. We will use following notations for the open, respectively closed, line segments in X with the endpoints x and y:

$$(x, y) = \{z \in X : z = x + t(y - x), t \in (0, 1)\},$$

$$[0, 1] = \{z \in X : z = x + t(y - x), t \in [0, 1]\}.$$

In [11], Definition 3.4, Luc has introduced the notion of so-called segment dense set. Let $V \subseteq X$ be a convex set. One says that the set $U \subseteq V$ is *segment dense* in V, if for each $x \in V$ there can be found $y \in U$ such that x is a cluster point of the set $[x, y] \cap U$.

Laszlo and Viorel [10] presented a denseness notion which is slightly different from the concept of the Luc [11] presented above.

Consider the sets $U \subseteq V \subseteq X$ and assume that V is convex. We say that U is *self-segment dense* in V, if U is dense in V and

$$\forall x, y \in U, \text{ the set } [x, y] \cap U \text{ is dense in } [x, y].$$

Example 2.8. (see [10]) Let V be the two dimensional Euclidean space $\mathbb{R}^2$ and define U to be the set

$$U := \{(p, q) \in \mathbb{R}^2 : p \in \mathbb{Q}, q \in \mathbb{Q}\}.$$

Then, it is clear that U is dense in $\mathbb{R}^2$. On the other hand U is not segment dense set in $\mathbb{R}^2$, since for $x = (0, \sqrt{2}) \in \mathbb{R}^2$ and for every $y = (p, q) \in U$, one has $[x, y] \cap U = \{y\}$. It can be easily observed that U is self-segment dense set in $\mathbb{R}^2$, since for every $x, y \in U, x =$

$(p,q), y=(r,s)$, we have $[x,y]\cap U=\{(p+t(r-p), q+t(s-q)) : t\in[0,1]\cap\mathbb{Q}\}$, which is obviously dense in [x,y].

Example 2.9. Consider $\mathbb{R}^2$ with usual topology and define V and U as follows:

$$V=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\right\} \text{ and } U=\left\{(p,q)\in\mathbb{Q}\times\mathbb{Q} : p^2+q^2\leq 1\right\}.$$

Then U is dense in V. Also, U is not segment dense in V. For $x=(0,\frac{1}{\sqrt{5}})\in V$ and for any $y=(p,q)\in U$, we have $[x,y]\cap U=\{y\}$. Further, U is self-segment dense set in V. For any $x=(p,q), y=(r,s)\in U$, the set $[x,y]\cap U$ is dense in $[x,y]$.

Example 2.10. Consider $\mathbb{R}^2$ with usual topology and define V and U as follows:

$$V=[0,1]\times[0,1]\setminus[(1,0),(1,1)] \text{ and } U=(0,1)\times(0,1).$$

Then, $U\subset V$ but U is not dense in V.

Clearly, U is segment dense in V. For any $x=(a_1,b_1)\in V$, there exists $y=(a_2,b_2)\in U$ such that x is a cluster point of $[x,y]\cap U$. Also U is not self-segment dense set as U is not dense in V.

3. Existence Results for Strong Vector Equilibrium Problems

In this section, we established the existence results for set-valued strong vector equilibrium problems with and without compactness assumptions by making use of KKM-mapping and Fan-Lemma 2.7.

Lemma 3.1. *Let X and Y be two Hausdorff topological vector spaces and $K\subset X$ be nonempty convex set. Let $F : K\times K\to 2^Y$ be a set-valued mapping satisfying the following conditions:*

(i) *for all $x\in K, F(x,x)=\{0\}$,*
(ii) *F is C-strongly pseudomonotone mapping,*
(iii) *for any fixed $x,y\in K$, the mapping $g(t):=F(ty+(1-t)x,y), t\in[0,1]$, is $(-C)$-l.s.c at $t=0^+$,*
(iv) *for all $x\in K$, the mapping $F(x,.) : K\to 2^Y$ is C-convex.*

Then, the following conditions are equivalent:

(I) *Find $x_0\in K$ such that $F(x_0,y)\nsubseteq -C\setminus\{0\}, \forall y\in K$.*
(II) *Find $x_0\in K$ such that $F(y,x_0)\subseteq -C, \forall y\in K$.*

