Thai Journal of Mathematics
Special Issue (Annual Meeting in Mathematics, 2008) : 59-68

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Symmetry of scalar second-order stochastic ordinary differential equations

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Abstract : In this manuscript, the definition of an admitted Lie group for stochastic differential equations given in [1] is applied to second-order ordinary stochastic differential equations. Admitted Lie group generators for a variety of equations are obtained.

Keywords : Stochastic process; Determining equations; Lie group of transformations

1 Introduction

One of the standard methods for finding exact solutions of differential equations is group analysis. A survey of this method can be found in [2], [3] and [4]. It involves the study of symmetries of equations. Symmetry means that any solution of a given system of equations is transformed by some local group of transformations to a solution of the same system. Moreover, symmetries allow finding new solutions of the system.

In contrast to deterministic differential equations, there have been only few attempts to apply symmetry techniques to stochastic differential equations. They fall into two groups as we review now.

The system of Itô equations

$$dX_i(t,\omega) = f_i(t, X(t,\omega))dt + g_{ik}(t, X(t,\omega))dB_k(t,\omega) \quad (i = 1, ..., n, \quad k = 1, ..., r) \tag{1.1}$$

with initial condition $X(0) = X^{(0)}$ is interpreted in the sense that

$$X_i(t,\omega) = X_i^{(0)}(\omega) + \int_0^t f_i(s, X(s,\omega))ds + \int_0^t g_{ik}(s, X(s,\omega))dB_k(s,\omega), \tag{1.2}$$

for almost all $\omega \in \Omega$ and each $t > 0$, where $f_i(t,X)$ is a drift vector, $g_{ik}(t,X)$ is a diffusion matrix and $B_k(k = 1, ..., r)$ are one-dimensional Brownian motions. The first integral in this equation is of Riemann type, while the second integral denotes a sum of Itô integrals with repeat index k.

The first approach [5, 6, 7, 8] considers fiber-preserving transformations

$$\overline{x}_i = \varphi(t, x, a), \quad \bar{t} = H(t, a) \quad (i = 1, ..., n), \tag{1.3}$$

and has been applied to stochastic dynamical systems [5, 6] and to the Fokker-Planck equation [7, 8]. Its weakness is that it can only be applied to fiber-preserving transformations which form a small subclass of all possibly transformations.

The second approach [10, 11, 12, 13, 14] deals with symmetry transformations including all the dependent variables in the transformation. This approach has been applied to scalar second-order stochastic ordinary differential equations [10, 11], to the Hamiltonian-Stratonovich dynamical control system [12] and to the Fokker-Planck equation [12, 13, 14]. There have also been attempts to involve Brownian motion in the transformation, without strict proof that Brownian motion is transformed to Brownian motion.

In [1] a new definition of an admitted Lie group of transformations for stochastic differential equations is given, including dependent as well as independent variables in the transformation. In particular, the transformation of Brownian motion is defined by transformation of the dependent and independent variables, and there is a strict proof that the transformed Brownian motion satisfies the properties of Brownian motion. This theory was applied to first-order stochastic ordinary differential equations in [1].

The present manuscript discusses how these idaes can be applied to scalar second-order stochastic ordinary differential equations.

2 Preliminaries

2.1 Lie group of transformations for stochastic differential equations

This section is devoted to reviewing the theory developed in [1]. We discuss transformations of stochastic processes and admitted Lie groups.

Assume that the set of transformations

$$\bar{t} = H(t, x, a), \quad \bar{x} = \varphi(t, x, a) \tag{2.1}$$

composes a Lie group. Let $h(t, x) = \frac{\partial H}{\partial a}(t, x, 0)$, $\xi(t, x) = \frac{\partial \varphi}{\partial a}(t, x, 0)$ be the coefficients of the infinitesimal generator

$$h(t, x)\partial_t + \xi(t, x)\partial_x.$$

According to Lie's theorem, the functions $H(t,x,a)$ and $\varphi(t,x,a)$ satisfy the Lie equations

$$\frac{\partial H}{\partial a} = h(H,\varphi), \quad \frac{\partial \varphi}{\partial a} = \xi(h,\varphi) \tag{2.2}$$

and the initial conditions for $a = 0$:

$$H = t, \quad \varphi = x. \tag{2.3}$$

Since $\frac{\partial H}{\partial t}(t,x,0) = 1$, then $\frac{\partial H}{\partial t}(t,x,a) > 0$ in a neighborhood of $a = 0$, where one can find a function $\eta(t,x,a)$ such that

