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Applications of Multivalued $\mathcal{F}_\delta$-Contraction with Stability Results

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Abstract The aim of this paper is to propose some new tripled coincidence and tripled fixed point theorems in the natural setting of metric spaces. First, we introduce the notion of multivalued almost $\mathcal{F}_\delta$-contraction endowed with suitable examples. Second, we utilize the established results to derive stability for the tripled coincidence point sets. Final section is devoted to the application part, where we apply our results to establish the existence of solution of matrix equations and integral inclusions so as to demonstrate the materiality and viability of our results, which is further garnished by a numerical experiment.

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1. Introduction and Basic Facts

In 1969, Nadler[14] introduced the concept of multivalued mappings and proved fixed point for such mappings in the framework of complete metric spaces. Inspired by the idea given in [14], Fisher[9] proved different type of fixed point results for multivalued cases with following notations:

Let (Y, σ) be a metric space (in short $m.s.$) and $CB(Y)$, the family of all non-empty closed and bounded subsets of Y, Consider $P, Q \in CB(Y)$,

$$\delta(P, Q) = \sup\{\sigma(a, b) : a \in P, b \in Q\}$$
$$D(a, Q) = \inf\{\sigma(a, b) : b \in Q\}.$$

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Berinde and Pacurar [5] defined Pompeiu-Housdorff distance H as follows:

$$H(P,Q) = \max\{\sup_{a\in Q} D(a,P), \sup_{a\in P} D(a,Q)\}. \tag{1.1}$$

Recently in 2012, Wardowski [23] described a new contraction called $\mathcal{F}$-contraction and acquired a fixed point result as a generalization of Banach contraction principle for a single valued mapping $S : Y \to Y$ as follows:

$$\forall a,b \in Y, \Big(\sigma(Sa,Sb) > 0 \implies \tau + \mathcal{F}(\sigma(Sa,Sb)) \le \mathcal{F}(\sigma(a,b))\Big),$$

where $\tau > 0$ and $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) $\mathcal{F}$ is strictly increasing, that is, for $\alpha,\beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$;

(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.

We denote by $\Im$, the set of all functions satisfying $(F1),(F2),(F3)$. For more synthesis on $\mathcal{F}$-contraction, we refer the reader to [10, 11, 21, 22] and the references therein.

Influenced by this innovative Wardowski-technique, Acar[2] enunciated some novel results for multivalued mappings by using the concept of δ and D-distances.

Definition 1.1. [2] Let (Y,σ) be a $m.s.$ and $B(Y)$ denote the family of all bounded subset of Y, then the mapping $S : Y \to B(Y)$ is called a multivalued almost $\mathcal{F}_\delta$-contraction if $\mathcal{F} \in \Im$ and there exists $\tau > 0$ and $K \ge 0$ such that

$$\tau + \mathcal{F}(\delta(Sa,Sb)) \le \mathcal{F}(m(a,b) + KD(b,Sa)) \text{ for all } a,b \in Y. \tag{1.2}$$

With $\min\{\delta(Sa,Sb),\sigma(a,b)\} > 0$, where

$$m(a,b) = \max\{\sigma(a,b), D(a,Sa), D(b,Sb), \frac{1}{2}[D(a,Sb) + D(b,Sa)]\}.$$

Theorem 1.2. [2] *Let (Y,σ) be a complete $m.s.$,$B(Y)$ denote the family of all bounded subset of Y and $S : Y \to B(Y)$ be a multivalued almost $\mathcal{F}_\delta$-contraction. If $\mathcal{F}$ is continuous and Sa is closed for all $a \in Y$, then S has a fixed point in Y.*

Following Concept was introduced in [18], which is necessary for the succeeding analysis.

Definition 1.3. [18] Let Y be a non-empty set and $\mathcal{F} : Y^N \to Y$ be a given mapping $(N \ge 2)$. An element $(a_1,a_2,...,a_N) \in Y^N$ is said to be fixed point of $N-$order of the mapping $\mathcal{F}$ if

$$\begin{aligned} a_1 &= \mathcal{F}(a_1,a_2,...,a_N)\\ a_2 &= \mathcal{F}(a_2,a_3,...,a_N,a_1)\\ &\vdots\\ a_N &= \mathcal{F}(a_N,a_1,...,a_{N-1}). \end{aligned} \tag{1.3}$$

If $N = 3$,then we have:

Definition 1.4. [18] Let Y be a non-empty set, an element $(a,b,c) \in Y^3$ is called a tripled fixed point(in short TFP) of $\mathcal{F} : Y^3 \to Y$ if

$$\mathcal{F}(a,b,c) = a, \;\; \mathcal{F}(b,c,a) = b, \;\; \mathcal{F}(c,a,b) = c.$$

In the situation of ordered sets with mixed monotone property, Berinde et.al.[4] defined differently the concept of TFP in a different way as below:

Definition 1.5. [4] Let Y be a non-empty set, an element $(a,b,c) \in Y^3$ is called a TFP of $\mathcal{F} : Y^3 \to Y$ if

$$\mathcal{F}(a,b,c) = a, \;\; \mathcal{F}(b,a,b) = b, \;\; \mathcal{F}(c,b,a) = c.$$

For multivalued case the concept of TFP was introduced in [1] as:

Definition 1.6. [1] Let Y be a non-empty set and $CL(Y)$ denotes the family of all colsed subsets of Y. Then, an element $(a,b,c) \in Y^3$ is called a TFP of $\mathcal{F} : Y^3 \to CL(Y)$ if

$$a \in \mathcal{F}(a,b,c), \;\; b \in \mathcal{F}(b,c,a), \;\; c \in \mathcal{F}(c,a,b).$$

Stability of fixed point sets has always been a core area of interest for many researchers. Some note worthy contribution can be seen in papers like [6], [7], [13]. In a paper [3] authors established fixed point results for generalized almost contractions. Radenovic et.al [17] propounded an alternative approach to fixed point results via simulation function. On the other hand, following Lemma is due to [15].

Lemma 1.7. [15] *Let (Y,σ) be metric space, $P,Q \in CB(Y)$ and $r > 1$. Then for each $p \in P$, there exists $q \in Q$, such that $\sigma(p,q) \leq rH(P,Q)$.*

Definition 1.8. [20] Let (Y,σ) be a $m.s.$, $\{S : Y \to CB(Y)\}$ be a multivalued mapping and $h : Y \to Y$ a single valued mapping. The pair of mappings (h,S) is said to be compatible, if $\lim\limits_{n\to\infty} D(hy_{n+1}, Shx_n) = 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in Y, such that $\lim\limits_{n\to\infty} hx_n = y_n = l$, for some l in Y, where $y_{n+1} \in Sx_n$ for $n = 1,2,3...$

Using the concept of δ-distance and $\mathcal{F}$-contraction, in this paper, we introduce some newfangled tripled coincidence point and tripled fixed point results concerning almost $\mathcal{F}_\delta$-contraction for multivalued mappings. Proved results are hosted by a series of good examples, thereby giving better understanding of the proposed results. Moreover, stability results for tripled coincidence point sets are also discussed followed by a suitable example. Another purpose to set up the fixed point results is that our results are utilized to establish the existence of solution of integral inclusions and matrix equations. At the end, for application point of view, we also propose an open problem for future scope of the study. In the rest of the paper $C(Y)$ denotes the family of non empty compact subsets of Y.

