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On Lupaş-Jain-Beta Operators

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Abstract In this paper, an integral variant of the Lupaş operators are investigated. The rate of convergence of these operators with the help of K-functional is discussed. Asymptotic formula, rate of convergence and weighted approximation results are established. At the end, we discussed better error estimation of these operators.

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1. Introduction

In recent years, Patel and Mishra [36] generalized Jain operators as a variant of the Lupaş operators [30] defined by

$$P_n^{[\beta]}(f,x)=\sum_{k=0}^{\infty}\frac{(nx+k\beta)_k}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right),\ x\geq 0,\ \ f:[0,\infty)\to\mathbb{R}, \tag{1.1}$$

where $(nx+k\beta)_0=1$, $(nx+k\beta)_1=nx$ and $(nx+k\beta)_k=nx(nx+k\beta+1)(nx+k\beta+2)\dots(nx+k\beta+k-1)$, $k\geq 2$.
By using analogous Abel and Jensen combinatorial formulas for factorial powers (see [40]). In [8], the authors modified the operators (1.1) into following sense

$$L_n^{[\beta]}(f,x)=\sum_{k=0}^{\infty}\frac{nx(nx+1+k\beta)_{k-1}}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right), \tag{1.2}$$

and $L_n^{[\beta]}(f,0)=f(0)$ for real valued bounded functions f on $[0,\infty)$, where $0\leq\beta<1$ and β depending only on n. The authors called this operators as Lupaş-Jain operators.

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Consider the weight function $\rho_\lambda : [0,\infty) \to [1,\infty)$, $\rho_\lambda(x) = 1 + x^{2+\lambda} (\lambda \geq 0)$ and $\rho(x) = \rho_0(x) = 1 + x^2$, we define the space

$$B_\rho\left([0,\infty)\right) = \{f : [0,\infty) \to \mathbb{R}, |f(x)| \leq M_f \rho(x), x \geq 0\},$$

where M_f is a constant depending on f.

$$C_\rho([0,\infty)) = \{f \in B_\rho\left([0,\infty)\right) : f \text{ is continous on } [0,\infty)\},$$

$$C_\rho^k([0,\infty)) = \left\{f \in C_\rho\left([0,\infty)\right) : \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k_f\right\},$$

where k_f is a constant depending on f. It is obvious that $C_\rho^k\left([0,\infty)\right) \subset C_\rho\left([0,\infty)\right) \subset B_\rho\left([0,\infty)\right)$. The space $B_\rho\left([0,\infty)\right)$ is normed linear space with with the norm $\|f\|_\rho = \sup_{x\geq 0} \dfrac{|f(x)|}{\rho(x)}$.

In order to approximate Lebesgue integrable functions, Durrmeyer [19] proposed an integral modification of Bernstein polynomials, which was later studied in different forms by many authors [7, 27, 31, 32, 41]. The Durrmeyer modification of Jain operators and their generalizations was studied in [4, 9, 26, 35, 42]. We now propose the Durrmeyer type integral modification of the operators (1.2) as follows: $x \geq 0$

$$D_n^{1/n}(f,x) = \sum_{k=1}^{\infty} \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \times \int_0^\infty \frac{f(t)}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + 2^{-nx} f(0), \tag{1.3}$$

where $n > 1$, $f \in C_{\rho_\lambda}([0,\infty))$. By using a bivariate kernel we can write (1.3) in a more compact form, as follows

$$D_n^{1/n}(f,x) = \int_0^\infty H_n(t,x) f(t) dt, n > 1, x \geq 0, \tag{1.4}$$

where

$$H_n(t,x) = \sum_{k=1}^{\infty} \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \frac{1}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}} + 2^{-nx}\delta(t).$$

In the above δ represents Dirac delta function for which

$$\int_0^\infty \delta(t) f(t) dt = f(0).$$

The operators defined by (1.3) are the integral modification of the Jain variant of Lupaş operators for the case $\beta = \dfrac{1}{n}$ having weight function of some beta basis function.
In the present manuscript, we introduce the operators (1.3) and estimate their moments. Also, studied local approximation results, rate of convergence, weighted approximation theorem.