$$\eta^2(t,x,a) = \frac{\partial H}{\partial t}(t,x,a).$$

Using the function $\eta(t,x,a)$, one can define a transformation of a stochastic process $X(t,\omega)$ by

$$\bar{X}(\bar{t},\omega) = \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a), \tag{2.4}$$

where

$$\beta(t) = \int_0^t \eta^2(s, X(s,\omega), a)ds, \quad t \geq 0,$$

and $\alpha(t)$ is the inverse function of $\beta(t)$. This gives an action of Lie group (2.1) on the set of stochastic processes. Replacing $\bar{t}$ by $\beta(t)$ in (2.4), one obtains

$$\bar{X}(\beta(t),\omega) = \varphi(t, X(t,\omega), a).$$

It is useful to introduce the function

$$\tau(t,x) = \frac{\partial \eta}{\partial a}(t,x,0).$$

Notice that the functions $h(t,x)$ and $\tau(t,x)$ are related by the formulae

$$\tau(t,x) = \frac{\frac{\partial h}{\partial t}(t,x)}{2}, \quad h(t,x) = 2\int_0^t \tau(s,x)ds.$$

We are now ready to present the notions of admitted Lie group and determining equations.

Definition 1. (see [1]) A Lie group of transformations (2.1) is called admitted by the stochastic differential equation (1.2), if for any solution $X(t,\omega)$ of (1.2) the functions $\xi(t,x)$ and $\tau(t,x)$ satisfy the following determining equations:

$$\begin{aligned}
&\xi_{i,t}(t,X(t,\omega)) + f_j\xi_{i,j}(t,X(t,\omega)) + \tfrac{1}{2}g_{jk}g_{lk}\xi_{i,jl}(t,X(t,\omega)) \\
&\qquad -2f_{i,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds \\
&\qquad\qquad -f_{i,j}\xi_j(t,X(t,\omega)) - 2f_i\tau(t,X(t,\omega)) = 0, \\
&g_{jk}\xi_{i,j}(t,X(t,\omega)) - 2g_{ik,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds \\
&\qquad\qquad -g_{ik}\tau(t,X(t,\omega)) - g_{ik,j}\xi_j(t,X(t,\omega)) = 0, \\
&(i = 1,...,n; k = 1,...,r),
\end{aligned} \tag{2.5}$$

where $(\bullet)_{,\bullet}$ stands for the partial derivative with respect to the coordinate appearing in the subscript, for example, $\xi_{i,t} = \frac{\partial \xi_i}{\partial t}$.

Note that the determining equations (2.5) were constructed under the assumption that the Lie group of transformations (2.1) transforms any solution of equation (1.2) to a solution of the same equation.

2.2 Determining equations for scalar second-order stochastic differential equations

This section is devoted to constructing determining equations of an admitted Lie group of transformations for a scalar second-order stochastic differential equation. Let us consider the Itô equation

$$\ddot{X}(t) = f(t, X(t), \dot{X}(t)) + g(t, X(t), \dot{X}(t))\frac{dB(t)}{dt}, \tag{2.6}$$

where f and g are given functions and B is a Brownian motion. Equation (2.6) describes, for example, the motion of a particle in a noise-perturbed force field and can be interpreted as the system of first-order stochastic ordinary differential equations

$$\begin{aligned} X(t,\omega) &= X(0,\omega) + \int_0^t Y(s,\omega)ds, \\ Y(t,\omega) &= Y(0,\omega) + \int_0^t f(s, X(s,\omega), Y(s,\omega))ds + \int_0^t g(s, X(s,\omega), Y(s,\omega))dB(s). \end{aligned} \tag{2.7}$$

According to definition 1 in the previous section, a Lie group of transformations (2.1) is called admitted by the stochastic differential equation (2.7), if for any solution $\Big(X(t,\omega), Y(t,\omega)\Big)$ of (2.7) the functions $\xi(t,x)$ and $\tau(t,x)$ satisfy the system of determining equations

