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A Class of Close-to-Convex Functions Satisfying a Differential Inequality

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Abstract Let $\mathcal{H}_\alpha^\phi(\beta)$ denote the class of functions f, analytic in the open unit disk $\mathbb{E}$, which satisfy the condition

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\left(2+\frac{zf''(z)}{f'(z)}-\frac{z\phi'(z)}{\phi(z)}\right)\right]>\beta,\ z\in\mathbb{E},$$

where $\alpha,\ \beta$ are pre-assigned real numbers and ϕ is a starlike function in $\mathbb{E}$. In the present paper, we prove that members of the class $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and hence univalent for real numbers $\alpha,\ \beta,\ \alpha\le\beta<1$ and for a starlike function ϕ.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in the open unit disk $\mathbb{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. A function $f\in\mathcal{A}$ is said to be starlike if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}.$$

We denote the class of starlike functions by $\mathcal{S}^*$.
Let $\mathcal{K}$ denote the class of convex functions f with $f'(0)\neq 0$ so that

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in\mathbb{E}.$$

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A function $f \in \mathcal{A}$ is said to be close-to-convex for a starlike function ϕ if

$$\Re\left(\frac{zf'(z)}{\phi(z)}\right) > 0,\ z \in \mathbb{E}.$$

This is well known that a close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [8], independently, proved that if an analytic function f satisfies $\Re(f'(z)) > 0$ for all z in $\mathbb{E}$, then f is univalent in $\mathbb{E}$.

For real numbers $\alpha,\ \beta$ and $f \in \mathcal{A},\ \phi \in \mathcal{S}^*$, we define the differential operator $I(\alpha; f, \phi)$ as

$$I(\alpha; f, \phi) = (1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right)$$

and a class $\mathcal{H}_\alpha^\phi(\beta)$ as under:

$$\mathcal{H}_\alpha^\phi(\beta) = \left\{f \in \mathcal{A} : \Re(I(\alpha; f, \phi)) > \beta,\ z \in \mathbb{E}\right\}.$$

We denote $\mathcal{H}_\alpha^z(\beta)$ simply by $\mathcal{H}_\alpha(\beta)$. In fact, the class $\mathcal{H}_\alpha(0)$ was first studied, in 1975, by Al-Amiri and Reade [2]. They proved that for $\alpha \leq 0$, each function in $\mathcal{H}_\alpha(0)$ satisfies $\Re(f'(z)) > 0$ in $\mathbb{E}$ and hence univalent in $\mathbb{E}$. They left the problem of univalence for $\alpha > 0$ (except for $\alpha = 1$, where f is convex, obviously) open. Ahuja and Silverman [1] observed that the convex function $f(z) = z/(1-z)$ is not in $\mathcal{H}_\alpha(0)$ for any real $\alpha,\ \alpha \neq 1$. Further this problem was pursued by Singh et al. [7] and they proved that for $0 < \alpha < 1$, the class $\mathcal{H}_\alpha(\alpha)$ consists of close-to-convex and hence univalent functions. In 2007, Singh et al. [5] studied the class $\mathcal{H}_\alpha(\beta)$. They proved that if $f \in \mathcal{H}_\alpha(\beta)$, then $\Re(f'(z)) > 0$ in $\mathbb{E}$ for all real numbers α, β satisfying $\alpha \leq \beta < 1$ and the result is best possible one in the sense that β cannot be replaced by a real number less than α. Their result contains the previous result of Singh et al. [7] and improves the result of Al-Amiri and Reade [2].

In the present paper, we study a more general class $\mathcal{H}_\alpha^\phi(\beta)$ and establish that the functions in $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and consequently univalent subject to the condition $\alpha \leq \beta < 1$, where α, β are real numbers and ϕ is a starlike function.

2. Preliminary

To prove our result, we shall need the following lemma by Miller [3].

Lemma 2.1. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ and let $\Phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let Φ satisfy the following conditions:*

(i) *$\Phi(u, v)$ is continuous in $\mathbb{D}$,*
(ii) *$(1, 0) \in \mathbb{D}$ and $\Re\{\Phi(1, 0)\} > 0$;*
(iii) *$\Re\{(iu_2, v_1)\} \leq 0$ for all $((iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1 + u_2^2)/2$.*

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\Phi(p(z), zp'(z))] > 0,\ z \in \mathbb{E},$$

then $\Re(p(z)) > 0$ in $\mathbb{E}$.

3. Univalence of Functions in $\mathcal{H}_\alpha^\phi(\beta)$

Theorem 3.1. *Let ϕ be a starlike function and $\alpha,\ \beta$ be real numbers such that $\alpha \leq \beta < 1$. If $f \in \mathcal{A}$ satisfies*

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right)\right] > \beta,\ \ z \in \mathbb{E}, \tag{3.1}$$

then $\Re\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ in $\mathbb{E}$. So f is close-to-convex and hence univalent in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (3.1) cannot be replaced by a constant smaller than α.