2. Tripled Fixed Point Results

We begin our work by introducing following definitions:

Definition 2.1. Consider (Y,σ) as a $m.s.$ and $S : Y \times Y \times Y \to C(Y)$ be a multivalued mapping. Then S is called almost $\mathcal{F}_\delta$-contraction, if $\mathcal{F} \in \Im$ and there exists $\tau > 0$ and $K \geq 0$, such that

$$\begin{aligned}&\tau + \mathcal{F}(\delta(S(a,b,c), S(p,q,r)) + \delta(S(b,c,a), S(q,r,p)) + \delta(S(c,a,b), S(r,p,q)))\\ &\leq \mathcal{F}(M(a,b,c,p,q,r) + KN(a,b,c,p,q,r)) \text{ for all } a,b,c,p,q,r \in Y. \end{aligned} \tag{2.1}$$

Where

$$\begin{aligned} M(a,b,c,p,q,r) &= \max\{\sigma(a,p)+\sigma(b,q)+\sigma(c,r), D(a,S(a,b,c)) \\ &\quad + D(b,S(b,c,a)) + D(c,S(c,a,b)), D(p,S(p,q,r)) \\ &\quad + D(q,S(q,r,p)) + D(r,S(r,p,q))\} \\ N(a,b,c,p,q,r) &= D(p,S(a,b,c)) + D(q,S(b,c,a)) + D(r,S(c,a,b)). \end{aligned}$$

Definition 2.2. Consider (Y,σ) as a $m.s.$ and $S: Y\times Y\times Y \to C(Y)$ be a multivalued mapping. Then S is said to be almost $\mathcal{F}_\delta$-contraction with respect to a self mapping $h: Y\to Y$, if $\mathcal{F}\in\Im$ and there exists $\tau>0$ and $K\geq 0$ such that

$$\begin{aligned} &\tau + \mathcal{F}(\delta(S(a,b,c),S(p,q,r)) + \delta(S(b,c,a),S(q,r,p)) + \delta(S(c,a,b),S(r,p,q))) \\ &\leq \mathcal{F}(M(a,b,c,p,q,r) + KN(a,b,c,p,q,r)) \text{ for all } a,b,c,p,q,r\in Y. \end{aligned} \tag{2.2}$$

Where

$$\begin{aligned} M(a,b,c,p,q,r) &= \max\{\sigma(ha,hp)+\sigma(hb,hq)+\sigma(hc,hr), D(ha,S(a,b,c)) \\ &\quad + D(hb,S(b,c,a)) + D(hc,S(c,a,b)), D(hp,S(p,q,r)) \\ &\quad + D(hq,S(q,r,p)) + D(hr,S(r,p,q))\} \\ N(a,b,c,p,q,r) &= D(hp,S(a,b,c)) + D(hq,S(b,c,a)) + D(hr,S(c,a,b)). \end{aligned}$$

Definition 2.3. Let Y be a nonempty set, $h: Y\to Y$ a single valued mapping and $S: Y\times Y\times Y\to C(Y)$ be a multivalued mapping. An element $(a,b,c)\in Y\times Y\times Y$ is called a tripled coincidence point (in short TCP) of h and S if $ha\in S(a,b,c)$, $hb\in S(b,c,a)$ and $hc\in S(c,a,b)$.The set of all coincidence points of h and S is denoted by $C(h,S)$.

Note that Definition 2.2 and Definition 2.3 coincide with Definition 2.1 and Definition 1.6 respectively if h taken to be an identity mapping.

Theorem 2.4. *Let (Y,σ) be a complete m.s., $h: Y\to Y$ a single valued mapping and $S: Y\times Y\times Y\to C(Y)$ be a multivalued almost $\mathcal{F}_\delta$-contraction with respect to h. Suppose that*

(i) $S(Y\times Y\times Y)\subseteq h(Y)$
(ii) *h is continuous*
(iii) *The pair of mappings (h,S) is compatible.*

Then, $C(h,S)$ is non-empty.

Proof. Let $(x_0,y_0,z_0)\in Y\times Y\times Y$, then there exists $x_1\in S(x_0,y_0,z_0)$, $y_1\in S(y_0,z_0,x_0)$, $z_1\in S(z_0,x_0,y_0)$.
Since $S(Y\times Y\times Y)\subseteq h(Y)$, there exists x_2,y_2,z_2 such that $hx_2=x_1, hy_2=y_1, hz_2=z_1$. Again for $(x_2,y_2,z_2)\in Y\times Y\times Y$ there exists $x_3\in S(x_2,y_2,z_2), y_3\in S(y_2,z_2,x_2), z_3\in S(z_2,x_2,y_2)$ and also $x_4,y_4,z_4\in Y$ such that $hx_4=x_3, hy_4=y_3, hz_4=z_3$. Continuing this process, we get sequences

$$x_{2n-1}=hx_{2n},\quad y_{2n-1}=hy_{2n},\quad z_{2n-1}=hz_{2n}$$

also

$$\begin{aligned} x_{2n-1} &= hx_{2n}\in S(x_{2n-2},y_{2n-2},z_{2n-2}), \\ y_{2n-1} &= hy_{2n}\in S(y_{2n-2},z_{2n-2},x_{2n-2}), \\ z_{2n-1} &= hz_{2n}\in S(z_{2n-2},x_{2n-2},y_{2n-2}). \end{aligned}$$

Consider

$$\begin{aligned}
&\tau + \mathcal{F}(\sigma(hx_2, hx_4) + \sigma(hy_2, hy_4) + \sigma(hz_2, hz_4)\\
&\le \tau + \mathcal{F}(\delta(S(x_0, y_0, z_0), S(x_2, y_2, z_2)) + \delta(S(y_0, z_0, x_0), S(y_2, z_2, x_2))\\
&\quad + \delta(S(z_0, x_0, y_0), S(z_2, x_2, y_2))).
\end{aligned} \tag{2.3}$$

