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Common Fixed Point Results for (α, β)-orbital-cyclic Admissible Triplet in Extended b-metric Spaces

Nehjamang Haokip[1,*], Nilakshi Goswami[1] and Binod Chandra Tripathy[2]

[1] *Department of Mathematics, Gauhati University, Guwahati - 14, India*
e-mail : mark02mm@yahoo.co.in (N. Haokip); nila_g2003@yahoo.co.in (N. Goswami)

[2] *Department of Mathematics, Tripura University, Agartala - 22, India*
e-mail : tripathybc@yahoo.com

Abstract In this paper, we derive a common fixed point result for a class of contractive mappings which is a generalization of the contractive condition given in [1]. A convergence result for an iteration scheme is also obtained for such mappings. Suitable examples have been provided in support of the results obtained.

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1. Introduction

There have been numerous attempts to generalize the *Banach Contraction Principle* of 1922 by either considering a more generalized space or a more generalized (or different) contractive condition or both (one may refer to [5, 7, 8, 10, 11, 14, 17, 19, 22, 26, 27, 30–32], and the references therein).

The notion of *b-metric spaces* is also a consequence of an attempt to generalize the Banach Contraction Principle to a more generalized space. Many authors have contributed to the fixed point theory in b-metric spaces (one may refer to [12, 13, 18, 23, 25] and the references therein).

In 2017, the notion of *extended b-metric*, which is a generalization of a b-metric was introduced by Kamran et al. [16]. In 2018, Alqahtani et al. [1] proved a common fixed point result for a pair of mappings in an extended b-metric space. In this paper, we obtain a common fixed point result for a triplet of mappings from which we derive the main result of [1] as a corollary. An iteration scheme for the triplet of mappings is also defined and its convergence is studied. Lastly, the rate of convergence of the new iteration is compared to a known iteration scheme to show that the new iteration scheme converge faster to the common fixed point.

*Corresponding author.

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2. Preliminaries

In this section, we reproduce and introduce some definitions which will be used in our main results.

Definition 2.1. [6] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, \infty)$ is called a b-metric if it satisfies the following properties.

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$; and
(3) $d(x, z) \leq s\{d(x, y) + d(y, z)\}$, for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space with coefficient s.

Definition 2.2. [20] Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence in X. Then we say $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$, denoted by $\lim_{n\to\infty} x_n = x$; $\{x_n\}$ is said to be a Cauchy sequence if and only if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$; and (X, d) is said to be complete if and only if every Cauchy sequence is convergent.

Definition 2.3. [20] Let (X, d) be a b-metric space. A mapping $T : X \longrightarrow X$ is said to be continuous at a point $x \in X$ if for every sequence $\{x_n\}$ converging to x, we have

$$\lim_{n\to\infty} Tx_n = Tx.$$

T is said to be continuous in X if it is continuous at all points of X.

A b-metric need not be continuous, one may refer to Example 2 in [25].

In 2017, Kamran et al. [16] introduced a generalized form of b-metric space and proved some fixed point theorems.

Definition 2.4. [16] Let X be a nonempty set and $\mu : X \times X \longrightarrow [1, \infty)$. An *extended b-metric* is a function $d_\mu : X \times X \longrightarrow [0, \infty)$ such that for all $x, y, z \in X$

($d_\mu 1$) $d_\mu(x, y) = 0$ if and only if $x = y$,
($d_\mu 2$) $d_\mu(x, y) = d_\mu(y, x)$,
($d_\mu 3$) $d_\mu(x, y) \leq \mu(x, y)\Big(d_\mu(x, z) + d_\mu(z, y)\Big)$.

The pair (X, d_μ) is called an *extended b-metric space.*

Example 2.5. Let $X = [-1, 1]$ and $\mu : X \times X \longrightarrow [1, \infty)$ be defined by $\mu(x, y) = \frac{1+x^2+y^2}{x^2+y^2}$. Define $d_\mu : X \times X \longrightarrow [0, \infty)$ as follows.

$$d_\mu(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

$$d_\mu(x, 0) = d_\mu(0, x) = \frac{1}{x^2} \quad \text{if} \quad x \neq 0,$$

$$d_\mu(x, y) = \frac{1}{x^2 y^2} \quad \text{if} \quad 0 \neq x \neq y \neq 0.$$

Then it can be easily checked that d_μ defines an extended b-metric on X.

We note here that if $\mu(x, y) = s$ for some $s \geq 1$, then we get the definition of a b-metric space. The notion of convergence, completeness and continuity in b-metric spaces can as well be extended to the extended b-metric space. Then regardless of the continuity of d_μ, a convergent sequence in a b-metric space have a unique limit. Then the result of [16] may be restated as follows.

Theorem 2.6. [16] *Let (X, d_μ) be an extended b-metric space and $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$*

$$d_\mu(Tx, Ty) \leq d_\mu(x, y) \tag{2.1}$$

where $0 \leq k < 1$ is such that, for each $x_0 \in X$, $\lim_{m,n\to\infty}(x_m, x_n) < \frac{1}{k}$ and $x_n = T^n x_0$, $n = 1, 2, \ldots$. Then T has a unique fixed point w in X. Moreover, for each $x \in X$, $T^n x \longrightarrow w$.

Following the definition of (α, β)-*orbital-cyclic admissible pair* given by Alqahtani et al. [1], we define the following.