$$\begin{aligned}
&\xi_{1,t}\Big(t, X(t,\omega), Y(t,\omega)\Big) + y\xi_{1,x}\Big(t, X(t,\omega), Y(t,\omega)\Big) + f\xi_{1,y}\Big(t, X(t,\omega), Y(t,\omega)\Big) \\
&\qquad + \tfrac{1}{2}g^2\xi_{1,yy}\Big(t, X(t,\omega), Y(t,\omega)\Big) - \xi_2\Big(t, X(t,\omega), Y(t,\omega)\Big) \\
&\qquad\qquad - 2y\tau\Big(t, X(t,\omega), Y(t,\omega)\Big) = 0, \\
&\xi_{2,t}\Big(t, X(t,\omega), Y(t,\omega)\Big) + y\xi_{2,x}\Big(t, X(t,\omega), Y(t,\omega)\Big) + f\xi_{2,y}\Big(t, X(t,\omega), Y(t,\omega)\Big) \\
&+ \tfrac{1}{2}g^2\xi_{2,yy}\Big(t, X(t,\omega), Y(t,\omega)\Big) - 2f_t\Big(t, X(t,\omega), Y(t,\omega)\Big)\int_0^t \tau\Big(s, X(s,\omega), Y(s,\omega)\Big)ds \\
&\qquad - f_j\xi_j\Big(t, X(t,\omega), Y(t,\omega)\Big) - 2f\tau\Big(t, X(t,\omega), Y(t,\omega)\Big) = 0, \\
&\xi_{1,y}\Big(t, X(t,\omega), Y(t,\omega)\Big) = 0, \\
&g\xi_{2,y}\Big(t, X(t,\omega), Y(t,\omega)\Big) - 2g_t\Big(t, X(t,\omega), Y(t,\omega)\Big)\int_0^t \tau\Big(s, X(s,\omega), Y(s,\omega)\Big)ds \\
&\qquad - g\tau\Big(t, X(t,\omega), Y(t,\omega)\Big) - g_j\xi_j\Big(t, X(t,\omega), Y(t,\omega)\Big) = 0.
\end{aligned} \tag{2.8}$$

3 Main Results

In the following, we present and solve the determining equations for a variety of scalar second-order stochastic differential equations.

3.1 Narrow-sense linear equation

Let μ, ν and σ be constants, and $\sigma \neq 0$. Consider the equation [15],

$$\ddot{X} = -\nu^2 X - \mu \dot{X} + \sigma \frac{dB}{dt},$$

called the narrow-sense linear equation. The functions f and g in equations (2.8) are $f = -\nu^2 x - \mu y$ and $g = \sigma$. The system of determining equations thus becomes

$$\begin{gathered}
\xi_{1,t} + y\xi_{1,x} - (\nu^2 x + \mu y)\xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2y\tau - \xi_2 = 0, \\
\xi_{2,t} + y\xi_{2,x} - (\nu^2 x + \mu y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} + 2(\nu^2 x + \mu y)\tau + \nu^2 \xi_1 + \mu \xi_2 = 0, \\
\xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0.
\end{gathered} \tag{3.1}$$

If $\mu^2 - 4\nu^2 \geq 0$, then the general solution of this system is

$$\xi_1 = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}, \quad \xi_2 = C_1 \gamma_1 e^{\gamma_1 t} + C_2 \gamma_2 e^{\gamma_2 t}, \quad \tau = 0, \tag{3.2}$$

where $\gamma_1 = -\frac{1}{2}(\mu + \sqrt{\mu^2 - 4\nu^2})$ and $\gamma_2 = -\frac{1}{2}(\mu - \sqrt{\mu^2 - 4\nu^2})$. For this solution, $h = 0$, and a basis of admitted generators corresponding to (3.2) is

$$e^{\gamma_1 t}\partial_x, \quad e^{\gamma_2 t}\partial_x.$$

If $\mu^2 - 4\nu^2 < 0$, then the general solution of determining equations (3.1) is

$$\begin{gathered}
\xi_1 = C_1 e^{-\frac{\mu}{2}t}\cos(\gamma_3 t) + C_2 e^{-\frac{\mu}{2}t}\sin(\gamma_3 t), \\
\xi_2 = C_1 \gamma_1 e^{-\frac{\mu}{2}t}\cos(\gamma_3 t) + C_2 \gamma_2 e^{-\frac{\mu}{2}t}\sin(\gamma_3 t), \quad \tau = 0,
\end{gathered} \tag{3.3}$$

where $\gamma_1 = -\frac{1}{2}(\mu + \sqrt{\mu^2 - 4\nu^2})$, $\gamma_2 = -\frac{1}{2}(\mu - \sqrt{\mu^2 - 4\nu^2})$ and $\gamma_3 = \sqrt{4\nu^2 - \mu^2}$. Here again, $h = 0$. Thus, a basis of admitted generators corresponding to (3.3) is

$$e^{-\frac{\mu}{2}t}\cos(\gamma_3 t)\partial_x, \quad e^{-\frac{\mu}{2}t}\sin(\gamma_3 t)\partial_x.$$