By applying (2.2), we get

$$\begin{aligned}
&\tau + \mathcal{F}(\sigma(hx_2, hx_4) + \sigma(hy_2, hy_4) + \sigma(hz_2, hz_4))\\
&\le \mathcal{F}(\max\{\sigma(hx_0, hx_2) + \sigma(hy_0, hy_2) + \sigma(hz_0, hz_2),\\
&D(hx_0, S(x_0, y_0, z_0)) + D(hy_0, S(y_0, z_0, x_0)) + D(hz_0, S(z_0, x_0, y_0)),\\
&D(hx_2, S(x_2, y_2, z_2)) + D(hy_2, S(y_2, z_2, x_2)) + D(hz_2, S(z_2, x_2, y_2))\}\\
&+ K\{D(hx_2, S(x_0, y_0, z_0)) + D(hy_2, S(y_0, z_0, x_0)) + D(hz_2, S(z_0, x_0, y_0))\})\\
&\le \mathcal{F}(\max\{\sigma(hx_0, hx_2) + \sigma(hy_0, hy_2) + \sigma(hz_0, hz_2),\\
&\sigma(hx_0, hx_2) + \sigma(hy_0, hy_2) + \sigma(hz_0, hz_2),\\
&\sigma(hx_2, hx_4) + \sigma(hy_2, hy_4) + \sigma(hz_2, hz_4)\}\\
&+ K\{\sigma(hx_2, hx_2) + \sigma(hy_2, hy_2) + \sigma(hz_2, hz_2)\}\\
&\le \mathcal{F}(\max\{\sigma(hx_0, hx_2) + \sigma(hy_0, hy_2) + \sigma(hz_0, hz_2),\\
&\sigma(hx_2, hx_4) + \sigma(hy_2, hy_4) + \sigma(hz_2, hz_4)\}).
\end{aligned} \tag{2.4}$$

$$Let \;\; \delta_{2n-2} = \sigma(hx_{2n-2}, hx_{2n}) + \sigma(hy_{2n-2}, hy_{2n}) + \sigma(hz_{2n-2}, hz_{2n}),$$

then $\tau + \mathcal{F}(\delta_2) \le \mathcal{F}(\max\{\delta_0, \delta_2\})$. Set $M = max\{\delta_0, \delta_2\}$.
If $M = \delta_2$, then $\tau + \mathcal{F}(\delta_2) \le \mathcal{F}(\delta_2)$, which is a contradiction.
Thus $M = \delta_0$, so

$$\tau + \mathcal{F}(\delta_2) \le \mathcal{F}(\delta_0)$$

$$\mathcal{F}(\delta_2) \le \mathcal{F}(\delta_0) - \tau.$$

Maintaining the above procedure, we get

$$\mathcal{F}(\delta_{2n}) \le \mathcal{F}(\delta_{2n-2}) - \tau \le \mathcal{F}(\delta_{2n-4}) - 2\tau ... \le \mathcal{F}(\delta_0) - n\tau. \tag{2.5}$$

From (2.5), we get $\lim_{n\to\infty} \mathcal{F}(\delta_{2n}) = -\infty$, and from (**F**2) we get $\lim_{n\to\infty} \delta_{2n} = 0$, from (**F**3), there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} \delta_{2n}^k F(\delta_{2n}) = 0$.
By (2.5), the following holds for all $n \in N$

$$\delta_{2n}^k \mathcal{F}(\delta_{2n}) - \delta_{2n}^k \mathcal{F}(\delta_0) \le -n\tau\delta_{2n}^k \le 0. \tag{2.6}$$

Letting $n \to \infty$ in (2.6), we obtain

$$\lim_{n\to\infty} \delta_{2n}^k = 0. \tag{2.7}$$

From (2.7), there exists $n_1 \in N$ such that $n\delta_{2n}^k \le 1$ for all $n \ge n_1$. So we have

$$\delta_{2n} \le \frac{1}{n^{\frac{1}{k}}} \;\text{ for all }\; n \ge n_1. \tag{2.8}$$

Take $m, n \in N$ such that $m > n \geq n_1$.
Using triangle inequality for the metric and from (2.8), we have

$$\begin{aligned}
&\sigma(hx_{2n}, hx_{2m}) + \sigma(hy_{2n}, hy_{2m}) + \sigma(hz_{2n}, hz_{2m}) \\
&\leq \sigma(hx_{2n}, hx_{2n+2}) + ... + \sigma(hx_{2m-2}, hx_{2m}) \\
&+ \sigma(hy_{2n}, hy_{2n+2}) + ... + \sigma(hy_{2m-2}, hy_{2m}) \\
&+ \sigma(hz_{2n}, hz_{2n+2}) + ... + \sigma(hz_{2m-2}, hz_{2m}) \\
&\leq \sigma(hx_{2n}, hx_{2n+2}) + \sigma(hy_{2n}, hy_{2n+2}) + \sigma(hz_{2n}, hz_{2n+2}) \\
&+ ... + \sigma(hx_{2m-2}, hx_{2m}) + \sigma(hy_{2m-2}, hy_{2m}) + \sigma(hz_{2m-2}, hz_{2m}) \\
&\leq \delta_{2n} + \delta_{2n+2} + ... + \delta_{2m-2} \\
&= \sum_{i=n}^{m-1} \delta_{2i} \\
&\leq \sum_{i=n}^{\infty} \delta_{2i} \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

With the convergence of $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$, we get

$$\sigma(hx_{2n}, hx_{2m}) + \sigma(hy_{2n}, hy_{2m}) + \sigma(hz_{2n}, hz_{2m}) \to 0 \text{ as } n \to \infty,$$

it gives

$$\sigma(hx_{2n}, hx_{2m}) \to 0, \sigma(hy_{2n}, hy_{2m}) \to 0, \sigma(hz_{2n}, hz_{2m}) \to 0.$$

Hence $\{hx_{2n}\}$,$\{hy_{2n}\}$ and $\{hz_{2n}\}$ are Cauchy sequences.
Completeness of X implies there exists $(x, y, z) \in Y \times Y \times Y$ such that $x_{2n-1} = hx_{2n} \to x$,$y_{2n-1} = hy_{2n} \to y$ and $z_{2n-1} = hz_{2n} \to z$ as $n \to \infty$.
By the compatibility of pair (h, S), we have

$$\begin{aligned}
&\lim_{n\to\infty} D(hx_{2n-1}, S(hx_{2n}, hy_{2n}, hz_{2n})) = 0 \\
&\lim_{n\to\infty} D(hy_{2n-1}, S(hy_{2n}, hz_{2n}, hx_{2n})) = 0 \\
&\lim_{n\to\infty} D(hz_{2n-1}, S(hz_{2n}, hx_{2n}, hy_{2n})) = 0.
\end{aligned}$$

Since D and h are continuous, we have

$$\begin{aligned}
&D(hx, S(x, y, z)) = 0 \\
&D(hy, S(y, z, x)) = 0 \\
&D(hz, S(z, x, y)) = 0.
\end{aligned}$$

Since $S(x, y, z), S(y, z, x), S(z, x, y)$ are Compact and hence closed.
i.e.

$$\bar{S}(x, y, z) = S(x, y, z), \bar{S}(y, z, x) = S(y, z, x), \bar{S}(z, x, y) = S(z, x, y),$$

implies

$$hx \in S(x, y, z), hy \in S(y, z, x), hz \in S(z, x, y)$$

i.e. $(x, y, z) \in C(h, S)$. Hence (x, y, z) is TCP of h and S.
This completes the proof. ∎

Example 2.5. Let $Y = [0, 1]$ with usual metric σ, be a complete $m.s.$, define a multivalued mapping $S : Y \times Y \times Y \to C(Y)$, by $S(x, y, z) = \{\frac{x}{3}\}$ with the self mapping $h : Y \to Y$ by $hx = x^2$and consider $\mathcal{F}(\alpha) = log\alpha$ with $K = 0$ and $0 < \tau < log3$. We claim that S is almost $\mathcal{F}_\delta$-contraction with respect to h and also satisfy all other conditions of Theorem 2.4. Hence $(0, 0, 0)$ is the TCP, of the pair (h, S), which is unique.