Definition 2.7. Let$f$, S and T be self maps on a complete extended b-metric space (X, d_μ) with f injective and $\alpha, \beta : X \times X \longrightarrow [0, \infty)$ be two mappings such that for any $x \in X$

$$\left.\begin{array}{l} \alpha(fx, fTx) \geq 1 \implies \beta(fTx, fSTx) \geq 1 \\ \beta(fx, fSx) \geq 1 \implies \alpha(fSx, fTSx) \geq 1 \end{array}\right\} \tag{2.2}$$

Then f, S and T are said to be (α, β)-*orbital-cyclic admissible triplet.*

The following notion of *Banach operator pair* will be used in this paper. It was first introduced by Subrahmanyam [28] and extended by Chen and Li [4] and, Öztürk and Başarir [21].

Definition 2.8. [21] Let f and T be self mappings on an extended b-metric space (X, d_μ). Then the pair (f, T) is said to be a *Banach operator pair* if for some $k \geq 0$,

$$d_\mu(fTx, Tx) \leq k d_\mu(Tx, x) \qquad \text{for all } x \in X.$$

We make a note here that if the pair (f, T) is a Banach operator pair, then f and T commutes on the set $F(T)$ of the fixed points of T (one may refer to Proposition 2.2 [4]).

We now state the following terminology for use in our main results.

Definition 2.9. Let T and f be self mappings on an extended b-metric space (X, d_μ). T is said to be Cauchy-commutative with respect to f, if for any sequence $\{x_n\}$ in X such that $\{fx_n\}$ is a Cauchy sequence, $fTx = Tfx$ for each x in $\{x_n\}$.

Definition 2.10. A function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is called a *subadditive altering distance function* if

(i) $\phi(x + y) \leq \phi(x) + \phi(y) \qquad \forall x, y \in [0, \infty)$
(ii) ϕ is an *altering distance function* [9], (i.e., ϕ is continuous, strictly increasing and $\phi(t) = 0$ if and only if $t = 0$)

Example 2.11. [13] It can be easily seen that the functions $\phi_1(x) = kx$ for some $k \geq 1$, $\phi_2(x) = \sqrt[n]{x}$, $n \in \mathbb{N}$, $\phi_3(x) = \log(1 + x)$, $x \geq 0$ and $\phi_4(x) = \tan^{-1} x$ are such subadditive altering distance functions.

3. Main Results

Assuming the continuity of d_μ, we first prove a generalization of the result obtained in [1]. The following lemma follows from the proof of Lemma 2 of [2] from the fact that ϕ is a subadditive altering distance function.

For a real number $\theta \geq 0$, let θ^* be the least integer $\geq \theta$.

Lemma 3.1. *Let (X, d_μ) be an extended b-metric space. Suppose there exists $q \in [0,1)$ such that for an arbitrary $x_0 \in X$, the sequence $\{x_n\}$ satisfies*

$$\lim_{n,m\to\infty} \mu(x_n, x_m)^* < \frac{1}{q} \quad \text{and} \quad \phi\big(d_\mu(x_{n+1}, x_n)\big) \le q\phi\big(d_\mu(x_n, x_{n-1})\big)$$

for all positive integer n, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 3.2. If ϕ is the identity mapping on $[0, \infty)$, then as seen in Lemma 2.1 of [1], the condition $\lim_{n,m\to\infty} \mu(x_n, x_m)^* < \frac{1}{q}$ in Lemma 3.1 (and consequently in the proof of the following theorems) may be replaced by $\lim_{n,m\to\infty} \mu(x_n, x_m) < \frac{1}{q}$.

Theorem 3.3. *Let (X, d_μ) be a complete extended b-metric space and $f, S, T : X \longrightarrow X$ be (α, β)-orbital-cyclic admissible triplet on X. Let (f, S) and (f, T) be Banach operator pairs such that for all $x, y \in X$*

$$\alpha(fx, fTx)\beta(fy, fSy)\phi\big(d_\mu(fTx, fSy)\big) \le k_1\phi\big(d_\mu(fx, fy)\big) + k_2\phi\big(d_\mu(fx, fTx)\big) + k_3\phi\big(d_\mu(fy, fSy)\big) \tag{3.1}$$

for some $k_1, k_3 \ge 0$, $k_2 > 0$ and $k_1 + k_2 + k_3 < 1$. Suppose that there exists $x_0 \in X$ such that $\alpha(fx_0, fTx_0) \ge 1$. Let for each $x_0 \in X$, $\lim_{n,m\to\infty} \mu(fx_n, fx_m)^* < \frac{1-k_3}{k_1+k_2}$, where $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$ for all positive integers n.

(a) f, S and T have a unique common fixed point, if S and T are continuous and Cauchy commutative with respect to f, and $\alpha(fz, fz) \ge 1$ for $z \in CF(S,T)$, where $CF(S,T)$ denotes the set of common fixed points of S and T.

(b) f, S and T have a unique common fixed point if f is continuous and $\{x_n\} \subseteq X$ is a sequence such that $\lim_{n\to\infty} x_n = z$, then $\alpha(fz, fTz) \ge 1$ and $\beta(fz, fSz) \ge 1$.