Proof. (I)⇒ (II). It follows from the C-strongly pseudomonotonicity of F.
(II)⇒ (I). Suppose that (II) holds, then there exists $x_0\in K$ such that

$$F(y,x_0)\subseteq -C, \forall y\in K.$$

To complete the proof, we need to show that there exists $x_0\in K$ such that

$$F(x_0,y)\nsubseteq -C\setminus\{0\}, \forall y\in K.$$

If possible, let us suppose that there exists $y_0\in K$ such that

$$F(x_0,y_0)\subseteq -C\setminus\{0\}. \tag{3.1}$$

Let $x_t=tx_0+(1-t)y_0$, where $t\in(0,1]$, then $x_t\in K$ and hence

$$F(x_t,x_0)\subseteq -C. \tag{3.2}$$

Since F is C-convex in the second argument, we have

$$\begin{aligned}\{0\} = F(x_t, x_t) &\subseteq tF(x_t, x_0) + (1-t)F(x_t, y_0) - C \\ &\subseteq -C + (1-t)F(x_t, y_0) - C \\ &\subseteq (1-t)F(x_t, y_0) - C \\ \Rightarrow F(x_t, y_0) &\subseteq C. \end{aligned} \tag{3.3}$$

Since F is $(-C)$-lower semicontinuous at $t = 0^+$ in the first argument and $x_t \to x_0$, we have $F(x_0, y_0) \subseteq C$. If not, then $F(x_0, y_0) \subseteq Y \setminus C$ and $Y \setminus C$ is an open neighborhood of $F(x_0, y_0)$, so there is a $\delta \in (0, 1]$ such that

$$F(x_t, y_0) \cap (Y \setminus C - C) \neq \phi, \forall t \in (0, \delta].$$

$$\Rightarrow F(x_t, y_0) \cap (Y \setminus C) \neq \phi, \forall t \in (0, \delta],$$

which is a contradiction to (3.3). Hence, $F(x_0, y_0) \nsubseteq -C \setminus \{0\}$.

Thus, there exists $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

■

Theorem 3.2. *Let X, Y be two Hausdorff topological vector spaces and $K \subset X$ be a nonempty convex compact set. Let $F : K \times K \to 2^Y$ be a set-valued map satisfying the following conditions:*

(i) *for all $x \in K, F(x, x) = \{0\}$,*
(ii) *for any fixed $x, y \in K$, the mapping $g(t) := F(ty + (1-t)x, y), t \in [0, 1]$, is $(-C)$ -l.s.c at $t = 0^+$,*
(iii) *F is $C-$ strongly pseudomonotone,*
(iv) *for all $x \in K, F(x, .)$ is C-convex and C-lower semicontinuous.*

Then, there exists $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

Proof. Define set-valued maps $A, B : K \to 2^K$ by

$$A(y) := \{x \in K : F(x, y) \nsubseteq -C \setminus \{0\}\}, \forall y \in K$$

and

$$B(y) := \{x \in K : F(y, x) \subseteq -C\}, \forall y \in K.$$

Clearly, for all $y \in K$ both $A(y)$ and $B(y)$ are nonempty.
Claim: A is a KKM-mapping.
If possible, let us suppose that there exist $\{y_1, y_2, .., y_n\} \subset K$ and $t_i \geq 0, 1 \leq i \leq n$, with $\sum_{i=1}^n t_i = 1$ such that

$$y = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n A(y_i).$$

$$\Rightarrow y \notin A(y_i), \forall i = 1, 2, .., n.$$

$$\Rightarrow F(y, y_i) \subseteq -C \setminus \{0\}, i = 1, 2, .., n. \tag{3.4}$$

Since F is C-convex in the second argument, we have

$$\begin{aligned}\{0\} = F(y,y) &\subseteq \sum_{i=1}^{n} t_i F(y,y_i) - C \\ &\subseteq -C \setminus \{0\} - C \\ &\subseteq -C \setminus \{0\},\end{aligned}$$

a contradiction. Hence A is a KKM-mapping.

Also, F is C-strongly pseudomonotone mapping, so $A(y) \subset B(y)$. Hence B is a KKM-mapping. By Lemma 3.1, we have $\bigcap_{y\in K} A(y) = \bigcap_{y\in K} B(y)$.