Example 2.6. If we consider $S(x, y, z) = \{\frac{x+y+z}{3}\}$, with the self mapping h and other parameters incuding metric space as in above example. We claim that S is almost $\mathcal{F}_\delta$-contraction with respect to h and also satisfy all other conditions of Theorem 2.4. Hence $(0, 0, 0)$ and $(1, 1, 1)$ are TCP of the pair (h, S). Thus our theorem gives the guarantee for TCP but not for uniqueness.

Theorem 2.7. *Let (Y, σ) be a complete $m.s.$ and $S : Y \times Y \times Y \to C(Y)$ be a multivalued almost $\mathcal{F}_\delta$-contraction, then S has a TFP.*

Proof. Let $(x_0, y_0, z_0) \in Y \times Y \times Y$, then there exists $x_1 \in S(x_0, y_0, z_0)$,$y_1 \in S(y_0, z_0, x_0)$, $z_1 \in S(z_0, x_0, y_0)$ and for $x_1, y_1, z_1 \in Y$, there exists $x_2, y_2, z_2 \in Y$ such that $x_2 \in S(x_1, y_1, z_1)$, $y_2 \in S(y_1, z_1, x_1)$, $z_2 \in S(z_1, x_1, y_1)$. Continuing this process, we get sequences

$$x_{n+1} \in S(x_n, y_n, z_n), \quad y_{n+1} \in S(y_n, z_n, x_n), \quad z_{n+1} \in S(z_n, x_n, y_n).$$

Maintaining the same process as in Theorem 2.4, we have $\{x_n\}$,$\{y_n\}$and $\{z_n\}$ are Cauchy sequences.
Completeness of Y implies that, there exists $(x, y, z) \in Y \times Y \times Y$, such that
$x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to \infty$.
Since, we have

$$D(x_{n+1}, S(x_n, y_n, z_n)) = 0,$$
$$D(y_{n+1}, S(y_n, z_n, x_n)) = 0,$$
$$D(z_{n+1}, S(z_n, x_n, y_n)) = 0.$$

Letting $n \to \infty$, we have

$$D(x, S(x, y, z)) = 0,$$
$$D(y, S(y, z, x)) = 0,$$
$$D(z, S(z, x, x)) = 0.$$

Since $S(x, y, z), S(y, z, x), S(z, x, y)$ are compact and hence closed.
i.e.

$$\bar{S}(x, y, z) = S(x, y, z), \bar{S}(y, z, x) = S(y, z, x), \bar{S}(z, x, y) = S(z, x, y)$$

implies

$$x \in S(x, y, z), y \in S(y, z, x), z \in S(z, x, y)$$

Hence (x, y, z) is TFP of S. This completes the proof. ∎

Remark 2.8. It is worth noting that as a consequence of Theorem 2.7 if we take $S : Y \times Y \times Y \to Y$ as a single valued mapping and if S satisfies the same inequality as in Theorem 2.7 with σ (metric) in palce of δ and D distances, then S has TFP.

3. Stability Results for Tripled Coincidence Point Sets

Lemma 3.1. *Let (Y,σ) be a m.s., $h:Y\to Y$ a single valued mapping and $\{S_n : Y\times Y\times Y\to C(Y)\}$ be a sequence of multivalued almost $\mathcal{F}_\delta$-contractions with respect to h, which is uniformly convergent to $S:Y\times Y\times Y\to C(Y)$, then S is also a multivalued almost $\mathcal{F}_\delta$-contraction with respect to h.*

Proof. Since S_n for all $n\geq 1$, is multivalued almost $\mathcal{F}_\delta$-contraction with respect to h, thus
$\tau+\mathcal{F}(\delta(S_n(x,y,z),S_n(u,v,w))+\delta(S_n(y,z,x),S_n(v,w,u))+\delta(S_n(z,x,y),S_n(w,u,v)))$

$$\begin{aligned}&\leq \mathcal{F}(\max\{\sigma(hx,hu)+\sigma(hy,hv)+\sigma(hz,hw),D(hx,S_n(x,y,z))\\&+D(hy,S_n(y,z,x))+D(hz,S_n(z,x,y)),D(hu,S_n(u,v,w))\\&+D(hv,S_n(v,w,u))+D(hw,S_n(w,u,v))\}+KD(hu,S_n(x,y,z))\\&+D(hv,S_n(y,z,x))+D(hw,S_n(z,x,y))),\end{aligned} \tag{3.1}$$

for all $x,y,z,u,v,w\in Y$.
Letting $n\to\infty$ and $\{S_n\}$ converges uniformly to S, then
$\tau+\mathcal{F}(\delta(S(x,y,z),S(u,v,w))+\delta(S(y,z,x),S(v,w,u))+\delta(S(z,x,y),S(w,u,v)))$

$$\begin{aligned}&\leq \mathcal{F}(\max\{\sigma(hx,hu)+\sigma(hy,hv)+\sigma(hz,hw),D(hx,S(x,y,z))\\&+D(hy,S(y,z,x))+D(hz,S(z,x,y)),D(hu,S(u,v,w))\\&+D(hv,S(v,w,u))+D(hw,S(w,u,v))\}+KD(hu,S(x,y,z))\\&+D(hv,S(y,z,x))+D(hw,S(z,x,y))).\end{aligned} \tag{3.2}$$

Hence S is almost $\mathcal{F}_\delta$-contraction with respect to h. ∎

Theorem 3.2. *Let (Y,σ) be a m.s., $h:Y\to Y$ a single valued mapping and $S_1,S_2: Y\times Y\times Y\to C(Y)$ be two multivalued mappings such that the pairs (S_1,h) and (S_2,h) satisfy all the conditions of Theorem 2.4 and let $M_i=sup\{\sigma(x,hx)+\sigma(y,hy)+\sigma(z,hz): (x,y,z)\in C(h,S_i)\}$, where i=1,2, exists. Then $H(C(h,S_1),C(h,S_2))\leq qk+R$, where $q>1, R=\max\Big\{M_i:i=1,2\}$,*
*$k=sup\{H(S_1(x,y,z),S_2(x,y,z))+H(S_1(y,z,x),S_2(y,z,x))+H(S_1(z,x,y),$
$S_2(z,x,y)):(x,y,z)\in Y\times Y\times Y\Big\}$.*