Proof. By the given condition, there exists x_0 in X with $\alpha(fx_0, fTx_0) \ge 1$. Taking $x_1 = Tx_0$, $x_2 = Sx_1$, and inductively we construct a sequence $\{x_n\}$, where

$$x_{2n-1} = Tx_{2n-2} \quad \text{and} \quad x_{2n} = Sx_{2n-1}, \qquad n = 1, 2, 3, \ldots \tag{3.2}$$

Since f, S and T are (α, β)-orbital-cyclic admissible triplet, we get (as in [1]),

$$\alpha(fx_{2n}, fx_{2n+1}) \ge 1 \quad \text{and} \quad \beta(fx_{2n+1}, fx_{2n+2}) \ge 1 \qquad n = 0, 1, 2, \ldots \tag{3.3}$$

We assume, without loss of generality, that $x_n \ne x_{n+1}$ for all non-negative integers n. Because, if $x_{n_0} = x_{n_0+1}$ for some non-negative integer n_0, then by our choice of the sequence $\{x_n\}$, we can show that $w = x_{n_0}$ is a common fixed point of f, S and T and the proof is complete.

For this, we consider the following two cases for n_0.

If n_0 is even, say, $n_0 = 2p$, then $x_{2p} = x_{2p+1} = Tx_{2p}$ and x_{2p} is a fixed point of T. Since (f, T) is a Banach operator pair, we have for some $k \ge 0$,

$$d_\mu\big(fx_{2p}, x_{2p}\big) = d_\mu\big(fTx_{2p}, Tx_{2p}\big) \le kd_\mu\big(Tx_{2p}, x_{2p}\big) = 0,$$

showing that x_{2p} is also a fixed point of f.

We also claim that $x_{2p} = x_{2p+1} = Tx_{2p} = Sx_{2p+1}$. Suppose the contrary that $Tx_{2p} \ne Sx_{2p+1}$. Then taking $x = x_{2p}$ and $y = x_{2p+1}$ in (3.1), and using (3.2) and (3.3) we have,

$$\phi\big(d_\mu(fTx_{2p}, fSx_{2p+1})\big) \le k_3\phi\big(d_\mu(fTx_{2p}, fSx_{2p+1})\big).$$

Since ϕ is subadditive altering distance function and $k_3 < 1$, this implies

$$d_\mu(fTx_{2p}, fSx_{2p+1}) < d_\mu(fTx_{2p}, fSx_{2p+1}),$$

a contradiction.

Hence $d_\mu(Tx_{2p}, Sx_{2p+1}) = 0$ and $x_{2p} = x_{2p+1} = Tx_{2p} = Sx_{2p+1}$ so that $x_{2p} = x_{2p+1} = w$ is a common fixed point of f, S and T.

Similarly, we get an analogous result for the case when n_0 is odd, that is, $n_0 = 2p+1$.

Thus we assume that $x_n \neq x_{n+1}$ for all non-negative integers n.

We now show that $\{fx_n\}$ is a Cauchy sequence. For doing the same, it is sufficient to consider the cases when $x = x_{2n}$, $y = x_{2n+1}$ and $x = x_{2n}$, $y = x_{2n-1}$.

Case (i) Let $x = x_{2n}$ and $y = x_{2n+1}$. By (3.1) and (3.3), we have,

$$\phi\big(d_\mu(fx_{2n+1}, fx_{2n+2})\big) = \phi\big(d_\mu(fTx_{2n}, fSx_{2n+1})\big) \le q\phi\big(d_\mu(fx_{2n}, fx_{2n+1})\big)$$

where $q = \frac{k_1+k_2}{1-k_3} < 1$ and $n = 0, 1, 2, \ldots$.

Case (ii) Let $x = x_{2n}$ and $y = x_{2n-1}$. Similarly, as in the above case, we get,

$$\phi\big(d_\mu(fx_{2n}, fx_{2n+1})\big) \le q\phi\big(d_\mu(fx_{2n-1}, fx_{2n})\big).$$

Therefore, from the above two cases we have, for all $n \in \mathbb{N}$,

$$\phi\big(d_\mu(fx_n, fx_{n+1})\big) \le q\phi\big(d_\mu(fx_{n-1}, fx_n)\big).$$

Hence by Lemma 3.1, $\{fx_n\}$ is a Cauchy sequence in X and there exists $w \in X$ such that $\lim_{n\to\infty} fx_n = w$, and consequently

$$\lim_{n\to\infty} fx_{2n} = w \qquad \text{and} \qquad \lim_{n\to\infty} fx_{2n+1} = w.$$

(a) Now, since S and T are continuous and Cauchy commutative with respect to f, we have,

$$Sw = S\left(\lim_{n\to\infty} fx_{2n-1}\right) = \lim_{n\to\infty} Sfx_{2n-1} = \lim_{n\to\infty} fSx_{2n-1} = \lim_{n\to\infty} fx_{2n} = w,$$
$$Tw = T\left(\lim_{n\to\infty} fx_{2n}\right) = \lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} fTx_{2n} = \lim_{n\to\infty} fx_{2n-1} = w,$$

showing that w is a common fixed point of S and T.

If w' is another common fixed point of S and T, then $w, w' \in CF(S,T)$. So, $\alpha(fw, fw) \ge 1$ and $\alpha(fw', fw') \ge 1$. Then $\alpha(fw, fTw) = \alpha(fw, fw) \ge 1$ and $\beta(fw', fSw') = \beta(fw', fw') \ge 1$ (since $\alpha(fx, fTx) \ge 1$ implies $\beta(fTx, fSTx) \ge 1$). And, using (3.1),

$$\phi\big(d_\mu(fw, fw')\big) = \phi\big(d_\mu(fw, fw')\big) \le k_1\phi\big(d_\mu(fw, fw')\big),$$

which is possible only if $fw = fw'$. Since f is injective, we have $w = w'$. Thus the common fixed point of S and T is unique.