Next, we claim that $B(y)$ is compact, for all $y \in K$. Since F is C-lower semicontinuous in the second argument. Hence $B(y)$ is closed by Lemma 2.5 , for all $y \in K$. Since $B(y)$ is closed and $B(y) \subset K$, it follows that $B(y)$ is compact for all $y \in K$. Hence, by Ky-Fan Lemma 2.7, we have

$$\bigcap_{y\in K} B(y) \neq \phi$$

and so,

$$\bigcap_{y\in K} A(y) \neq \phi.$$

Hence, there exists $x_0 \in K$ such that

$$F(x_0,y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

■

Theorem 3.3. *Let X, Y be two Hausdorff topological vector spaces and $K \subset X$ be a nonempty closed convex set. Let $F : K \times K \to 2^Y$ be a set-valued map satisfying the following conditions:*

(i) *for all $x \in K, F(x,x) = \{0\}$,*
(ii) *for any fixed $x, y \in K$, the mapping $g(t) := F(ty + (1-t)x, y), t \in [0,1]$, is $(-C)$-l.s.c at $t = 0^+$,*
(iii) *F is C-strongly pseudomonotone, ı*
(iv) *for all $x \in K, F(x,.)$ is C-convex and C-lower semicontinuous,*
(v) *there exists a nonempty compact convex subset $K_0 \subset X$ such that, for each $x_1 \in K \setminus K_0$, there exists some $y_1 \in K_0$ such that $F(y_1,x_1) \subseteq C \setminus \{0\}$.*

Then, there exists $x_0 \in K_0$ such that

$$F(x_0,y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

Proof. Define set-valued maps $A, B : K \to 2^K$ by

$$A(y) := \{x \in K : F(x,y) \nsubseteq -C \setminus \{0\}\}, \forall y \in K$$

and

$$B(y) := \{x \in K : F(y,x) \subseteq -C\}, \forall y \in K.$$

Clearly, for all $y \in K$ both $A(y)$ and $B(y)$ are nonempty.

Hence, by the proof of Theorem 3.2, both A and B are KKM-mappings. Also by Lemma 3.1, we have,

$$\bigcap_{y\in K} A(y) = \bigcap_{y\in K} B(y). \tag{3.5}$$

First, we show that $B(y)$ is closed, for all $y \in K$. For this, since F is C-lower semicontinuous in the second argument and so by Lemma 2.5, $B(y)$ is closed, $\forall y \in K$.
Claim: $B(y_1) \subset K_0$, where $y_1 \in K_0$, as in assumption (v). If possible, let us suppose that $B(y_1) \not\subset K_0$. Then, there exists $x_1 \in B(y_1)$ such that $x_1 \notin K_0$. Now $x_1 \in B(y_1)$ implies that $F(y_1, x_1) \subseteq -C$, and $x_1 \notin K_0$, a contradiction to assumption (v). Hence $B(y_1) \subset K_0$ and K_0 is compact implies that $B(y_1)$ is compact. Since $B(y)$ is closed for all $y \in K$ and compact for some $y_0 \in K$.

Hence, by Fan-Lemma 2.7, we have $\bigcap_{y\in K} B(y) \neq \phi$.

Thus, from (3.5), we have

$$\bigcap_{y\in K} A(y) \neq \phi.$$

Hence, there exists $x_0 \in K_0$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

■

4. Self-segment-dense Set and Strong Vector Equilibrium Problems

In this section, we obtained the existence results for set-valued strong vector equilibrium problem with and without compactness assumptions, by making use of self-segment-dense set.

Lemma 4.1. [9, Lemma 3.1]*Let X be a Hausdorff locally convex convex topological vector space, let $V \subseteq X$ be a convex set and let $U \subseteq V$ a self-segment-dense in V. Then, for all finite subset $\{u_1, u_2 \cdots u_n\} \subseteq U$ one has*

$$cl(co\{u_1, u_2 \cdots u_n\} \cap D) = co\{u_1, u_2 \cdots u_n\}.$$

Theorem 4.2. *Let X, Y be two Hausdorff locally convex topological vector spaces and K be a nonempty convex compact subset of X. Let $D \subset K$ be a self segment dense set. Let $F : K \times K \to 2^Y$ be set-valued map satisfying the following conditions:*

- (i) *for all $x \in D$, $F(x, x) = \{0\}$,*
- (ii) *F is C-strongly pseudomonotone,*
- (iii) *for any fixed $x, y \in K$, the mapping $g(t) := F(ty + (1-t)x, y), t \in [0, 1]$, is $(-C)$-l.s.c at $t = 0^+$,*
- (iv) *for all $x \in D$, $F(x, .)$ is C-convex on K,*
- (v) *for all $y \in D, F(., y)$ C-lower semicontinuous on K,*
- (vi) *for all $y \in K, F(., y)$ C-lower semicontinuous on $K \setminus D$.*