Proof. By applying Theorem 2.4, we have $C(S_1,h)$ and $C(S_2,h)$ are non-empty.
Let $(x_0,y_0,z_0)\in C(S_1,h)$, that is $hx_0\in S_1(x_0,y_0,z_0), hy_0\in S_1(y_0,z_0,x_0), hz_0\in S_1(z_0,x_0,y_0)$.
Applying the Lemma1.7, for every $hx_0\in S_1(x_0,y_0,z_0)$ there exists $x_1=hx_2\in S_2(x_0,y_0,z_0)$, for every $hy_0\in S_1(y_0,z_0,x_0)$ there exists $y_1=hy_2\in S_2(y_0,z_0,x_0)$ and for every $hz_0\in S_1(z_0,x_0,y_0)$ there exists $z_1=hz_2\in S_2(z_0,x_0,y_0)$, such that

$$\sigma(hx_0,hx_2)\leq qH(S_1(x_0,y_0,z_0),S_2(x_0,y_0,z_0))$$

$$\sigma(hy_0,hy_2)\leq qH(S_1(y_0,z_0,x_0),S_2(y_0,z_0,x_0))$$

$$\sigma(hz_0,hz_2)\leq qH(S_1(z_0,x_0,y_0),S_2(z_0,x_0,y_0))$$

$$\begin{aligned}\sigma(hx_0,hx_2)+\sigma(hy_0,hy_2)+\sigma(hz_0,hz_2)\leq q\{&H(S_1(x_0,y_0,z_0),S_2(x_0,y_0,z_0))\\&+H(S_1(y_0,z_0,x_0),S_2(y_0,z_0,x_0))\\&+H(S_1(z_0,x_0,y_0),S_2(z_0,x_0,y_0))\}.\end{aligned} \tag{3.3}$$

Again, for every $hx_2 \in S_2(x_0, y_0, z_0)$, there exists $x_3 = hx_4 \in S_2(x_2, y_2, z_2)$, for every $hy_2 \in S_2(x_0, y_0, z_0)$, there exists $y_3 = hy_4 \in S_2(y_2, z_2, x_2)$, and for every $hz_2 \in S_2(x_0, y_0, z_0)$, there exists $z_3 = hz_4 \in S_2(z_2, x_2, y_2)$, such that

$$\sigma(hx_2, hx_4) \le qH(S_2(x_0, y_0, z_0), S_2(x_2, y_2, z_2))$$

$$\sigma(hy_2, hy_4) \le qH(S_2(y_0, z_0, x_0), S_2(y_2, z_2, x_2))$$

$$\sigma(hz_2, hz_4) \le qH(S_2(z_0, x_0, y_0), S_2(z_2, x_2, y_2)).$$

In this way we can construct the sequences

$$x_{2n-1} = hx_{2n} \in S_2(x_{2n-2}, y_{2n-2}, z_{2n-2})$$

$$y_{2n-1} = hy_{2n} \in S_2(y_{2n-2}, z_{2n-2}, x_{2n-2})$$

$$z_{2n-1} = hz_{2n} \in S_2(z_{2n-2}, x_{2n-2}, y_{2n-2}).$$

By processing on the same line to Theorem 2.4 we can establish that $\{hx_{2n}\}, \{hy_{2n}\}$ and $\{hz_{2n}\}$ are Cauchy sequences.
Completeness of Y ensures that there exists $(x, y, z) \in Y \times Y \times Y$,
such that $x_{2n-1} = hx_{2n} \to x$,$y_{2n-1} = hy_{2n} \to y$, $z_{2n-1} = hz_{2n} \to z$ as $n \to \infty$.
Using compatibility of the pair (S_2, h)

$$\lim_{n\to\infty} D(hx_{2n-1}, S_2(hx_{2n}, hy_{2n}, hz_{2n})) = 0$$

$$\lim_{n\to\infty} D(hy_{2n-1}, S_2(hy_{2n}, hz_{2n}, hx_{2n})) = 0$$

$$\lim_{n\to\infty} D(hz_{2n-1}, S_2(hz_{2n}, hx_{2n}, hy_{2n})) = 0.$$

Since h is Continuous, we have

$$D(hx, S_2(x, y, z)) = 0$$

$$D(hy, S_2(y, z, x)) = 0$$

$$D(hz, S_2(z, x, y)) = 0.$$

Since $S_2(x, y, z), S_2(y, z, x), S_2(z, x, y)$ are Compact and hence closed.
i.e.

$$\bar{S}_2(x, y, z) = S_2(x, y, z), \bar{S}_2(y, z, x) = S_2(y, z, x), \bar{S}_2(z, x, y) = S_2(z, x, y)$$

implies

$$hx \in S_2(x, y, z), hy \in S_2(y, z, x), hz \in S_2(z, x, y)$$

i.e. $(x, y, z) \in C(S_2, h)$.
Consider

$$\begin{aligned}\sigma(hx_0, x) + \sigma(hy_0, y) + \sigma(hz_0, z) \le \sum_{i=0}^{n} &\sigma(hx_{2i}, hx_{2i+2}) + \sigma(hy_{2i}, hy_{2i+2}) \\ &+ \sigma(hz_{2i}, hz_{2i+2}) + \sigma(hx_{2n+2}, x) \\ &+ \sigma(hy_{2n+2}, y) + \sigma(hz_{2n+2}, z).\end{aligned}$$

Letting $n \to \infty$, we acquire

$$\begin{aligned}\sigma(hx_0,x)+\sigma(hy_0,y)+\sigma(hz_0,z) &\leq \sigma(hx_0,hx_2)+\sigma(hy_0,hy_2)+\sigma(hz_0,hz_2)\\ &\quad+\sum_{i=1}^{\infty}\sigma(hx_{2i},hx_{2i+2})+\sigma(hy_{2i},hy_{2i+2})\\ &\quad+\sigma(hz_{2i},hz_{2i+2})\\ &\leq qk+\sum_{i=1}^{\infty}\delta_{2i}\\ &\leq qk+\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}\\ &\leq qk,\ as\ i\to\infty.\end{aligned}$$

Again,

$$\begin{aligned}\sigma(x_0,x)+\sigma(y_0,y)+\sigma(z_0,z) \leq&\sigma(x_0,hx_0)+\sigma(y_0,hy_0)+\sigma(z_0,hz_0)\\ &+\sigma(hx_0,x)+\sigma(hy_0,y)+\sigma(hz_0,z)\\ \sigma(x_0,x)+\sigma(y_0,y)+\sigma(z_0,z) \leq&qk+R.\end{aligned} \tag{3.4}$$

Thus, for each $(x_0,y_0,z_0)\in C(h,S_1)$, there exists $(x,y,z)\in C(h,S_2)$, such that $\sigma(x_0,x)+\sigma(y_0,y)+\sigma(z_0,z)\leq qk+R$
Similarly for any arbitrary $(u_0,v_0,w_0)\in C(h,S_2)$, there exists $(u,v,w)\in C(h,S_1)$ such that $\sigma(u_0,u)+\sigma(v_0,v)+\sigma(w_0,w)\leq qk+R$.
Hence conclude that $H(C(h,S_1),C(h,S_2))\leq qk+R$. ■

Theorem 3.3. *Let (Y,σ) be a m.s., $h:Y\to Y$ a single valued mapping and $\{S_n : Y\times Y\times Y\to C(Y)\}$ be a sequence of multivalued almost $\mathcal{F}_\delta$-contractions with respect to h, which is uniformly convergent to $S:Y\times Y\times Y\to C(Y)$, such that the pairs (S_n,h) satisfy all the conditions of theorem 2.4, and also (i) $S(Y\times Y\times Y)\subseteq h(Y)$ (ii)The pair of mappings (h,S) is compatible.*
Let $M_n = sup\{\sigma(x,hx)+\sigma(y,hy)+\sigma(z,hz) : (x,y,z)\in C(h,S_n)\cup C(h,S)\}$ and $M_n\to 0, n\to\infty$. Then $\lim_{n\to\infty} H(C(h,S_n),C(h,S))=0$.
Hence the coincidence point sets of the sequence $\{(h,S_n)\}$ of pair of mappings are stable.