2. Some Moments

The moment of the operators (1.2) are studied in [8]. For the particular case $\beta = \frac{1}{n}$, $n \in \{2, 3, \ldots\}$, we obtain

$L_n^{[\frac{1}{n}]}(1, x) = 1$; $L_n^{[\frac{1}{n}]}(t, x) = \frac{nx}{n-1}$; $L_n^{[\frac{1}{n}]}(t^2, x) = \frac{n^2x^2}{(n-1)^2} + \frac{2n^2x}{(n-1)^3}$.

Using the process given in [8] and [36], we calculate the following 3rd and 4th moments of the operators (1.2) as

$$L_n^{[\frac{1}{n}]}(t^3, x) = \frac{n^3x^3}{(n-1)^3} + \frac{6n^3x^2}{(n-1)^4} + \frac{6n^2(1+n)x}{(n-1)^5}$$

and

$$L_n^{[\frac{1}{n}]}(t^4, x) = \frac{n^4x^4}{(n-1)^4} + \frac{12n^4x^3}{(n-1)^5} + \frac{12n^3(2+3n)x^2}{(n-1)^6} + \frac{2n^2\left(13+34n+13n^2\right)x}{(n-1)^7}.$$

Lemma 2.1. *The following equalities hold.*

(1) $D_n^{1/n}(1, x) = 1$,

(2) $D_n^{1/n}(t, x) = \frac{nx}{n-1}$,

(3) $D_n^{1/n}(t^2, x) = \frac{n^3x^2}{(n-1)^3} + \frac{n\left(3n^2-2n+1\right)x}{(n-1)^4}$.

Proof. Using $L_n^{[\frac{1}{n}]}(1, x) = 1$, we have

$$\begin{aligned} D_n^{1/n}(t, x) &= \int_0^\infty H_n(t, x)t\,dt \\ &= \sum_{k=1}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \frac{B(n, k+1)}{B(n+1, k)} \\ &= \sum_{k=0}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \frac{k}{n} = L_n^{[\frac{1}{n}]}(t, x) = \frac{nx}{n-1}. \end{aligned}$$

Lastly,

$$\begin{aligned} D_n^{1/n}(t^2, x) &= \int_0^\infty H_n(t, x)t^2\,dt \\ &= \sum_{k=1}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \\ &\quad \times \int_0^\infty \frac{1}{B(n+1, k)} \frac{t^{k+1}}{(1+t)^{n+k+1}}dt \\ &= \sum_{k=1}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(nx+k\frac{1}{n}\right)} \frac{B(n-1, k+2)}{B(n+1, k)} \\ &= \frac{n}{n-1} L_n^{[\frac{1}{n}]}(t^2, x) + \frac{1}{n-1} L_n^{[\frac{1}{n}]}(t, x) = \frac{n^3x^2}{(n-1)^3} \end{aligned}$$

$$+\frac{n\left(3n^2-2n+1\right)x}{(n-1)^4}.$$

■

Further, the 3rd and 4th moments are obtained as follows

$$\begin{aligned}
D_n^{1/n}(t^3,x) &= \int_0^\infty H_n(t,x)t^3dt\\
&= \sum_{k=1}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!}2^{-\left(nx+k\frac{1}{n}\right)}\int_0^\infty \frac{1}{B(n+1,k)}\frac{t^{k+2}}{(1+t)^{n+k+1}}dt\\
&= \frac{n^2}{(n-1)(n-2)}L_n^{[\frac{1}{n}]}(t^3,x)+\frac{3n}{(n-1)(n-2)}L_n^{[\frac{1}{n}]}(t^2,x)\\
&\quad +\frac{2}{(n-1)(n-2)}L_n^{[\frac{1}{n}]}(t,x)\\
&= \frac{n^5x^3}{(n-2)(n-1)^4}+\frac{3n^3\left(1-2n+3n^2\right)x^2}{(n-2)(n-1)^5}\\
&\quad +\frac{2n\left(1-4n+9n^2-7n^3+7n^4\right)x}{(n-2)(n-1)^6}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_n^{1/n}(t^4,x) &= \int_0^\infty H_n(t,x)t^4dt\\
&= \sum_{k=1}^\infty \frac{nx\left(nx+1+k\frac{1}{n}\right)_{k-1}}{2^k k!}2^{-\left(nx+k\frac{1}{n}\right)}\int_0^\infty \frac{1}{B(n+1,k)}\frac{t^{k+3}}{(1+t)^{n+k+1}}dt\\
&= \frac{n^3}{(n-1)(n-2)(n-3)}L_n^{[\frac{1}{n}]}(t^4,x)+\frac{6n^2}{(n-1)(n-2)(n-3)}L_n^{[\frac{1}{n}]}(t^3,x)\\
&\quad +\frac{11n}{(n-1)(n-2)(n-3)}L_n^{[\frac{1}{n}]}(t^2,x)+\frac{6}{(n-1)(n-2)(n-3)}L_n^{[\frac{1}{n}]}(t,x)\\
&= \frac{n^7x^4}{(n-3)(n-2)(n-1)^5}+\frac{6n^5\left(1-2n+3n^2\right)x^3}{(n-3)(n-2)(n-1)^6}\\
&\quad +\frac{n^3\left(11-44n+102n^2-92n^3+83n^4\right)x^2}{(n-3)(n-2)(n-1)^7}\\
&\quad +\frac{2n^2\left(3-7n+19n^2+n^3+17n^4+24n^5+3n^6\right)x}{(n-3)(n-2)(n-1)^8}.
\end{aligned}$$