Then, there exists $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

Proof. Define set-valued maps $A, B : D \to 2^K$ by

$$A(y) := \{x \in D : F(x,y) \nsubseteq -C \setminus \{0\}\}, \forall y \in D$$

and

$$B(y) := \{x \in D : F(y,x) \subseteq -C\}, \forall y \in D.$$

Clearly, for all $y \in D$ both $A(y)$ and $B(y)$ are nonempty.
Claim: A is a KKM-mapping.
That is, for any $\{y_1, y_2, ..y_n\} \subset D$, we have

$$co\{y_1, y_2, ..y_n\} \cap D \subseteq \bigcup_{i=1}^{n} A(y_i).$$

If possible, let us suppose that there exist $\{y_1, y_2, .., y_n\} \subset D$ and $t_i \geq 0, 1 \leq i \leq n$, with $\sum_{i=1}^{n} t_i = 1$ such that

$$y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} A(y_i).$$

$$\Rightarrow y \notin A(y_i), \forall i = 1, 2, .., n.$$

$$\Rightarrow F(y, y_i) \subseteq -C \setminus \{0\}, i = 1, 2, .., n. \tag{4.1}$$

Since F is C-convex in the second argument, it follows that

$$\begin{aligned}\{0\} = F(y,y) &\subseteq \sum_{i=1}^{n} t_i F(y, y_i) - C \\ &\subseteq -C \setminus \{0\} - C \\ &\subseteq -C \setminus \{0\},\end{aligned}$$

a contradiction. Thus, for any $\{y_1, y_2, ..y_n\} \subset D$, we have

$$co\{y_1, y_2, ..y_n\} \cap D \subseteq \bigcup_{i=1}^{n} A(y_i).$$

$$\Rightarrow cl\,(co\{y_1, y_2, ..y_n\} \cap D) \subseteq cl(\bigcup_{i=1}^{n} A(y_i)).$$

By Lemma 3.1, $cl(co\{y_1, y_2, ..y_n\} \cap D) = co\{y_1, y_2, ..y_n\}$ and $cl(\bigcup_{i=1}^{n} A(y_i)) = \bigcup_{i=1}^{n} A(y_i)$, we have

$$co\{y_1, y_2, ..y_n\} \subseteq \bigcup_{i=1}^{n} A(y_i).$$

Hence, A is a KKM-mapping.

Also, F is C-strongly pseudomonotone mapping, so $A(y) \subset B(y)$ and hence B is a KKM-mapping. By Lemma 3.1, we have

$$\bigcap_{y \in D} A(y) = \bigcap_{y \in D} B(y).$$

Next, we claim that $B(y)$ is compact for all $y \in D$. Since F is lower semicontinuous in the first argument and hence by Lemma 2.5, $B(y)$ is closed for all $y \in D$. Since $B(y)$ is closed and $B(y) \subset K$, it follows that $B(y)$ is compact for all $y \in D$. Hence, by Fan-Lemma 2.7, we have

$$\bigcap_{y \in D} B(y) \neq \phi.$$

So,

$$\bigcap_{y \in D} A(y) \neq \phi.$$

Hence, there exists $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in D.$$

By Lemma 3.1, there exists $x_0 \in K$ such that

$$F(y, x_0) \subseteq -C, \forall y \in D. \tag{4.2}$$

To complete the proof, we need to show that there exists $x_0 \in K$ such that

$$F(y, x_0) \subseteq -C, \forall y \in K.$$

If possible, let us suppose that there exists $y_0 \in K \setminus D$ such that

$$F(y_0, x_0) \nsubseteq -C$$

$$\Rightarrow F(y_0, x_0) \subseteq Y \setminus -C.$$

Since F is lower semicontinuous in the first argument and $Y \setminus -C$ is open with $F(y_0, x_0) \subseteq Y \setminus -C$, so there exists an open neighborhood U of y_0 in K such that

$$F(y, x_0) \cap (Y \setminus -C) \neq \phi, \forall y \in U.$$

As D is dense in K, there exists $y_1 \in D \cap U$ such that $F(y_1, x_0) \cap (Y \setminus -C) \neq \phi$, a contradiction to (4.2). Hence, there exists $x_0 \in K$ such that

$$F(x_0, y) \subseteq -C, \forall y \in K.$$

By Lemma 3.1, there exists $x_0 \in K$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