Proof. Let $p(z) = \frac{zf'(z)}{\phi(z)}$ where $p(0) = 1$, is analytic in $\mathbb{E}$. Then,

$$(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right) = (1-\alpha)p(z) + \alpha\left(1 + \frac{zp'(z)}{p(z)}\right)$$

Thus, condition (3.1) is equivalent to

$$\Re\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha-\beta}{1-\beta}\right) > 0,\ \ z \in \mathbb{E}. \tag{3.2}$$

If $\mathbb{D} = \mathbb{C}\setminus\{0\}\times\mathbb{C}$, define $\Phi(u,v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u,v) = \frac{1-\alpha}{1-\beta}u + \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha-\beta}{1-\beta}.$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0) \in \mathbb{D}$ and $\Re(\Phi(1,0)) = 1 > 0$. Further, in view of (3.2), $\Re(\Phi(p(z), zp'(z))) > 0,\ z \in \mathbb{E}$. Let $u = u_1 + iu_2,\ v = v_1 + iv_2$ where $u_1,\ u_2,\ v_1$ *and* v_2 are all reals. Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\frac{1+u_2^2}{2}$, we have

$$\Re[\Phi(iu_2, v_1)] = \Re\left[\frac{1-\alpha}{1-\beta}u_2 i + \frac{\alpha}{1-\beta}\frac{v_1}{u_2 i} + \frac{\alpha-\beta}{1-\beta}\right]$$

$$= \frac{\alpha-\beta}{1-\beta} \leq 0.$$

In view of Lemma 2.1, proof now follows.

To show that the constant β on the right hand side of (3.1) cannot be replaced by a real number smaller than α, we select the function $f(z) = ze^z$ which is a member of class $\mathcal{A}$, and a starlike function $\phi(z) = \frac{z}{(1-z)^2}$. On taking $\alpha = -1$, we have plotted the image of unit disk $\mathbb{E}$ under the operator $(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right) = 2(1+z)(1-z)^2e^z + \frac{4z^2+z^3-1}{1-z^2}$ in Figure 1. We have noticed that

$$\Re\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right)\right)$$

is smaller than -1 (chosen value of α). In Figure 2, we have plotted image of unit disc $\mathbb{E}$ under $\frac{zf'(z)}{\phi(z)} = (1+z)(1-z)^2e^z$ and observed that $\Re\left(\frac{zf'(z)}{\phi(z)}\right) \not> 0$ for all $z \in \mathbb{E}$.

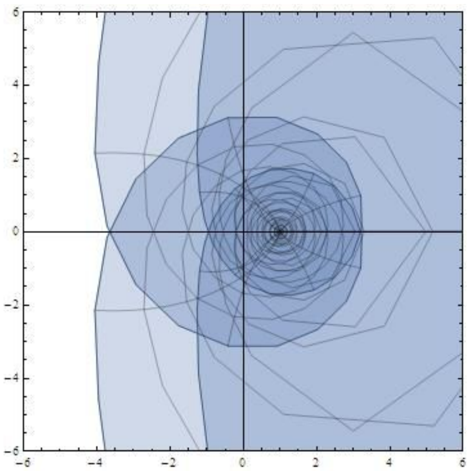

Figure 1

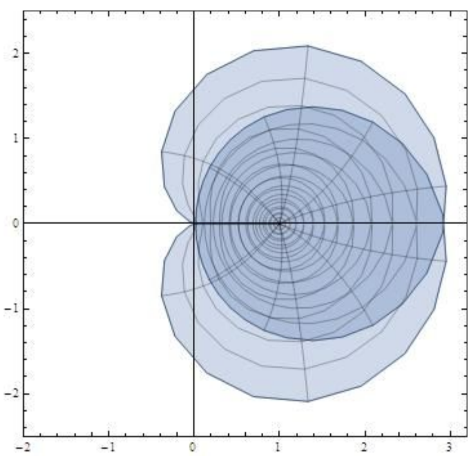

Figure 2

■

We illustrate the above result with the help of following example:

Example 3.2. On selecting $\phi(z) = \dfrac{z}{(1-z)^2}$ and $f(z) = \dfrac{z}{(1-z)^2}$ in Theorem 3.1, we can easily check that for $\alpha = -1 = \beta$,

$$\Re\left(\frac{1+2z+3z^2}{1-z^2}\right) > -1$$

implies that $\Re\left(\dfrac{zf'(z)}{\phi(z)}\right) = \Re\left(\dfrac{1+z}{1-z}\right) > 0$, thus f is close-to-convex and hence univalent in $\mathbb{E}$.

Theorem 3.3. *Let ϕ be a starlike function and α, β be real numbers such that $\alpha \geq \beta > 1$. If $f \in \mathcal{A}$ satisfies*

$$\Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(2 + \frac{zf''(z)}{f'(z)} - \frac{z\phi'(z)}{\phi(z)}\right)\right] < \beta, \ z \in \mathbb{E}, \tag{3.3}$$

then $\Re\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ in $\mathbb{E}$. So f is close-to-convex and hence univalent in $\mathbb{E}$.

Proof. Write $\frac{zf'(z)}{\phi(z)} = p(z)$ and note that $1-\beta < 0$, condition (3.3) reduces to

$$\Re\left[\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha-\beta}{1-\beta}\right] > 0, \ z \in \mathbb{E}.$$

The proof can now be completed on the same lines as in Theorem 3.1. ■

In a special case, when $\phi(z) = z$ in Theorem 3.1, we obtain the following result of Singh et al. [5].

Theorem 3.4. *Let α and β be real numbers such that $\alpha \leq \beta < 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies*

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, \ z \in \mathbb{E}. \tag{3.4}$$

Then $\Re\ f'(z) > 0$ in $\mathbb{E}$. So, f is close-to-convex and hence univalent in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (3.4) cannot be replaced by a constant smaller than α.

Taking, $\phi(z) = z$ in Theorem 3.3, we obtain the following result of Singh et al. [6].

Theorem 3.5. *Let α and β be real numbers such that $\alpha \geq \beta > 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies*

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < \beta, \ z \in \mathbb{E}.$$

Then $\Re\ f'(z) > 0$ in $\mathbb{E}$. So, f is close-to-convex and hence univalent in $\mathbb{E}$.

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