In the following lemma, we obtain central moments of the operators $D_n^{1/n}(\cdot,x)$.

Lemma 2.2. *The following equalities hold.*

(1) $D_n^{1/n}(t-x,x)=\dfrac{x}{n-1}$,

(2) $D_n^{1/n}((t-x)^2,x)=\dfrac{n\left(3n^2-2n+1\right)x}{(n-1)^4}+\dfrac{\left(n^2+n-1\right)x^2}{(n-1)^3}$,

(3) $D_n^{1/n}((t-x)^4,x) = \dfrac{\left(3n^5+40n^4-77n^3+30n^2+11n-6\right)x^4}{(n-3)(n-2)(n-1)^5}$

$$+\frac{6x^3\left(3n^6+31n^5-72n^4+63n^3-29n^2+6n\right)}{(n-3)(n-2)(n-1)^6}$$
$$+\frac{nx^2\left(27n^6+188n^5-362n^4+444n^3-341n^2+128n-24\right)}{(n-3)(n-2)(n-1)^7}$$
$$+\frac{2n^2x\left(3-7n+19n^2+n^3+17n^4+24n^5+3n^6\right)}{(n-3)(n-2)(n-1)^8}.$$

The proof follows from the linearity of $D_n^{1/n}$.

Remark 2.3. Let f be a continuous and bounded function on $[0,\infty)$. For $n\to\infty$, the sequence $\{D_n^{1/n}(f,x)\}$ converges uniformly to $f(x)$ in $[a,b]\subset[0,\infty)$, which follows from the well known Bohman-Korovkin theorem.

Theorem 2.4. *For each $f\in C_\rho^k([0,\infty))$, we have*

$$\lim_{n\to\infty}\|D_n^{1/n}(f,\cdot)-f\|_\rho=0.$$

Proof. Using the theorem in [24], in order to prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty}\|D_n^{1/n}(t^i,\cdot)-x^i\|_\rho=0,\ i=0,1,2.$$

Since $D_n^{1/n}(1,\cdot)=1$. The above conditions hold for $i=0$, we can write

$$\lim_{n\to\infty}\|D_n^{1/n}(t,\cdot)-x\|_\rho\le\lim_{n\to\infty}\sup_{x\ge0}\frac{1}{1+x^2}\frac{x}{n-1}=0.$$

Finally

$$\begin{aligned}\|D_n^{1/n}(t^2,\cdot)-x^2\|_\rho&\le\sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{n^3x^2}{(n-1)^3}+\frac{n\left(3n^2-2n+1\right)x}{(n-1)^4}-x^2\right|\\&=\sup_{x\ge0}\frac{1}{1+x^2}\left|\frac{(n-2n^2+3n^3)x}{(n-1)^4}+\frac{(1-3n+3n^2)x^2}{(n-1)^3}\right|,\end{aligned}$$

which gives

$$\lim_{n\to\infty}\|D_n^{1/n}(t^2,\cdot)-x^2\|_\rho=0.$$

This completes the proof of the theorem. ∎

3. Main Results

Theorem 3.1. *Let $f\in C_\rho([0,\infty))$ and has second derivative at a point $x\in(0,\infty)$. Then*

$$\lim_{n\to\infty}n\left[D_n^{1/n}(f,x)-f(x)\right]=xf'(x)+\frac{x(x+3)}{2}f''(x).$$