■

Theorem 4.3. *Let X, Y be two Hausdorff locally convex topological vector spaces and K be a nonempty closed convex subset of X. Let $D \subset K$ be a self segment dense set. Let $F : K \times K \to 2^Y$ be set-valued map satisfying the following conditions:*

(i) *for all $x \in D$, $F(x, x) = \{0\}$,*
(ii) *F is C-strongly pseudomonotone,*
(iii) *for any fixed $x, y \in K$, the mapping $g(t) := F(ty + (1-t)x, y), t \in [0, 1]$, is $(-C)$-l.s.c at $t = 0^+$,*
(iv) *for all $x \in D$, $F(x, .)$ is C-convex on K,*
(v) *for all $y \in D, F(., y)$ C-lower semicontinuous on K,*
(vi) *for all $x \in K, F(x, .)$ C-lower semicontinuous on $K \setminus D$,*
(vii) *there exists a nonempty compact convex subset $K_0 \subset X$ such that, for each $x_0 \in K \setminus K_0$, there exists some $y_0 \in D \cap K_0$ such that $F(y_0, x_0) \subseteq C \setminus \{0\}$.*

Then, there exists $x_0 \in K$ *such that*

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

Proof. Define set-valued maps $A, B : D \to 2^K$ by

$$A(y) := \{x \in K : F(x, y) \nsubseteq -C \setminus \{0\}\}, \forall y \in D.$$

and

$$B(y) := \{x \in K : F(y, x) \subseteq -C\}, \forall y \in D.$$

Clearly, for all $y \in D$ both $A(y)$ and $B(y)$ are nonempty.
Thus, by Theorem 4.2, both A and B are KKM-mapping. Also by Lemma 3.1, we have

$$\bigcap_{y \in D} A(y) = \bigcap_{y \in D} B(y). \tag{4.3}$$

Since F is C-lower semicontinuous in the first argument and hence by Lemma 2.5, we conclude that $B(y)$ is closed, $\forall y \in D$.
Claim: $B(y_0) \subset K_0$, where $y_0 \in K_0$, as in assumption (vii). If possible, let us suppose that $B(y_0) \not\subset K_0$. Then, there exists $x_0 \in B(y_0)$ such that $x_0 \notin K_0$. It means that $F(y_0, x_0) \subseteq -C, x_0 \notin K_0$, a contradiction to assumption (vii). Hence $B(y_0) \subset K_0$ and so $B(y_0)$ is compact. Since $B(y)$ is closed for all $y \in K$ and compact for some $y_0 \in K$. Hence by Fan Lemma 2.7, we have

$$\bigcap_{y \in D} B(y) \neq \phi.$$

From (4.3), we have

$$\bigcap_{y \in D} A(y) \neq \phi.$$

Hence, there exists $x_0 \in K_0$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in D.$$

By Lemma 3.1, there exists $x_0 \in K_0$ such that

$$F(y, x_0) \subseteq -C, \forall y \in D. \tag{4.4}$$

To complete the proof, we need to show that there exists $x_0 \in K_0$ such that

$$F(y, x_0) \subseteq -C, \forall y \in K.$$

If possible, let us assume that there exists $y_0 \in K \setminus D$ such that

$$F(y_0, x_0) \nsubseteq -C.$$

$$\Rightarrow F(y_0, x_0) \subseteq Y \setminus -C.$$

Since F is C-lower semicontinuous in the first argument and $Y \setminus -C$ is open in Y. Therefore, there exists a neighborhood U of y_0 such that

$$F(y, x_0) \cap (Y \setminus -C + C) \neq \phi, \forall y \in U.$$

$$\Rightarrow F(y, x_0) \cap (Y \setminus -C) \neq \phi, \forall y \in U.$$

Since D is dense in K, so there exists $y_1 \in U \cap D$ such that $F(y_1, x_0) \cap (Y \setminus -C) \neq \phi$, a contradiction to (4.4). Thus, there exists $x_0 \in K_0$ such that

$$F(y, x_0) \subseteq -C, \forall y \in K.$$

Hence, by Lemma 3.1, there exists $x_0 \in K_0$ such that

$$F(x_0, y) \nsubseteq -C \setminus \{0\}, \forall y \in K.$$

■

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