Since (f,S) and (f,T) are Banach operator pairs, f commute with S and T at the fixed points of S and T, respectively. This implies $fSw = Sfw$ for $w \in F(S)$, that is, $fw = Sfw$, showing that fw is another fixed point of S. Similarly, fw is also another fixed point of T, and hence a common fixed point of S and T.

Since the common fixed point of S and T is unique, we have $fw = w$, showing that w is a fixed point of f. Thus f, S and T have a common fixed point w, which is unique.

(b) Now considering the alternate hypothesis that $\{x_n\} \subseteq X$ be a sequence with $\lim_{n\to\infty} x_n = z$ implies $\alpha(fz, fTz) \ge 1$ and $\beta(fz, fSz) \ge 1$, we shall show that z is a unique common fixed point of f, S and T.

Taking $x = z$ and $y = x_{2n+1}$ in (3.1) and using (3.3), we get,

$$\phi\big(d_\mu(fTz, fx_{2n+2})\big) \le k_1\phi\big(d_\mu(fz, fx_{2n+1})\big) + k_2\phi\big(d_\mu(fz, fTz)\big) + k_3\phi\big(d_\mu(fx_{2n+1}, fx_{2n+2})\big).$$

Taking the limit as $n \to \infty$, we get $\phi\big(d_\mu(fTz, fz)\big) \le k_2\phi\big(d_\mu(fz, fTz)\big)$ and since $k_2 < 1$, this implies $fz = fTz$ or $z = Tz$.

In a similar manner, taking $x = x_{2n}$ and $y = z$ in (3.1) and using (3.3), we get $z = Sz$, showing that z is a common fixed point of S and T. The rest is analogous to the above proof of (a). ■

Taking $k_1 = 0$, $k_2 = k_3 = k$, and ϕ and f as the identity mapping in Theorem 3.3, we get a similar result given in Theorem 2.1 of [1] as a corollary.

Corollary 3.4. [1] *Let S, T be two self-mappings on a complete extended b-metric space (X, d_μ) such that the pair S, T forms an (α, β)-orbital-cyclic admissible pair satisfying*

$$\alpha(x, Tx)\beta(y, Sy)d_\mu(Tx, Sy) \le k\Big\{d_\mu(x, Tx) + d_\mu(y, Sy)\Big\}$$

for some $0 < k < \frac{1}{2}$. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Let for each $x_0 \in X$, $\lim_{n,m\to\infty} \mu(x_n, x_m)^* < \frac{1-k}{k}$, where $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$ for all positive integers n.

(a) *S and T have a unique common fixed point, if S and T are continuous and for $z \in CF(S, T)$, $\alpha(z, z) \ge 1$, where $CF(S, T)$ denotes the set of common fixed points of S and T.*

(b) *S and T have a unique common fixed point, if $\{x_n\} \subseteq X$ is a sequence with $\lim_{n\to\infty} x_n = w$ implies $\alpha(w, Tw) \ge 1$ and $\beta(w, Sw) \ge 1$.*

Example 3.5. Let $X = [0, 1]$ and $\mu : X \times X \longrightarrow [1, \infty)$ and $d_\mu : X \times X \longrightarrow [0, \infty)$ be defined by

$$\mu(x, y) = \begin{cases} \frac{1+x+y}{x+y}, & x + y \neq 0 \\ 1, & x + y = 0 \end{cases} \quad \text{and} \quad d_\mu(x, y) = \begin{cases} 0, & x = y \\ d_\mu(x, 0) = \frac{1}{x}, & x \neq 0 \\ \frac{1}{xy}, & xy \neq 0 \end{cases}$$

respectively. Then (X, d_μ) is an extended b-metric space [1].

Consider the mappings $f, S, T : X \longrightarrow X$ defined by $fx = x$,

$$Sx = \begin{cases} 1, & \text{if } \ x = \frac{1}{4}, \frac{3}{4} \\ \frac{x+1}{2}, & \text{otherwise} \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & \text{if } \ x = \frac{1}{2}, \frac{3}{4}, \\ \frac{2x+1}{3}, & \text{otherwise} \end{cases}$$

respectively. Also, let $\alpha, \beta : X \times X \longrightarrow [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Big\{(1, 1), \left(\frac{3}{4}, 1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{4}, \frac{1}{2}\right)\Big\} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Big\{(1, 1), \left(\frac{3}{4}, 1\right), \left(\frac{1}{4}, 1\right), \left(\frac{1}{2}, \frac{3}{4}\right)\Big\} \\ 0, & \text{otherwise} \end{cases}$$

respectively.

Then it can be easily checked that f, S and T form an (α, β)-orbital-cyclic admissible triplet. We also note that $x = 1$ is the only common fixed point of S and T, with $\alpha(f1, fT1) \geq 1$ and $\beta(f1, fS1) \geq 1$.

Let ϕ be the identity mapping. If $x_0 = 1, \frac{1}{2}$ or $\frac{3}{4}$, then $x_n = 1$ for all n, and so,

$$\lim_{n,m\to\infty} \mu(fx_n, fx_m) = \frac{3}{2} < \frac{8}{5} = \frac{1-k_3}{k_1+k_2}$$

where $k_1 = \frac{1}{16}$, $k_2 = \frac{1}{4}$ and $k_3 = \frac{1}{2}$.