Proof. By Taylor's expansion of f, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{3.1}$$

where $r(t,x)$ is the Peano form of the remainder, continuous and bounded on $[0,\infty)$ and $\lim_{t\to x} r(t,x)=0$. Operating $D_n^{1/n}$ to the equation (3.1), we obtain

$$D_n^{1/n}(f,x)-f(x)=D_n^{1/n}(t-x,x)f'(x)+D_n^{1/n}((t-x)^2,x)\frac{f''(x)}{2}+D_n^{1/n}(r(t,x)(t-x)^2,x).$$

Using the Cauchy-Schwarz inequality, we have

$$D_n^{1/n}(r(t,x)(t-x)^2,x)\le\sqrt{D_n^{1/n}(r^2(t,x),x)}\sqrt{D_n^{1/n}((t-x)^4,x)}. \tag{3.2}$$

We have $r^2(x,x)=0$ and $r^2(t,x)\in C_\rho^k[0,\infty)$

$$\lim_{n\to\infty} D_n^{1/n}(r^2(t,x),x)=r^2(x,x)=0. \tag{3.3}$$

uniformly with respect to $x\in[0,A]$. Now from (3.2), (3.3) and from Lemma 2.2, we get

$$\lim_{n\to\infty} D_n^{1/n}(r^2(t,x)(t-x)^2,x)=0.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} n\left[D_n^{1/n}(f,x)-f(x)\right] &= \lim_{n\to\infty} n\left[D_n^{1/n}(t-x,x)f'(x)\right.\\ &\quad+\frac{1}{2}f''(x)D_n^{1/n}((t-x)^2,x)\\ &\quad\left.+D_n^{1/n}(r^2(t,x)(t-x)^2,x)\right]\\ &= xf'(x)+\frac{x(x+3)}{2}f''(x).\end{aligned}$$

■

By $C_B([0,\infty))$, we denote the class of real valued, continuous and bounded functions $f(x)$ for $x\in[0,\infty)$ with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. For $f\in C_B([0,\infty))$ and $\delta>0$ the m^{th} order modulus of continuity is defined as

$$\omega_m(f,\delta)=\sup_{0\le h\le\delta}\sup_{x\in[0,\infty)}|\Delta_h^m f(x)|,$$

where Δ_h is the forward difference. In case $m=1$, we mean the usual modulus of continuity denoted by $\omega(f,\delta)$ and defined as

$$\omega(f,\delta)=\sup_{0<|x-y|<\delta}|f(x)-f(y)|.$$

The Peetre's K-functional is defined as

$$K_2(f,\delta)=\inf_{g\in C_B^2([0,\infty))}\{\|f-g\|+\delta\|g''\|\},$$

where $C_B^2([0,\infty))=\{g\in C_B([0,\infty)):g',g''\in C_B([0,\infty))\}$.

Theorem 3.2. *Let $f\in C_B[0,\infty)$. Then*

$$|D_n^{1/n}(f,x)-f(x)|\le C\omega_2\left(f,\sqrt{\delta_n}\right)+\omega\left(f,\frac{x}{n-1}\right),$$

where $\delta_n=\dfrac{n\left(3n^2-2n+1\right)x}{(n-1)^4}+\dfrac{\left(n^2+2n-2\right)x^2}{(n-1)^3}$, $n\in\{2,3,\ldots\}$.

Proof. We introduce the auxiliary operators $\tilde{D}_n^{1/n} : C_B([0,\infty)) \to C_B([0,\infty))$ as follows

$$\tilde{D}_n^{1/n}(f,x) = D_n^{1/n}(f,x) - f\left(\frac{nx}{n-1}\right) + f(x). \tag{3.4}$$

These operators are linear and preserve the linear functions in view of Lemma 2.2. Let $g \in C_B^2([0,\infty))$ and $x,t \in [0,\infty)$. By Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we have

$$\begin{aligned}
|\tilde{D}_n^{1/n}(g,x) - g(x)| &\leq \tilde{D}_n^{1/n}\left(\left|\int_x^t (t-u)g''(u)du\right|, x\right) \\
&\leq D_n^{1/n}\left(\left|\int_x^t (t-u)g''(u)du\right|, x\right) \\
&\quad + \left|\int_x^{\frac{nx}{n-1}} \left(\frac{nx}{n-1} - u\right) g''(u)du\right| \\
&\leq D_n^{1/n}\left((t-x)^2, x\right) \|g''\| + \left|\int_x^{\frac{nx}{n-1}} \left(\frac{x}{n-1}\right) du\right| \|g''\|.
\end{aligned}$$