3.2 Ornstein-Uhlenbeck process

A better model of Brownian movement is provided by the Ornstein-Uhlenbeck equation [16]

$$\ddot{X} = -b\dot{X} + \sigma \frac{dB}{dt}, \tag{3.4}$$

where $b > 0$ is the friction coefficient, and $\sigma \neq 0$ is the diffusion coefficient. The functions f and g in equations (2.8) are $f = -by$ and $g = \sigma$. Thus, the system of determining equations becomes

$$\begin{gathered}
\xi_{1,t} + y\xi_{1,x} - by\xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2y\tau - \xi_2 = 0, \\
\xi_{2,t} + y\xi_{2,x} - by\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} + 2y\tau + b\xi_2 = 0, \\
\xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0.
\end{gathered}$$

The general solution of this system is

$$\xi_1 = C_1 + C_2 e^{-bt}, \quad \xi_2 = -C_2 b e^{-bt}, \quad \tau = 0. \tag{3.5}$$

A basis of admitted generators corresponding to (3.5) is

$$\partial_x, \quad e^{-bt}\partial_x.$$

3.3 Mass-Spring linear oscillator

The response of a mass-spring linear oscillator to white-noise is described by the equation [11]

$$\ddot{X} = -b^2 X + \sigma \frac{dB}{dt},$$

where $b^2 = \frac{k}{m}$, m is the mass, k is the characteristic coefficient of the spring and $\sigma \neq 0$ is constant. The functions appearing in equations (2.8) are $f = -b^2 x$ and $g = \sigma$, and the system of determining equations becomes

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} - b^2 x \xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2y\tau - \xi_2 = 0, \\ \xi_{2,t} + y\xi_{2,x} - b^2 x \xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} + 2b^2 x\tau + b^2 \xi_1 = 0, \\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0. \end{gathered} \tag{3.6}$$

If $b = 0$, the general solution of determining equations (3.6) is

$$\xi_1 = 3C_1 x + C_2 t + C_3, \quad \xi_2 = C_1 y + C_2, \quad \tau = C_1. \tag{3.7}$$

For this solution $h(t,x) = 2\int_0^t \tau(s,x)ds = 2C_1 t$. Thus, a basis of admitted generators corresponding to (3.7) is

$$3x\partial_x + 2t\partial_t, \quad t\partial_x, \quad \partial_x.$$

If $b \neq 0$, the general solution of determining equations (3.6) is

$$\xi_1 = C_1 \sin(bt) + C_2 \cos(bt), \quad \xi_2 = C_1 b \cos(bt) - C_2 b \sin(bt), \quad \tau = 0. \tag{3.8}$$

A basis of admitted generators corresponding to (3.8) is

$$\sin(bt)\partial_x, \quad \cos(bt)\partial_x.$$

3.4 Nonlinear equation

Consider the equation [11]

$$\ddot{X} = a\exp(-\dot{X}) + b\exp(-\frac{\dot{X}}{2})\frac{dB}{dt}, \tag{3.9}$$

where a, $b \neq 0$ are constant.The functions appearing in equations (2.8) are $f = ae^{-y}$ and $g = be^{-\frac{y}{2}}$. The system of determining equations becomes

$$\begin{gathered}\xi_{1,t} + y\xi_{1,x} + ae^{-y}\xi_{1,y} + \tfrac{1}{2}b^2e^{-y}\xi_{1,yy} - 2y\tau - \xi_2 = 0,\\ \xi_{2,t} + y\xi_{2,x} + ae^{-y}\xi_{2,y} + \tfrac{1}{2}b^2e^{-y}\xi_{2,yy} - 2ae^{-y}\tau - ae^{-y}\xi_1 = 0,\\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau + \tfrac{1}{2}\xi_2 = 0.\end{gathered}$$

Its general solution is

$$\xi_1 = C_1 + C_2(x+t), \quad \xi_2 = C_2, \quad \tau = \frac{1}{2}C_2. \tag{3.10}$$

For this solution, $h(t,x) = 2\int_0^t \tau(s,x)ds = C_1t$. A basis of admitted generators corresponding to (3.10) is

$$\partial_x, \quad (x+t)\partial_x + t\partial_t.$$

In the above examples, all transformations were fiber-preserving. In this case, the proof that a Lie group of transformations transforms every solution of the equation into a solution of the same equation is easy.

