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The General Iterative Methods for a Split Feasibility Problem and a Mixed Equilibrium Problem in a Hilbert Space

Kiattisak Rattanaseeha

Division of Mathematics, Department of Science, Faculty of Science and Technology, Loei Rajabhat University, Loei 42000, Thailand
e-mail : kiattisakrat@live.com

Abstract In this paper, we introduce the general iterative method for finding a common element of the solution set of a split feasibility problem and the set of a mixed equilibrium problem and fixed point problem in a real Hilbert space. Under appropriate conditions imposed on the parameters, the strong convergence theorems are obtained.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2, and $A : H_1 \to H_2$ is a linear and bounded operator. The split feasibility problem (for short, SFP) is to find $x^* \in H_1$ such that

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1}$$

Throughout this paper, we denote the set of solutions of SFP (1.1) by Γ, i.e., $\Gamma = \{x \in H_1 : x^* \in C \text{ and } Ax^* \in Q\}$ and assume that Γ is nonempty. For related works, please refer to [1–4].

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H. Let $S : C \to C$ be a nonlinear mapping. The fixed point problem is to find $x \in C$ such that $Sx = x$. We denote the set of solutions of fixed point problem by $\mathrm{Fix}(S)$, i.e., $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$. The mappings S is said to be *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \forall x, y \in C. \tag{1.2}$$

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Following, let F_1, F_2 be two bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then we consider the mixed equilibrium problem (for short, MEP): finding $x \in C$ such that

$$F_1(x,y) + F_2(x,y) + \langle Ax, x-y\rangle \geq 0, \forall y \in C, \tag{1.3}$$

where A is nonlinear mapping from C into H. The set of solutions of the MEP (1.3) is denoted by $\text{MEP}(F_1, F_2, A)$. If $A = 0$, we denote $\text{MEP}(F_1, F_2, 0)$ by $\text{MEP}(F_1, F_2)$. If $A = 0$ and $F_2 = 0$, then the MEP (1.3) becomes the following equilibrium problem (for short, EP): finding $x \in C$ such that

$$F_1(x,y) \geq 0, \forall y \in C. \tag{1.4}$$

The set of solutions of the EP (1.4) is denoted by $\text{EP}(F_1)$. Let $F_1(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$. Then $z \in \text{EP}(F_1)$ if and only if $\langle Az, y-z\rangle \geq 0$ for all $y \in C$. Numerous problem in physics, optimization and economics reduce to find a solution of (1.4).

Recently, many authors considered the iterative methods for finding a common element of the set of solutions to the problem (1.4) and of the set of fixed points of nonexpansive mappings; see, for example, [5, 6] and the references therein.

Next, let $A : C \to H$ be a nonlinear mapping. We recall the following definitions:

(1) A is said to be *monotone*, if

$$\langle Ax - Ay, x-y\rangle \geq 0, \ \ \forall x, y \in C.$$

(2) A is said to be *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x-y\rangle \geq \alpha\|x-y\|^2, \ \ \forall x, y \in C.$$

In such a case, A is said to be *α-strongly monotone.*

(3) A is said to be inverse-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2, \ \ \forall x, y \in C.$$

In such a case, A is said to be *α-inverse-strongly monotone.*

Recall that the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \forall v \in C. \tag{1.5}$$

We denote the set of solutions of the problem (1.5) by $\text{VI}(C, A)$. One can easily see that the variational inequality problem is equivalent to a fixed point problem. $u \in C$ is a solution to the problem (1.5) if and only if u is a fixed point of the mapping $P_C(I - \lambda)T$, where $\lambda > 0$ is a constant. The variational inequalities have been widely studied in the literature; see, for example, the work of Kumam and Jaiboon [7]and the references therein.

Recently, Ceng, Wang and Yao [8] considered an iterative method for the system of variational inequalities(1.5). They got a strongly convergence theorem for the problem (1.5) and a fixed point problem for a single nonexpansive mapping; see [8]for more details.

On the other hand, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings (see [10] for further developments in both Hilbert and Banach spaces).

A mapping $f : C \to C$ is called α-contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x-y\|, \forall x, y \in C. \tag{1.6}$$

Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1-\sigma_n)Tx_n + \sigma_n f(x_n), \ n \geq 0, \tag{1.7}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [9, 10] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I-f)q, p-q\rangle \geq 0, p \in C.$$

Let A be a strongly positive linear bounded operator on a Hilbert space H with constant $\bar{\gamma}$; that is there exists $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \forall x \in H. \tag{1.8}$$

Recently, Marino and Xu [11] introduced the following general iterative method:

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n \gamma f(x_n), n \geq 0, \tag{1.9}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, x \in C, \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for $\gamma f (i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

In 2007, Takahashi and Takahashi [12] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.4) and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequence $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (1-\alpha_n)Su_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{1.11}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \text{Fix}(S) \cap \text{EP}(F)$ where $z = P_{\text{Fix}(S)\cap\text{EP}(F)} f(z)$.

Next, Plubtieng and Punpaeng, [13] introduced an iterative scheme by the general iterative method for finding a common element of the set of solutions (1.4) and the set of fixed points of nonexpansive mappings in Hilbert spaces.

Let $S : H \to H$ be a nonexpansive mapping. Starting with an arbitrary $x_1 \in H$, define sequence $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I-\alpha_n A)Su_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{1.12}$$

They proved that if the sequence $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ generated by (1.12) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \ \ \forall x \in \text{Fix}(S) \cap \text{EP}(F), \tag{1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x\in \text{Fix}(S)\cap \text{EP}(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for $\gamma f(i.e., h'(x) = \gamma f(x)$ for $x \in H)$.