Next, using central moments of operators, we have

$$\begin{aligned}
|\tilde{D}_n^{1/n}(g,x) - g(x)| &\leq \left[D_n^{1/n}\left((t-x)^2, x\right) + \left(\frac{x}{n-1}\right)^2\right] \|g''\| \\
&\leq \left[\frac{n\left(3n^2-2n+1\right)x}{(n-1)^4} + \frac{\left(n^2+n-1\right)x^2}{(n-1)^3} + \left(\frac{x}{n-1}\right)^2\right] \|g''\| \\
&\leq \left[\frac{n\left(3n^2-2n+1\right)x}{(n-1)^4} + \frac{\left(n^2+2n-2\right)x^2}{(n-1)^3}\right] \|g''\| \\
&= \delta_n \|g''\|.
\end{aligned} \tag{3.5}$$

Since

$$|D_n^{1/n}(f,x)| \leq \int_0^\infty H_n(t,x)|f(t)|dt \leq \|f\|.$$

Now, for the operators $\tilde{D}_n^{1/n}$, we have

$$\|\tilde{D}_n^{1/n}(\cdot,x)\| \leq \|D_n^{1/n}(\cdot,x)\| + 2\|f\| \leq 3\|f\|, \; f \in C_B\left([0,\infty)\right). \tag{3.6}$$

Using (3.5) and (3.6), we have

$$\begin{aligned}
|D_n^{1/n}(f,x) - f(x)| &\leq |\tilde{D}_n^{1/n}(f-g,x) - (f-g)(x)| + |\tilde{D}_n^{1/n}(g,x) - g(x)| \\
&\quad + \left|f\left(\frac{nx}{n-1}\right) - f(x)\right| \\
&\leq 4\|f-g\| + \delta_n\|g''\| + \left|f(x) - f\left(\frac{nx}{n-1}\right)\right|
\end{aligned}$$

$$\leq 5C \left\{ \|f-g\| + \delta_n \|g''\| \right\} + \omega \left(f, \frac{x}{n-1} \right).$$

Taking infimum over all $g \in C_B^2([0,\infty)]$, and using the inequality $K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta})$, $\delta > 0$ due to [14], we get the desired assertion. ■

4. A-Statistical Approximation

In this section, we present the A-statistical approximation properties of generalized Lupaş-Jain-beta operators on the weighted spaces.
Let us recall the concept of A-statistical convergence. Let $A = (a_{jn})$ be a summability matrix and let $x = (x_n)$ be a sequence. We say that $Ax := \left\{ (Ax)_j \right\}$ is the A-transformation of x, if the series

$$(Ax)_j := \sum_n a_{jn} x_n$$

is convergent for each j. Further, we say that x is A-summable to L if the sequence Ax converges to a number L. A summability matrix A is said to be regular if $\lim_j (Ax)_j = L$ whenever $\lim_n x_n = L$ [10]. Let $A = (a_{jn})$ be a non-negative regular summability matrix and let K be a subset of positive integer. Then K is said to have A-density $\delta_A(K)$ if the limit

$$\delta_A(K) := \lim_j \sum_n a_{jn}$$

exists [11, 12, 22]. The sequence $x = (x_n)$ is said to be A-statistically convergent to real number α if for any $\varepsilon > 0$

$$\lim_j \sum_{n: |x_n - \alpha| \geq \varepsilon} a_{jn} = 0.$$

In this case, we write $st_A - \lim x = \alpha$ [20, 21]. If A is the identity matrix I, then I-statistical convergence reduces to ordinary convergence, and, if $A = C_1$, the Cesàro matrix of order one, then it coincides with statistical convergence. Many authors have studied Korovkin type approximation properties of the statistical convergence for several operators by following work of Gadjiev and Orhan [23]. (see, for instance [1, 15, 29, 33, 34, 38, 39])