On the other hand, if $x_0 \neq 1, \frac{1}{2}$ or $\frac{3}{4}$, then for all $n = 1, 2, 3, \ldots$

$$x_{2n-1} = \frac{1}{3^n}(2x_0 + 1) + 2\sum_{k=1}^{n-1}\left(\frac{1}{3}\right)^k \text{ and } x_{2n} = \frac{1}{3^n}\left(x_0 + \frac{1}{2}\right) + \frac{1}{2} + \sum_{k=1}^{n-1}\left(\frac{1}{3}\right)^k$$

Since $\lim_{n\to\infty} x_{2n-1} = 1$ and $\lim_{n\to\infty} x_{2n} = 1$, that is, $\lim_{n\to\infty} x_n = 1$ for all $n = 1, 2, 3, \ldots$, we get in this case too,

$$\lim_{n,m\to\infty} \mu(fx_n, fx_m) = \frac{3}{2} < \frac{1-k_3}{k_1+k_2}.$$

We also note that for any x_0 in X, the sequence $\{x_n\}$ as defined above is such that $\lim_{n\to\infty} x_n = 1$ with $\alpha(f1, fT1) \geq 1$ and $\beta(f1, fS1) \geq 1$.

Since $\alpha(x, y) = 0$ except at the points $(1, 1)$, $\left(\frac{3}{4}, 1\right)$, $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$; and $\beta(x, y) = 0$ except at the points $(1, 1)$, $\left(\frac{3}{4}, 1\right)$, $\left(\frac{1}{4}, 1\right)$ and $\left(\frac{1}{2}, \frac{3}{4}\right)$, one can easily check that f, S and T satisfy (3.1) and thus by Theorem 3.3, f, S and T have a unique common fixed point, $x = 1$.

When $S = T$ in Definition 2.7, T is then said to be (α, β)-orbital-cyclic admissible mapping with respect to f.

Corollary 3.6. *Let (X, d_μ) be a complete extended b-metric space and $T : X \longrightarrow X$ be (α, β)-orbital-cyclic admissible mapping with respect to f, and (f, T) be a Banach operator pair such that for all $x, y \in X$*

$$\alpha(fx, fTx)\beta(fy, fTy)\phi\big(d_\mu(fTx, fTy)\big) \leq k_1\phi\big(d_\mu(fx, fy)\big) + k_2\phi\big(d_\mu(fx, fTx)\big) + k_3\phi\big(d_\mu(fy, fTy)\big) \tag{3.4}$$

for some $k_1, k_3 \geq 0$, $k_2 > 0$ and $k_1 + k_2 + k_3 < 1$. Suppose that there exists $x_0 \in X$ such that $\alpha(fx_0, fTx_0) \geq 1$. Let for each $x_0 \in X$, $\lim_{n,m\to\infty} d_\mu(fx_n, fx_m)^* < \frac{1-k_3}{k_1+k_2}$, where $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$ for positive integers n.

(a) f and T have a unique common fixed point, if T is continuous and Cauchy commutative with respect to f, and for $z \in F(T)$, $\alpha(fz, fz) \geq 1$, where $F(T)$ denotes the set of fixed points of T.

(b) f and T have a unique common fixed point if f is continuous and $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} x_n = z$ implies $\alpha(fz, fTz) \geq 1$.

Proof. The proof follows from Theorem 3.3 by taking $S = T$. ∎

Example 3.7. Let $X = [0, 1]$ and define $d_\mu : X \times X \longrightarrow [0, \infty)$ and $\mu : X \times X \longrightarrow [1, \infty)$ by $d_\mu(x, y) = |x - y|$ and $\mu(x, y) = 1 + x + y$, respectively for all $x, y \in X$. Consider the mappings $f, T : X \longrightarrow X$ defined respectively by

$$fx = x \quad \text{and} \quad Tx = \frac{x}{4}, \quad \text{for all} \quad x \in X.$$

Consider the mappings $\alpha, \beta : X \times X \longrightarrow [0, \infty)$ defined by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1, & x,y \in \left[0, \frac{1}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

Then T is an (α, β)-orbital-cyclic admissible pair with respect to f. To see this, let $x \in X$ be such that $\alpha(fx, fTx) \geq 1$ and $\beta(fx, fTx) \geq 1$. Then $fx, fTx \in \left[0, \frac{1}{2}\right]$, i.e., x must be in $\left[0, \frac{1}{2}\right]$. Now, if $x \in \left[0, \frac{1}{2}\right]$, then $fTx = \frac{x}{4} \leq \frac{1}{2}$ and $fT^2x = \frac{x}{16} \leq \frac{1}{2}$, which implies $\alpha(fTx, fT^2x) = 1$ and $\beta(fTx, fT^2x) = 1$.

Clearly, for $x_0 = \frac{1}{3}$, say, $\alpha\left(fx_0, fTx_0\right) = \alpha\left(\frac{1}{3}, \frac{1}{12}\right) = 1$.

Again, for each $x_0 \in X$, $x_n = T^n x_0 = \frac{x_0}{4^n}$ for all positive integers n. So, we have,

$$\lim_{n,m \to \infty} \mu(x_n, x_m)^* = 1 < \frac{1 - k_3}{k_1 + k_2},$$

where $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{3}$ and $k_3 = \frac{1}{3}$.