3.5 Non fiber-preserving transformations

Consider the equation [11]

$$\ddot{X} = \sigma X\frac{dB}{dt}, \tag{3.11}$$

where $\sigma \neq 0$ is constant. Here the functions appearing in equations (2.8) are $f = 0$ and $g = \sigma x$. The system of determining equations becomes

$$\begin{gathered}\xi_{1,t} + y\xi_{1,x} + \tfrac{1}{2}\sigma^2x^2\xi_{1,yy} - 2y\tau - \xi_2 = 0,\\ \xi_{2,t} + y\xi_{2,x} + \tfrac{1}{2}\sigma^2x^2\xi_{2,yy} = 0,\\ \xi_{1,y} = 0, \quad x\xi_{2,y} - x\tau - \xi_1 = 0.\end{gathered}$$

and its general soilution is

$$\xi_1 = C_1x + C_2x^{-2}, \quad \xi_2 = C_1y, \quad \tau = -C_2x^{-3}. \tag{3.12}$$

For this solution, $h(t,x) = 2\int_0^t \tau(s,x)ds = -2C_2x^{-3}t$, and a basis of generators corresponding to (3.12) is

$$x\partial_x, \quad x^{-2}\partial_x - 2x^{-3}t\partial_t.$$

For finding the Lie group of transformations corresponding to the second generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = -2\varphi_1^{-3}H, \quad \frac{\partial \varphi_1}{\partial a} = \varphi_1^{-2},$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variable x,

$$\bar{t} = H = t(1 + 3ax^{-3})^{-\frac{2}{3}}, \quad \bar{x} = \varphi_1 = (x^3 + 3a)^{\frac{1}{3}}. \tag{3.13}$$

Hence $\eta^2 = (1 + 3ax^{-3})^{-\frac{2}{3}}$.

Let us show that the Lie group of transformations (3.13) transforms every solution of equation (3.11) into a solution of the same equation. Assume that $X(t)$ is a solution of equation (3.11). In [1], it was proven that the Brownian motion $B(t)$ is transformed to the Brownian motion

$$\bar{B}(t) = \int_0^{\alpha(t)} (1 + 3aX^{-3}(s))^{-\frac{1}{3}} dB(s), \tag{3.14}$$

where

$$\beta(t) = \int_0^t (1 + 3aX^{-3}(s))^{-\frac{2}{3}} ds, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = (x^3 + 3a)^{\frac{1}{3}}$ and $\varphi_2(t, x, y, a) = y$, one has

$$\begin{aligned} \varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^t Y(s)X^2(s)(X^3(s) + 3a)^{-\frac{2}{3}} ds \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^t \sigma X(s) dB(s). \end{aligned} \tag{3.15}$$

Changing the variable $s = \alpha(\bar{s})$ in the Riemann integral in (3.15), it becomes

$$\int_0^t Y(s)X^2(s)(X^3(s) + 3a)^{-\frac{2}{3}} ds = \int_0^{\beta(t)} Y(\alpha(\bar{s})) d\bar{s}.$$

Because of the transformation of the Brownian motions (3.14), the Itô integral in (3.15) becomes

$$\int_0^t \sigma X(s) dB(s) = \int_0^{\beta(t)} \sigma X(\alpha(\bar{s}))(1 + 3aX^{-3}(\alpha(\bar{s})))^{\frac{1}{3}} d\bar{B}(\bar{s}).$$

Since $X(\alpha(\bar{t}))\big(1 + 3aX^{-3}(\alpha(\bar{t}))\big)^{\frac{1}{3}} = \bar{X}(\bar{t}, \omega)$ and $Y(\alpha(\bar{t})) = \bar{Y}(\bar{t}, \omega)$, one obtains

$$\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \varphi_1(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \bar{Y}(s,\omega) ds,$$

$$\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \varphi_2(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \sigma \bar{X}(s,\omega) d\bar{B}(s).$$

Because $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t),\omega)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} \bar{Y}(s,\omega) ds,$$

$$\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) + \int_0^{\beta(t)} \sigma \bar{X}(s,\omega) d\bar{B}(s),$$

which is equivalent to

$$\ddot{\bar{X}} = \sigma \bar{X} \frac{d\bar{B}}{dt}.$$

This confirms that the Lie group of transformations (3.13) indeed transforms every solution of equation (3.11) into a solution of the same equation.

4 Conclusion

The new definition of an admitted Lie group of transformations for stochastic differential equations given in [1] was applied to scalar second-order stochastic differential equations. This approach includes the dependent and independent variables in the transformation. The transformation of Brownian motion is defined by the transformation of dependent and independent variables. The developed theory was applied to five scalar second-order stochastic differential equations: an equation representing an Ornstein-Uhlenbeck process, an equation describing a mass-spring linear oscillator to a white-noise random, the narrow-sense linear equation, a linear Itô equation and a nonlinear Itô equation.

Acknowledgement(s) : The author is thankful to S.V. Meleshko and E. Schulz for very fruitful discussions.

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(Received 13 May 2008)

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