Now, we consider the following class of positive linear operators which includes the operators given by (1.2);

$$D_n^\beta(f,x) = \sum_{k=1}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} \times \int_0^\infty \frac{f(t)}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + 2^{-nx} f(0), \tag{4.1}$$

where $n > 1$, $0 \leq \beta < 1$, $f \in C_\rho^k([0,\infty))$. If we take $\beta = \frac{1}{n}$, we obtain the operators $D_n^{1/n}$. From Lemma 1, we can easily obtain that

$$D_n^\beta(1;x) = 1, \tag{4.2}$$

$$D_n^\beta(t;x) = \frac{x}{1-\beta}, \tag{4.3}$$

$$D_n^{\beta}\left(t^2;x\right)=\frac{x^2}{(1-\beta)^3}+\frac{2x}{n(1-\beta)^4}+\frac{\beta x}{(1-\beta)^2}. \tag{4.4}$$

In order to get an approximation result, we consider β as a sequence of positive real numbers such that $0\leq\beta<1$, $\beta=\beta_n$ and $st_A-\lim_{n\to\infty}\beta_n=0$.

Here, we recall the weighted Korovkin type approximation theorem for the A-statistical convergence given by Duman and Orhan in [16]

Theorem 4.1. *[16] Let A be a non-negative regular summability matrix and let $\bar{\rho}_1$,$\bar{\rho}_2$ be weight functions such that $\overline{\rho_1}$*

$$\lim_{|x|\to\infty}\frac{\overline{\rho_1}(x)}{\overline{\rho_2}(x)}=0. \tag{4.5}$$

Assume that $(T_n)_{n\geq1}$ is a sequence of positive linear operators from $C_{\overline{\rho_1}}(\mathbb{R})$ into $B_{\overline{\rho_2}}(\mathbb{R})$. One has

$$st_A-\lim_n\|T_nf-f\|_{\overline{\rho_2}}=0,$$

for all $f\in C_{\overline{\rho_1}}(\mathbb{R})$ if and only if

$$st_A-\lim_n\|T_nF_v-F_v\|_{\overline{\rho_1}}=0,\qquad v=0,1,2,$$

where

$$F_v(x)=\frac{x^v\overline{\rho_1}(x)}{1+x^2},\qquad v=0,1,2.$$

By using this theorem, we present the following result for (D_n^{β}) :

Theorem 4.2. *Let $A=(a_{jn})$ be a non-negative regular summability matrix and let (β_n) be sequence of positive numbers such that, $0\leq\beta_n<1$, $st_A-\lim_{n\to\infty}\beta_n=0$. Then for each $f\in C_\rho^k[0,\infty)$, we have*

$$st_A-\lim_{n\to\infty}\left\|D_n^{\beta_n}f-f\right\|_{\rho_\lambda}=0,$$

where $\rho_\lambda(x)=1+x^{2+\lambda}$, $\lambda\geq0$.

Proof. Using Theorem 4.1, it is sufficient to prove that the operators $\left(D_n^{\beta_n}\right)$ verify the conditions given in (4.5). Indeed, from (4.2), it is clear that

$$st_A-\lim_n\left\|D_n^{\beta_n}(F_0)-F_0\right\|_\rho=0.$$

From (4.3), we get

$$\left\|D_n^{\beta_n}(F_1)-F_1\right\|_\rho\leq\sup_{x\geq0}\left\{\left|\frac{x}{1+x^2}\left(\frac{\beta_n}{1-\beta_n}\right)\right|\right\}\leq\frac{\beta_n}{1-\beta_n}.$$

Since $st_A-\lim_{n\to\infty}\beta_n=0$, we have

$$st_A-\lim_n\left\|D_n^{\beta_n}(F_1)-F_1\right\|_\rho=0.$$

Now, using (4.4), one can have

$$\left\|D_n^{\beta_n}(F_2)-F_2\right\|_{\rho_0}\leq\sup_{x\geq0}\left\{\left|\frac{x^2}{1+x^2}\frac{(\beta_n^3-3\beta_n^2+3\beta_n)}{(1-\beta_n)^3}\right|+\left|\frac{2x}{1+x^2}\frac{1}{n(1-\beta_n)^4}\right|\right.$$

$$+\frac{x}{1+x^2}\frac{\beta_n}{(1-\beta_n)^2}\Bigg\},$$

which implies

$$\begin{aligned}\left\|D_n^{\beta_n}(F_2)-F_2\right\|_{\rho_0} &\le \frac{\beta_n^3-3\beta_n^2+3\beta_n}{(1-\beta_n)^3}+\frac{2}{n(1-\beta_n)^4}+\frac{\beta_n}{(1-\beta_n)^2} \\ &= K_n.\end{aligned}$$