Now, for $x, y \in \left[0, \frac{1}{2}\right]$, with $\phi(x) = x$ we have,

$$\begin{aligned}\alpha(fx, fTx)\beta(fy, fTy)\phi\big(d_\mu(fTx, fTy)\big) &= \frac{1}{4}|x - y| \leq \frac{1}{4}|x - y| + \frac{|x|}{4} + \frac{|y|}{4} \\ &= k_1\phi\big(d_\mu(fx, fy)\big) + k_2\phi\big(d_\mu(fx, fTx)\big) + k_3\phi\big(d_\mu(fy, fTy)\big).\end{aligned}$$

For $x, y \in \left(\frac{1}{2}, 1\right]$, $\alpha(fx, fTx) = \alpha\left(x, \frac{x}{4}\right) = 0$ and the inequality (3.4) follows trivially.

Again, for $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, $\beta(fy, fTy) = \beta\left(y, \frac{y}{4}\right) = 0$ and the inequality (3.4) follows trivially.

Hence by Corollary 3.6, T has a unique fixed point $x = 0$.

Corollary 3.8. *Let (X, d_μ) be a complete extended b-metric space and $T : X \longrightarrow X$ be a mapping satisfying*

$$\phi\big(d_\mu(Tx, Ty)\big) \leq k_1\phi\big(d_\mu(x, y)\big) + k_2\phi\big(d_\mu(x, Tx)\big) + k_3\phi\big(d_\mu(y, Ty)\big) \tag{3.5}$$

for some $k_1, k_3 \geq 0$, $k_2 > 0$ and $k_1 + k_2 + k_3 < 1$.

Let for each $x_0 \in X$, $\lim_{n,m\to\infty} d_\mu(x_n, x_m)^* < \frac{1-k_3}{k_1+k_2}$, where $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. Then T have a unique fixed point.

Proof. The proof follows from Corollary 3.6 (b) by taking $\alpha(x, y) = \beta(x, y) = 1$ and $fx = x$ for all $x \in X$. ∎

In [21], Öztürk and Başarir defined a self map T on a cone metric space X to have the *property P* if $F(T) = F\big(T^n\big)$ for all $n \in \mathbb{N}$. The same notion can as well be defined for an extended b-metric space.

Theorem 3.9. *Let (X, d_μ) be a complete extended b-metric space and $T : X \longrightarrow X$ be (α, β)-orbital-cyclic admissible map with respect to f satisfying (3.4) for some $k_1, k_3 \geq 0$, $k_2 > 0$ and $k_1 + k_2 + k_3 < 1$. If $\alpha(fx, fTx) \geq 1$ and $\beta(fx, fTx) \geq 1$ for all $x \in F(T)$, then T has property P.*

Proof. Since $Tu = u$ implies $T^n u = u$ for all $n \in \mathbb{N}$, it is sufficient to show that $F(T^n) \subseteq F(T)$. Let $w \in F(T^n)$, then it is clear that $Tw \in F(T^n)$.

Let if possible, $Tw \neq w$. Then using (3.4),

$$\phi\big(d_\mu(fT^{n+1}w, fT^{n+2}w)\big) \leq k'\phi\big(d_\mu(fT^n w, fT^{n+1}w)\big)$$

where $k' = \frac{k_1+k_2}{1-k_3} < 1$.

Since ϕ is subadditive altering distance function, for some $k < 1$ we get,

$$d_\mu(fT^{n+1}w, fT^{n+2}w) \leq kd_\mu(fT^n w, fT^{n+1}w).$$

But then,

$$d_\mu(fTw, fT^2w) = d_\mu(fT^{n+1}w, fT^{n+2}w) \leq \cdots \leq k^{n+1}d_\mu\left(fTw, fT^2w\right) < d_\mu(fTw, fT^2w),$$

a contradiction. ∎

3.1. Convergence of Iteration

In 1970, Takahashi [29] introduced the following concept of convex structure in a metric space.

Definition 3.10. [29] Let (X, d) be a metric space. A mapping $\mathcal{W} : X^2 \times [0,1] \longrightarrow X$ satisfying

$$d\left(z, \mathcal{W}(x, y, \alpha)\right) \leq \alpha d(z, x) + (1-\alpha)d(z, y)$$

for all $x, y, z \in X$ and $\alpha \in [0,1]$ is called a *convex structure* on X.

The above notion of convex structure can as well be adopted naturally in extended b-metric space (X, d_μ) with the condition

$$\mu(x, y)d_\mu\left(z, \mathcal{W}(x, y, \alpha)\right) \leq \alpha d_\mu(z, x) + (1-\alpha)d_\mu(z, y). \tag{3.6}$$

We now define an iteration process in a convex extended b-metric space and derive a strong convergence result for it.