Proof. Since $\{S_n : Y\times Y\times Y\to C(Y)\}$ be a sequence of multivalued almost $\mathcal{F}_\delta$-contractions with respect to h, which is uniformly convergent to $S:Y\times Y\times Y\to C(Y)$. So S is continuous and by Lemma 3.1, also S is multivalued almost $\mathcal{F}_\delta$-contractions with respect to h, and given that $S(Y\times Y\times Y)\subseteq h(Y)$ and the pair of mappings (h,S) is compatible.
Let $k_n = sup\{H(S_n(x,y,z),S(x,y,z))+H(S_n(y,z,x),S(y,z,x))+H(S_n(z,x,y), S(z,x,y)) : (x,y,z)\in Y\times Y\times Y\}$.
By uniformly convergence of sequence $\{S_n\}$ to S.
$\lim_{n\to\infty} k_n = \lim_{n\to\infty} sup\{H(S_n(x,y,z),S(x,y,z))+H(S_n(y,z,x),S(y,z,x)) + H(S_n(z,x,y),S(z,x,y)) : (x,y,z)\in Y\times Y\times Y\}=0$.
Using Theorem 3.2, we get $H(C(h,S_n),C(h,S))\leq qk_n+M_n$.
Hence $\lim_{n\to\infty} H(C(h,S_n),C(h,S))\leq \lim_{n\to\infty}(qk_n+M_n)=0$.
That is $\lim_{n\to\infty} H(C(h,S_n),C(h,S))=0$. ■

Example 3.4. Let $Y = [0,1]$ with usual metric σ be a complete $m.s.$, define a sequence of multivalued mappings $S_n : Y \times Y \times Y \to C(Y)$, by $S_n(x,y,z) = [\frac{1}{3^n}, \frac{x+y+z}{3}]$, with the self mapping $h : Y \to Y$ by $hx = x^2$and consider $\mathcal{F}(\alpha) = log\alpha$ with $K = 0$,and $0 < \tau < log3$. We claim that S_n for all $n \geq 1$ are almost $\mathcal{F}_\delta$-contractions with respect to h and also satisfy all other conditions of Theorem 2.4,and it observe that $S_n(x,y,z) = [\frac{x+y+z}{3}, \frac{1}{3^n}]$ is uniformly convergent to $S(x,y,z) = [0, \frac{x+y+z}{3}]$.
Also $S(Y \times Y \times Y) \subseteq h(Y)$ and the pair (h,S) is compatible, then $\lim_{n\to\infty} H(C(h,S_n), C(h,S)) = 0$. Hence coincidence point sets of the sequence $\{(h,S_n)\}$ of pair of mappings are stable.

4. Applications

4.1. Application to Integral Inclusions

Take the following system of integral inclusions

$$\begin{aligned} a(t) &\in p(t) + \int_0^l \gamma(t,s)[F^*(s,a(s)) + G^*(s,b(s)) + H^*(s,c(s))]ds \\ b(t) &\in p(t) + \int_0^l \gamma(t,s)[F^*(s,b(s)) + G^*(s,c(s)) + H^*(s,a(s))]ds \\ c(t) &\in p(t) + \int_0^l \gamma(t,s)[F^*(s,c(s)) + G^*(s,a(s)) + H^*(s,b(s))]ds, \end{aligned} \tag{4.1}$$

where
(i) $F^*, G^*, H^* : [0,l] \times R \to C(R)$(Family of compact subsets of R) are continuous,
(ii) $p, a, b, c : [0,l] \to R$ are continuous,
(iii) $\gamma : [0,l] \times R \to [0,\infty)$ is continuous.

Theorem 4.1. *Consider the system of integral inclusions 4.1, with the mappings $f, g, h : [0,l] \times R \to R$,such that for all $f(s,a(s)) \in F^*(s,a(s))$and $f(s,b(s)) \in F^*(s,b(s))$ $g(s,a(s)) \in G^*(s,a(s))$and $g(s,b(s)) \in G^*(s,b(s))$ $h(s,a(s)) \in H^*(s,a(s))$and $h(s,b(s)) \in H^*(s,b(s))$implies*

$$\begin{aligned} |f(s,a(s)) - f(s,b(s))| &\leq |a(s) - b(s)|\, e^{-\tau} \\ |g(s,a(s)) - g(s,b(s))| &\leq |a(s) - b(s)|\, e^{-\tau} \\ |h(s,a(s)) - h(s,b(s))| &\leq |a(s) - b(s)|\, e^{-\tau} \end{aligned}$$

and

$$\max_{t\in[0,l]} \int_0^l \gamma(t,s)ds \leq \frac{1}{3l}.$$

Then, the system of integral inclusion has a solution.

Proof. Let us consider, the space $Y = C([0,l], R)$ of continuous functions on$[0,l]$, with

$$\sigma(a,b) = \max_{t\in[0,l]} |a(t) - b(t)|\,; a,b \in Y.$$

Then, obviously (Y,σ) be a complete $m.s.$.
Now define, a mapping $S : Y \times Y \times Y \to C(Y)$,by

$$
\begin{aligned}
S(a(t),b(t),c(t)) &= p(t) + \int_0^l \gamma(t,s)[F^*(s,a(s)) + G^*(s,b(s)) + H^*(s,c(s))]ds \\
S(b(t),c(t),a(t)) &= p(t) + \int_0^l \gamma(t,s)[F^*(s,b(s)) + G^*(s,c(s)) + H^*(s,a(s))]ds \\
S(c(t),a(t),b(t)) &= p(t) + \int_0^l \gamma(t,s)[F^*(s,c(s)) + G^*(s,a(s)) + H^*(s,b(s))]ds.
\end{aligned}
\tag{4.2}
$$

Now, for $S(a_1(t),b_1(t),c_1(t)), S(a_2(t),b_2(t),c_2(t)) \in C(Y)$, we have

$$
\begin{aligned}
&\delta(S(a_1(t),b_1(t),c_1(t)), S(a_2(t),b_2(t),c_2(t))) \\
&\quad = sup\{\sigma(a,b) : a \in S(a_1(t),b_1(t),c_1(t)), b \in S(a_2(t),b_2(t),c_2(t))\}
\end{aligned}
\tag{4.3}
$$

where $a(t) = p(t) + \int_0^l \gamma(t,s)[f(s,a_1(s)) + g(s,b_1(s)) + h(s,c_1(s))]ds$
$b(t) = p(t) + \int_0^l \gamma(t,s)[f(s,b_2(s)) + g(s,c_2(s)) + h(s,a_2(s))]ds$
for some $f(s,a_1(s)) \in F^*(s,a_1(s))$and $f(s,a_2(s)) \in F^*(s,a_2(s))$ $g(s,b_1(s)) \in G^*(s,b_1(s))$ and $g(s,b_2(s)) \in G^*(s,b_2(s)), h(s,c_1(s)) \in H^*(s,c_1(s))$and $h(s,c_2(s)) \in H^*(s,c_2(s))$.
Now,

$$
\begin{aligned}
|a(t)-b(t)| &= \left|\int_0^l \gamma(t,s)[f(s,a_1(s)) - f(s,a_2(s)) + g(s,b_1(s)) - g(s,b_2(s))\right. \\
&\quad + h(s,c_1(s)) - h(s,c_2(s))]ds| \\
&\leq \int_0^l |\gamma(t,s)|\,ds. \int_0^l [|f(s,a_1(s)) - f(s,a_2(s))| \\
&\quad + |g(s,b_1(s)) - g(s,b_2(s))| + |h(s,c_1(s)) - h(s,c_2(s))|]ds \\
&\leq \int_0^l |\gamma(t,s)|\,ds. \int_0^l [|a_1(s)-a_2(s)|\,e^{-\tau} + |b_1(s)-b_2(s)|\,e^{-\tau} \\
&\quad + |c_1(s)-c_2(s)|\,e^{-\tau}]ds.
\end{aligned}
\tag{4.4}
$$