Now, for a given $\epsilon > 0$, let us define the following sets:

$$M := \left\{n : \left\|D_n^{\beta_n}(F_2)-F_2\right\|_\rho \ge \epsilon\right\},$$

$$M_1 := \left\{n : \frac{\beta_n^3-3\beta_n^2+3\beta_n}{(1-\beta_n)^3} \ge \frac{\epsilon}{3}\right\},$$

$$M_2 := \left\{n : \frac{2}{n(1-\beta_n)^4} \ge \frac{\epsilon}{3}\right\},$$

$$M_3 := \left\{n : \frac{\beta_n}{(1-\beta_n)^2} \ge \frac{\epsilon}{3}\right\}.$$

Then, we see that $M \subseteq M_1 \cup M_2 \cup M_3$. Therefore, we get

$$\sum_{n:\left\|D_n^{\beta}(F_2)-F_2\right\|_\rho \ge\epsilon} a_{j,n} \le \sum_{n\in M_1} a_{j,n} + \sum_{n\in M_2} a_{j,n} + \sum_{n\in M_3} a_{j,n} \tag{4.6}$$

and, taking the limit $j \to \infty$ in (4.6), we have

$$st_A - \lim_n \left\|D_n^{\beta}(F_2)-F_2\right\|_\rho = 0.$$

This proves the theorem. ∎

Remark 4.3. Theorem 4.2 may be useful when the Theorem 2.4 doesn't work. Indeed, if β_n does not converge to zero as $n \to \infty$, then, we can consider Theorem 4.2.

The following example shows that there exists a sequence (β_n) such that A-statistical convergence holds but ordinary convergence does not hold for (β_n).

Example 4.4. Let (β_n) be the sequence defined by

$$\beta_n = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a perfect square} \\ \frac{1}{n}, & \text{otherwise.} \end{cases}$$

It is easy to see that (β_n) is not convergent but statistically convergent, i.e., C_1-statistically convergent.

5. King Type Operators

In recent years, the modification of King type which preserve the test functions has been a significant subject of research. The early one is due to King [28] which consider the modified Bernstein operators preserving the constant and the test function x^2 on the interval $[0,1]$. Later some researches increased in this field and some classical operators and several new sequences of operators which were constructed before were studied in this direction.

In 2007, Duman and Özarslan [17], introduced Szász-Mirakjan type operators preserving the test function constant and x^2 on the interval $[0,\infty)$ and showed the better error estimation for the modified operators. One may see some of the results in [2, 3, 5, 6, 13, 17, 25, 37] etc.

In this section, as in [18] for Szász-Mirakjan Kantorovich operators, we consider a similar modification of the Lupaş-Jain-beta operators which preserve the test functions e_0 and e_1. We introduce the operators for $x \in [0,\infty)$, $\left\{\widetilde{D}_n^{1/n} : C[0,\infty) \to C[0,\infty)\right\}_{n>1}$

$$\begin{aligned} \widetilde{D}_n^{1/n}(f;x) &= \sum_{k=1}^{\infty} \frac{n\lambda_n(x)\left(n\lambda_n(x)+1+k\frac{1}{n}\right)_{k-1}}{2^k k!} 2^{-\left(n\lambda_n(x)+k\frac{1}{n}\right)} \\ &\quad \times \int_0^{\infty} \frac{f(t)}{B(n+1,k)} \frac{t^{k-1}}{(1+t)^{n+k+1}} dt + 2^{-n\lambda_n(x)} f(0). \end{aligned} \tag{5.1}$$

Let's find the function λ_n, satisfying the condition

$$\widetilde{D}_n^{1/n}(t;x) = x$$

for all $x \in [0,\infty)$ and $n > 1$. From the definition of King-type Lupaş-Jain-beta operators, we can write

$$\lambda_n(x) = \left(1 - \frac{1}{n}\right) x, \qquad n > 1, \quad x \in [0,\infty),$$

where $\{\lambda_n : [0,\infty) \to \mathbb{R}\}_{n>1}$ is the sequence of functions.