Let (X, d_μ) be a convex extended b-metric space and f, S and T be self mappings on X. For $x_0 \in X$, we define

$$\begin{cases} fx_{n+1} = \mathcal{W}\left(fz_n, fTy_n, \alpha_n\right), \\ fy_n = \mathcal{W}\left(fTz_n, fSz_n, \beta_n\right), \\ fz_n = fSx_n \end{cases} \tag{3.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

Theorem 3.11. *Let (X, d_μ) be a complete convex extended b-metric space and $f, S, T : X \longrightarrow X$ be self mappings on X satisfying the conditions of Theorem 3.3 for some $k_1 \geq 0$, $k_2, k_3 > 0$ with $k_1 + k_2 + k_3 < 1$, so that f, S and T have a unique common fixed point. Let for all $x \in X$, $\mu(fx, fTx)^* < \frac{1-k_1}{2k_2}$ and $\mu(fx, fSx)^* < \frac{1-k_1}{2k_3}$. If in addition, $\alpha(fx, fTx) \geq 1$ and $\beta(fx, fSx) \geq 1$ for all $x \in X$, then the sequence $\{fx_n\}$ generated by (3.7) converges strongly to the common fixed point of f, S and T.*

Proof. Using (3.1), we get,

$$\phi\big(d_\mu(fTx, w)\big) \leq k\phi\big(d_\mu(fx, w)\big),$$

where $k = \frac{k_1+k_2\mu(fx,fTx)^*}{1-k_2\mu(fx,fTx)^*} < 1$.

This implies $d_\mu(fTx, w) \leq k'd_\mu(fx, w)$, for some $k' < 1$ (since ϕ is subadditive altering distance function).

Similarly, $\phi\big(d_\mu(fSx, w)\big) \leq l\phi\big(d_\mu(fx, w)\big)$, where $l = \frac{k_1+k_3\mu(fx,fSx)^*}{1-k_3\mu(fx,fSx)^*} < 1$ and as above, for some $l' < 1$, $d_\mu(fSx, w) \leq l'd_\mu(fx, w)$.

Now, using (3.7) we get,

$$d_\mu(fx_{n+1}, w) = d_\mu\big(\mathcal{W}(fz_n, fTy_n, \alpha_n), w\big) \le l' d_\mu(fx_n, w).$$

Inductively, for any positive integer n, we get, $d_\mu(fx_n, w) \le l'^n d_\mu(fx_0, w)$ and hence, in the limit as $n \to \infty$, $\lim_{n\to\infty} d_\mu(fx_n, w) = 0$, as required. ∎

Example 3.12. Consider the mappings f, S and T as given in Example 3.5 with $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{3}$, $k_3 = \frac{1}{3}$ and the extended b-metric space (X, d_μ) where $X = [0, 1]$ and $d_\mu(x, y) = |x - y|$, the usual metric with $\mu(x, y) = 1$.

We note that (X, d_μ) is a convex extended b-metric space satisfying the conditions of Theorem 3.3. So, by Theorem 3.3, we get $x = 1$ as the unique common fixed point of f, S and T.

Now, $\mathcal{W} : X^3 \longrightarrow X$ given by

$$\mathcal{W}(x, y, t) = tx + (1 - t)y$$

for all x, y and t in X defines a convex structure on X.

TABLE 1. Sequences generated by (3.8) with $x_0 = 0.65, 0.45$ and 0.05

$\boldsymbol{x_0}$	$x_1 = 0.90277$	$x_2 = 0.96836$	$x_3 = 0.98870$	$x_4 = 0.99571$	$x_5 = 0.9983$
$\mathbf{= 0.65}$	$x_6 = 0.99930$	$x_7 = 0.99970$	$x_8 = 0.99987$	$x_9 = 0.99994$	$x_{10} = 0.99997$
$\boldsymbol{x_0}$	$x_1 = 0.84722$	$x_2 = 0.95028$	$x_3 = 0.98224$	$x_4 = 0.99326$	$x_5 = 0.99732$
$\mathbf{= 0.45}$	$x_6 = 0.99890$	$x_7 = 0.99954$	$x_8 = 0.99980$	$x_9 = 0.99991$	$x_{10} = 0.99996$
$\boldsymbol{x_0}$	$x_1 = 0.73611$	$x_2 = 0.91413$	$x_3 = 0.96933$	$x_4 = 0.98835$	$x_5 = 0.99538$
$\mathbf{= 0.05}$	$x_6 = 0.99811$	$x_7 = 0.99920$	$x_8 = 0.99966$	$x_9 = 0.99985$	$x_{10} = 0.99993$

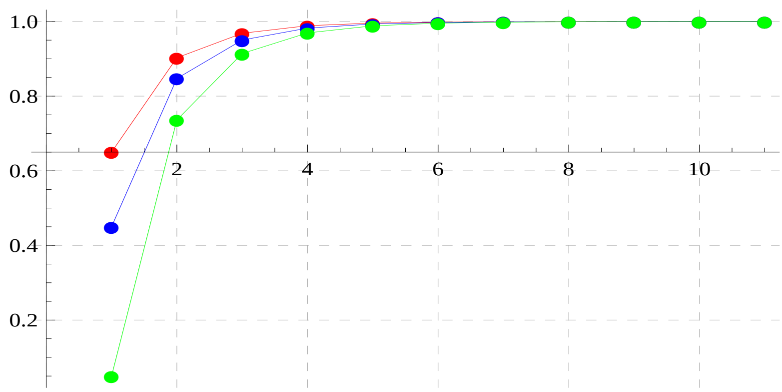

FIGURE 1. Sequences generated by (3.8) with $x_0 = 0.65$, 0.45 and 0.05

The iteration scheme (3.7) then reduce to

$$x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T\Big(\beta_n T S x_n + (1 - \beta_n) S^2 x_n\Big) \tag{3.8}$$

Here, $\mu(x, y) = 1 < \frac{1-k_1}{2k_2} = \frac{1-k_1}{2k_3} = \frac{9}{8}$. Taking $\alpha_n = \frac{n+1}{n+5}$ and $\beta_n = \frac{n+2}{n+6}$, the sequences of iterates generated by the iteration scheme (3.8) are given in Table 1.