Hence, we have

$$
\begin{aligned}
\max_{t\in[0,l]} |a(t)-b(t)| \leq \max_{t\in[0,l]} \int_0^l |\gamma(t,s)|\,ds. \int_0^l \max_{t\in[0,l]} [|a_1(t)-a_2(t))|\,e^{-\tau} \\
+ |b_1(t)-b_2(t)|\,e^{-\tau} + |c_1(t)-c_2(t))|\,e^{-\tau}]ds.
\end{aligned}
$$

This amounts to say that

$$
\begin{aligned}
\sigma(a,b) &\leq \frac{1}{3}\max_{t\in[0,l]}[|a_1(t)-a_2(t))| + |b_1(t)-b_2(t)| + |c_1(t)-c_2(t))|]e^{-\tau} \\
&= \frac{1}{3}[\sigma(a_1,a_2) + \sigma(b_1,b_2) + \sigma(c_1,c_2)]e^{-\tau}.
\end{aligned}
$$

Utilizing hypothesis of our theorem, we have

$$\begin{aligned}\sup\{\sigma(a,b) : a \in S(a_1(t), b_1(t), c_1(t)), b \in S(a_2(t), b_2(t), c_2(t))\} \leq \frac{1}{3}[&\sigma(a_1, a_2) \\ &+ \sigma(b_1, b_2) \\ &+ \sigma(c_1, c_2)]e^{-\tau}.\end{aligned}$$

$$\begin{aligned}\delta(S(a_1(t), b_1(t), c_1(t)), S(a_2(t), b_2(t), c_2(t))) \leq \frac{1}{3}[&\sigma(a_1, a_2) + \sigma(b_1, b_2) \\ &+ \sigma(c_1, c_2)]e^{-\tau}.\end{aligned}$$

By the similar calculations, we have

$$\begin{aligned}\delta(S(b_1(t), c_1(t), a_1(t)), S(b_2(t), c_2(t), a_2(t))) \leq \frac{1}{3}[&\sigma(b_1, b_2) + \sigma(c_1, c_2) \\ &+ \sigma(a_1, a_2)]e^{-\tau}.\end{aligned}$$

$$\begin{aligned}\delta(S(c_1(t), a_1(t), b_1(t)), S(c_2(t), a_2(t), b_2(t))) \leq \frac{1}{3}[&\sigma(c_1, c_2) + \sigma(a_1, a_2) \\ &+ \sigma(b_1, b_2)]e^{-\tau}.\end{aligned}$$

Taking these three inequalities into account, we arrive at

$$\begin{aligned}&\delta(S(a_1(t), b_1(t), c_1(t)), S(a_2(t), b_2(t), c_2(t))) + \delta(S(b_1(t), c_1(t), a_1(t)), \\ &S(b_2(t), c_2(t), a_2(t))) + \delta(S(c_1(t), a_1(t), b_1(t)), S(c_2(t), a_2(t), b_2(t))) \\ &\leq [\sigma(a_1, a_2) + \sigma(b_1, b_2) + \sigma(c_1, c_2)]e^{-\tau} \\ &\leq M(a_1, b_1, c_1, a_2, b_2, c_2)e^{-\tau}.\end{aligned} \tag{4.5}$$

Consequently, passing to logarithms, we get

$$\begin{aligned}\tau + log(&\delta(S(a_1(t), b_1(t), c_1(t)), S(a_2(t), b_2(t), c_2(t))) \\ &+ \delta(S(b_1(t), c_1(t), a_1(t)), S(b_2(t), c_2(t), a_2(t))) \\ &+ \delta(S(c_1(t), a_1(t), b_1(t)), S(c_2(t), a_2(t), b_2(t)))) \\ &\leq log(M(a_1, b_1, c_1, a_2, b_2, c_2)).\end{aligned} \tag{4.6}$$

Consequently, we arrive at

$$\begin{aligned}\tau + \mathcal{F}(&\delta(S(a_1(t), b_1(t), c_1(t)), S(a_2(t), b_2(t), c_2(t))) + \delta(S(b_1(t), c_1(t), a_1(t)), \\ &S(b_2(t), c_2(t), a_2(t))) + \delta(S(c_1(t), a_1(t), b_1(t)), S(c_2(t), a_2(t), b_2(t)))) \\ &\qquad\qquad\qquad \leq \mathcal{F}(M(a_1, b_1, c_1, a_2, b_2, c_2)),\end{aligned} \tag{4.7}$$

for $\mathcal{F}(t) = logt, t > 0$. Thus, S is almost $\mathcal{F}_\delta-$contraction, so by theorem 2.7, we conclude that S has TFP.
i.e. $a(t) \in S(a(t), b(t), c(t)), b(t) \in S(b(t), c(t), a(t)), c(t) \in S(c(t), a(t), b(t))$.
Thus integral inclusion defined in (4.1) has a solution. ∎

4.2. Application to Solution for Matrix Equation

Motivated by Fan et.al.[8],use the single valued case of Theorem 2.7 (Remark 2.8) to discuss the existence of solution for the matrix equations:

$$X^p - A^*XA + B^*XB - C^*XC = P, p > 1, \tag{4.8}$$

where $X \in H(m)$,the set of all Hermitian positive define matrices, P is an $m \times m$ positive define matrix. A, B, C are $m \times m$ non singular matrices, A^*, B^*, C^* denote the conjugate transpose of the matrices A, B, C respectively.
Following are some important results which play important role for the rest of our analysis. Thompson metric $\sigma : H(m) \times H(m) \to H(m)$ in [19] is defined as follows:

$$\sigma(A, B) = max\{\ln W(A/B), \ln W(B/A)\} = \left\|\ln(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\right\|, \tag{4.9}$$

where $W(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_{max}(B^{-\frac{1}{2}} A B^{-\frac{1}{2}})$. Note that $(H(m), \sigma)$ is a complete metric space (see [16]).