Noting the fact that when $n \to \infty$, $\lambda_n \to x$, $\widetilde{D}_n^{1/n}$ reduces to the classical Lupaş-Jain-beta operators. That is, classical Lupaş-Jain-beta operators turn out to be a limit element of $\widetilde{D}_n^{1/n}$.

Lemma 5.1. *For each $x \in [0,\infty)$ and $n > 1$, we have*

$$\begin{aligned} \widetilde{D}_n^{1/n}(1;x) &= 1, \\ \widetilde{D}_n^{1/n}(t;x) &= x, \\ \widetilde{D}_n^{1/n}(t^2;x) &= \frac{n}{(n-1)} x^2 + \frac{3n^2-2n+1}{(n-1)^3} x. \end{aligned} \tag{5.2}$$

Proof. Using the results for the Lupaş-Jain-beta operators (5.1), it is found that

$$\begin{aligned} \widetilde{D}_n^{1/n}(1;x) &= 1, \\ \widetilde{D}_n^{1/n}(t;x) &= x, \\ \widetilde{D}_n^{1/n}(t^2;x) &= \frac{n^3}{(n-1)^3}(\lambda_n)^2 + \frac{n(3n^2-2n+1)}{(n-1)^4}(\lambda_n). \end{aligned}$$

∎

In view of the definition of λ_n, we can obtain the moments of the operators.

Lemma 5.2. *For every $x \geq 0$, we have*

$$\widetilde{D}_n^{1/n}(t-x;x) = 0,$$
$$\widetilde{D}_n^{1/n}((t-x)^2;x) = \frac{1}{n-1}x^2 + \frac{3n^2-2n+1}{(n-1)^3}x.$$

6. Better Error Estimation

The aim of the constructing new type operators concerning King type modification is to deal with the best approximation. In this section, we concern with the rate of convergence of the operators $\widetilde{D}_n^{1/n}$ defined by (5.1). Then we will demonstrate that the operators (5.1) have a better error estimation on the interval $[0,\infty)$ than the classical operators given by (1.2).

We have the following estimates for $\widetilde{D}_n^{1/n}$ and $D_n^{1/n}$ in terms of the modulus of continuity $\omega(f,\delta)$.

Theorem 6.1. *For every $f \in C_B[0,\infty)$ and $x \in [0,\infty)$, we have*

$$\left|\widetilde{D}_n^{1/n}(f;x) - f(x)\right| \leq 2\omega(f,\rho_{n,x}),$$

where $\rho_{n,x}^2 = \frac{1}{n-1}x^2 + \frac{3n^2-2n+1}{(n-1)^3}x$.

Remark 6.2. For the classical Lupaş-Jain-beta operators satisfy

$$\left|D_n^{1/n}(f;x) - f(x)\right| \leq 2\omega(f,\eta_{n,x}),$$

where $\eta_{n,x}^2 = \frac{n^2+n-1}{(n-1)^3}x^2 + \frac{n\left(3n^2-2n+1\right)}{(n-1)^4}x$.

In the following theorem, we present analogues theorem for $\widetilde{D}_n^{1/n}$ to show a better order of approximation.

Theorem 6.3. *For every $f \in C_B[0,\infty)$, $x \in [0,\infty)$ and $n > 1$, we have*

$$\rho_{n,x} \leq \eta_{n,x}$$

and one can get the best approximation using $\widetilde{D}_n^{1/n}$.

Proof. The order of approximation to a function $f \in C_B[0,\infty)$, given by the sequence $\widetilde{D}_n^{1/n}$ will be at least as good as of $D_n^{1/n}$ whenever

$$\frac{1}{n-1}x^2 + \frac{3n^2-2n+1}{(n-1)^3}x \leq \frac{n^2+n-1}{(n-1)^3}x^2 + \frac{n\left(3n^2-2n+1\right)}{(n-1)^4}x.$$

Let

$$K_n(x) = \frac{3n-2}{(n-1)^3}x^2 + \frac{3n^2-2n+1}{(n-1)^4}x.$$

For all $x \in [0,\infty)$ and $n > 1$, we can always write $\rho_{n,x} \leq \eta_{n,x}$ and this guarantees that error estimations of King type operators are better than the classical operators. ∎

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