From the given tabulation and figure, it is clear that the sequence $\{x_n\}$ generated by (3.8) converges to 1.

Similarly, taking the points $x_0 = 0.45$, 0.05 and generating the sequence defined by (3.8), we can see a convergence to the common fixed point 1.

3.2. Rate of Convergence

In 2002, following the works of Rhoades [24], Berinde [3] compared the rate of convergence between two iteration schemes as given below.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive real numbers converging to α and β, respectively. Suppose that

$$\lim_{n\to\infty} \frac{d(\alpha_n, \alpha)}{d(\beta_n, \beta)} = l.$$

(i) If $l = 0$, then the sequence $\{\alpha_n\}$ is said to converge to α faster than that of the sequence $\{\beta_n\}$ to $\{\beta\}$.

(ii) If $0 < l < \infty$, then the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are said to have the same rate of convergence.

For a nonempty convex subset K of a complete extended b-metric space X, if $\{x_n\}$ and $\{u_n\}$ are two iterations both of which converge to a p of X, then $\{x_n\}$ converges faster than $\{u_n\}$ to p if

$$\lim_{n\to\infty} \frac{d(x_n, p)}{d(u_n, p)} = 0.$$

We compare the rate of convergence of the iteration scheme (3.7) against that of the iteration scheme (3.2) following a similar method employed by Kadioglu and Yildirim [15]. The mapping $\mathcal{W} : X^2 \times [0, 1] \longrightarrow X$ given by

$$\mathcal{W}(x, y, t) = tx + (1 - t)y$$

defines a convex structure on X.

Theorem 3.13. *Let (X, d_μ) be a complete extended b-metric space and f, S and T be self mappings on X satisfying the conditions of Theorem 3.11 with ϕ and f as the identity mapping. Then the iteration scheme given by (3.7) with $0 < \alpha \le \alpha_n, \beta_n < \beta \le \frac{1}{2}$ converges faster than that of the iteration given by (3.2) if $m \le \frac{1}{\sqrt{3}}$, where $m = \max\{k(x), l(x) : x \in X\}$, $k(x) = \frac{k_1+k_2\mu(fx,fTx)}{1-k_2\mu(fx,fTx)}$ and $l(x) = \frac{k_1+k_3\mu(fx,fSx)}{1-k_3\mu(fx,fSx)}$.*

Proof. Since ϕ is the identity mapping, from Theorem 3.11, if w is a common fixed point of f, S and T, we have,

$$d_\mu(fTx, w) \le k(x)d_\mu(fx, w) \quad \text{and} \quad d_\mu(fSx, w) \le l(x)d_\mu(fx, w)$$

where $k(x) = \frac{k_1+k_2\mu(fx,fTx)}{1-k_2\mu(fx,fTx)} < 1$ and $l(x) = \frac{k_1+k_3\mu(fx,fSx)}{1-k_3\mu(fx,fSx)} < 1$.

Let $m = \max\{k(x), l(x) : x \in X\}$. Now if $\{x_n\}$ (that is, $\{fx_n\}$) is a sequence generated by (3.7), then

$$\begin{aligned} d_\mu(fx_{n+1}, w) &\le \alpha_n d_\mu(fz_n.w) + (1 - \alpha_n)d_\mu(fTy_n, w) \\ &\le m^n\Big\{\beta + m^2(1 - \alpha)\beta + m^2(1 - \alpha)^2\Big\}^n d_\mu(fx_0.w). \end{aligned}$$

Also, if $\{u_n\}$ (that is, $\{fu_n\}$) is a sequence generated by (3.2), then for $n = 2k+1$, using (3.1) we get,

$$d_\mu(fu_{2k+1}, w) \leq k'(u_{2k})d_\mu(fu_{2k}, w),$$

where $k'(u_{2k}) = \frac{k_1+k_2\mu(fu_{2k},fu_{2k+1})}{1-k_2\mu(fu_{2k},fu_{2k+1})} < 1$.

Similarly, if $n = 2k$, using (3.1) we get,

$$d_\mu(fu_{2k+1}, w) \leq l'(u_{2k})d_\mu(fu_{2k}, w),$$

where $l'(u_{2k}) = \frac{k_1+k_3\mu(fu_{2k},fu_{2k+1})}{1-k_3\mu(fu_{2k},fu_{2k+1})} < 1$.

Therefore, for all non-negative integers n,

$$\begin{aligned} d_\mu(fu_{n+1}, w) &\leq \max\big\{k'(u_{2k}), l'(u_{2k})\big\}d_\mu(fu_n, w) \leq md_\mu(fu_n, w) \\ &\leq \cdots \leq m^n d_\mu(fu_0, w). \end{aligned}$$

Hence in the limit as $n \to \infty$, the ratio $\frac{d(fx_n,w)}{d(fu_n,w)} \longrightarrow 0$ if

$$m\big\{\beta + m^2(1-\alpha)\beta + m^2(1-\alpha)^2\big\} < m.$$

But since $m \leq \frac{1}{\sqrt{3}}$ and $0 < \alpha, \beta \leq \frac{1}{2}$, we have,

$$\beta + m^2(1-\alpha)\big\{\beta + (1-\alpha)\big\} < \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}+1\right) = 1,$$

and the proof is complete. ∎

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