Lemma 4.2. [12] *Let $\sigma : H(m) \times H(m) \to H(m)$ be a Thompson metric on the open convex cone $H(m)$, then for any $A, B \in H(m)$ and a non singular matrix N, we have that the following conditions hold:*

$$\sigma(A, B) = \sigma(A^{-1}, B^{-1}) = \sigma(N^* A N, N^* B N), \tag{4.10}$$

where A^{-1}, B^{-1} are the inversion of matrices A and B,respectively;

$$\begin{aligned} \sigma(A^p, B^p) &\leq p\sigma(A, B), p \in [0, 1]; \\ \sigma(N^* A^p N, N^* B^p N) &\leq |p|\, \sigma(A, B), p \in [-1, 1]. \end{aligned} \tag{4.11}$$

Lemma 4.3. [12] *For any $A, B, C, D \in H(m)$,*

$$\sigma(A + B, C + D) \leq max\{\sigma(A, C), \sigma(B, D)\}. \tag{4.12}$$

Especially,

$$\sigma(A + B, A + C) \leq \sigma(B, C). \tag{4.13}$$

Theorem 4.4. *Let $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in H(m)$, with*

$$\begin{aligned} \sigma\Big(\frac{A^* X_1 A - B^* X_2 B + C^* X_3 C}{2}, \frac{A^* Y_1 A - B^* Y_2 B + C^* Y_3 C}{2}\Big) \\ \leq \sigma(X_1, Y_1) e^{-\tau}, \tau > 0 \end{aligned} \tag{4.14}$$

then the matrix equations (4.8) possess a solution.

Proof. Let $S : H(m) \times H(m) \times H(m) \to H(m)$ be a single valued mapping, defined by

$$(X_1, X_2, X_3) = (P + A^* X_1 A - B^* X_2 B + C^* X_3 C)^{\frac{1}{p}}.$$

Using Lemma4.2 and Lemma4.3 for $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in H(m)$

$$\begin{aligned}
&\sigma(S(X_1, X_2, X_3), S(Y_1, Y_2, Y_3)) \\
&= \sigma((P + A^*X_1A - B^*X_2B + C^*X_3C)^{\frac{1}{p}}, (P + A^*Y_1A - B^*Y_2B + C^*Y_3C)^{\frac{1}{p}}) \\
&\leq \frac{1}{p}\sigma((P + A^*X_1A - B^*X_2B + C^*X_3C), (P + A^*Y_1A - B^*Y_2B + C^*Y_3C)) \\
&\leq \frac{1}{p}\sigma((A^*X_1A - B^*X_2B + C^*X_3C), (A^*Y_1A - B^*Y_2B + C^*Y_3C)) \\
&\leq \sigma((A^*X_1A - B^*X_2B + C^*X_3C), (A^*Y_1A - B^*Y_2B + C^*Y_3C)) \\
&\leq \sigma(\frac{A^*X_1A - B^*X_2B + C^*X_3C}{2}, \frac{A^*Y_1A - B^*Y_2B + C^*Y_3C)}{2} \\
&\leq \sigma(X_1, Y_1)e^{-\tau} \\
&\sigma(S(X_1, X_2, X_3), S(Y_1, Y_2, Y_3)) \\
&\leq \sigma(X_1, Y_1)e^{-\tau}.
\end{aligned} \tag{4.15}$$

Similarly,

$$\sigma(S(X_2, X_3, X_1), S(Y_2, Y_3, Y_1)) \leq \sigma(X_2, Y_2)e^{-\tau}$$

and

$$\sigma(S(X_3, X_1, X_2), S(Y_3, Y_1, Y_2)) \leq \sigma(X_3, Y_3)e^{-\tau}$$

Combining these inequalities, we get

$$\begin{aligned}
&\sigma(S(X_1, X_2, X_3), S(Y_1, Y_2, Y_3)) + \sigma(S(X_2, X_3, X_1), S(Y_2, Y_3, Y_1)) \\
&\qquad\qquad + \sigma(S(X_3, X_1, X_2), S(Y_3, Y_1, Y_2)) \\
&\quad \leq (\sigma(X_1, Y_1) + \sigma(X_2, Y_2) + \sigma(X_3, Y_3))e^{-\tau} \\
&\quad \leq [M(X_1, X_2, X_3, Y_1, Y_2, Y_3) + KN(X_1, X_2, X_3, Y_1, Y_2, Y_3)e^{-\tau}
\end{aligned} \tag{4.16}$$

Passing to logarithms, above inequality becomes

$$\begin{aligned}
&log[\sigma(S(X_1, X_2, X_3), S(Y_1, Y_2, Y_3)) + \sigma(S(X_2, X_3, X_1), S(Y_2, Y_3, Y_1)) \\
&\qquad\qquad + \sigma(S(X_3, X_1, X_2), S(Y_3, Y_1, Y_2))] \\
&\qquad \leq \log[M(X_1, X_2, X_3, Y_1, Y_2, Y_3) + KN(X_1, X_2, X_3, Y_1, Y_2, Y_3)] \\
&\qquad\qquad + \log e^{-\tau},
\end{aligned} \tag{4.17}$$

and ultimately, we get

$$\begin{aligned}
&\tau + F(\sigma(S(X_1, X_2, X_3), S(Y_1, Y_2, Y_3)) + \sigma(S(X_2, X_3, X_1), S(Y_2, Y_3, Y_1)) \\
&\qquad\qquad + \sigma(S(X_3, X_1, X_2), S(Y_3, Y_1, Y_2))) \\
&\qquad \leq F(M(X_1, X_2, X_3, Y_1, Y_2, Y_3) + KN(X_1, X_2, X_3, Y_1, Y_2, Y_3)),
\end{aligned} \tag{4.18}$$

for $F(t) = logt, t > 0$.

Hence we conclude that there exists $X_1, X_2, X_3 \in H(m)$, such that $X_1 = S(X_1, X_2, X_3), X_2 = S(X_2, X_3, X_1), X_3 = S(X_3, X_1, X_2)$. This shows the existence of the solution of Matrix equation (4.8). ∎

- **Numerical experiment**

Example 4.5. Let

$$X_1 = X_2 = X_3 = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$$

and

$$Y_1 = Y_2 = Y_3 = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

are Hermitian positive definite matrixes, with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are 2×2 non singular matrices. Then

$$A^*X_1A - B^*X_2B + C^*X_3C = \begin{bmatrix} 0 & 2 \\ 2 & -7 \end{bmatrix}$$

$$A^*Y_1A - B^*Y_2B + C^*Y_3C = \begin{bmatrix} 6 & 2 \\ 2 & -7 \end{bmatrix}$$

satisfy the condition of Theorem 4.4.
Hence the matrix equation (4.8) has a solution, which is

$$X_1 = \begin{bmatrix} 1 & -i \\ i & 2 \end{bmatrix}$$

since, it can easy to verify that the matrix

$$X_1^2 - A^*X_1A - B^*X_1B + C^*X_1C = \begin{bmatrix} 2 & -3i-2 \\ 3i+2 & 12 \end{bmatrix}$$

is a positive definite matrix.

Open Problem: For future reading, as an application, an open problem is suggested as follows:
A discretized population balance for continuous systems at steady state can be modeled by the following integral equation

$$f(t) = \frac{a}{2(1+2a)} \int_0^t f(t-x)f(x)dx + e^{-t}.$$

Whether the existence of solution of the above integral equation can be derived from results established in this note?

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