Furthermore, Bin-Chao Deang, Tong Chen and Qiao-Li Dong [14] introduced two viscosity iteration algorithms (one implicit and one explicit) for finding a common element of the solution set $\text{MEP}(F_1, F_2)$ of a mixed equilibrium problem and the set Γ of a split feasibility problem in a real Hilbert space. They derive the strong convergence of a viscosity iteration to an element of $\text{MEP}(F_1, F_2) \cap \Gamma$ under mild assumpions.

Motivated by this result, we introduce the general iterative method for finding a common element of the solution set of a split feasibility problem and the set of a mixed equilibrium problem in a real Hilbert space.

2. Preliminaries

Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. We call $f : C \to H$ is an α-contraction if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in C.$$

Let A be a strongly positive linear bounded operator on a Hilbert space H with constant $\bar{\gamma}$; that is there exists $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \ \forall x \in H.$$

Next,we denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. A space X is said to satisfy Opials condition [15] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \ \forall y \in X, y \neq x.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| \leq \|x - y\|, \ \ \forall y \in C.$$

P_C is called the (nearest point or metric) projection of H onto C. In addition, P_Cx is characterized by the following properties: $P_Cx \in C$ and

$$\langle x - P_Cx, y - P_Cx\rangle \geq 0, \tag{2.1}$$

$$\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \ \ \forall x \in H, y \in C. \tag{2.2}$$

Recall that a mapping $T : H \to H$ is said to be firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \ \ \forall x, y \in H.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy\rangle, \ \ \forall x, y \in H. \tag{2.3}$$

If A an $\alpha-$inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha} - Lipschitz$ continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|x-y-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2-2\lambda\langle Ax-Ay, x-y\rangle+\lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2 \end{aligned} \tag{2.4}$$

Thus, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.2. ([16]) *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\beta \in [0,1]$, we have*

$$\|\beta x+(1-\beta)y\|^2 = \beta\|x\|^2+(1-\beta)\|y\|^2-\beta(1-\beta)\|x-y\|^2.$$

Lemma 2.3. ([17]) *Assume that $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:*

$(A1)$ *$F(x,x) = 0$ for all $x \in C$;*
$(A2)$ *F is monotone, i.e, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;*
$(A3)$ *for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;*
$(A4)$ *for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Lemma 2.4. ([17]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \ \forall y \in C\},$$

for all $z \in H$. Then, the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{EP}(F)$;
(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.5. ([11]) *Let H be a Hilbert space, C be a nonempty closed convex subset of H, and $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\langle x-y, (A-\gamma f)x-(A-\gamma f)y\rangle \geq (\bar{\gamma}-\gamma\alpha)\|x-y\|^2,\ \ x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.6. ([11]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.7. ([18]) *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction such that*

$(f1)$ *$F_1(x,x) = 0$ for all $x \in C$;*

(*f*2) $F_1(x,\cdot)$ *is monotone and super hemicontinuous;*
(*f*3) $F_1(\cdot,x)$ *is lower semicontinuous and convex.*
Let $F_2 : C \times C \to \mathbb{R}$ *be a bifunction such that*
(*h*1) $F_2(x,x) = 0$ *for all* $x \in C$*;*
(*h*2) $F_2(x,\cdot)$ *is monotone and upper semicontinuous;*
(*h*3) $F_2(\cdot,x)$ *is convex.*
Moreover, let us suppose that
(H) *for fixed* $r > 0$ *and* $x \in C$*, there exists a bounded set* $k \subset C$ *and* $a \in K$ *such that for all* $z \in C \setminus K$*,* $-F_1(a,z) + F_2(z,a) + \frac{1}{r_n}\langle a - z_n, z - x\rangle < 0$*, for* $r > 0$ *and* $x \in H$*. Let* $T_r : H \to C$ *be a mapping defined by*

$$T_r(x) = \{z \in C : F_1(z,y) + F_2(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \ \forall y \in C\}, \tag{2.5}$$

called a resolvent of F_1 *and* F_2*. for all* $z \in H$*.*
Then

(i) $T_r \neq \emptyset$;
(ii) T_r *is asingle-valued;*
(iii) T_r *is firmly nonexpansive;*
(iv) $\mathrm{MEP}(F_1, F_2) = \mathrm{Fix}(T_r)$ *and it is closed and convex.*

Definition 2.8. Let C be a nonempty closed convex subset of a Hilbert space H and $f : H \to H$ be a function.

(i) Minimization problem:

$$\min_{x\in C} f(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2,$$

(ii) Tikhonov's regularization problem:

$$\min_{x\in C} f_\alpha(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2 \tag{2.6}$$

where $\alpha > 0$ is the regularization parameter.

Proposition 2.9. ([19]) *If the* SFP *is consistent, then the strong* $\lim_{\alpha\to 0} x_\alpha$ *exists and is the minimum-norm solution of the* SFP*.*

Proposition 2.10. ([19]) *A necessary and sufficient condition for* x_α *to converge in norm as* $\alpha \to 0$ *is that the minimization*

$$\lim_{u\in\overline{A(C)}} dist(u,Q) = \min_{u\in\overline{A(C)}} \|u - P_Q u\| \tag{2.7}$$

is attained at a point in the set $A(C)$*.*

Remark 2.11. Assume that the SFP is consistent, and let $x_{\min}$ be its minimum-norm solution, namely $x_{\min} \in \Gamma$ has the property

$$\|x_{\min}\| = \min\{\|x^*\| : x^* \in \Gamma\}.$$

From (2.6), observing that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$$

is an $(\alpha + \|A\|^2)$-Lipschitzian and α-strongly monotone mapping, the mapping $P_C(I - \lambda\nabla f_\alpha)$ is a contraction with the coefficient

$$\sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)} \leq 1 - \frac{1}{2}\alpha\lambda,$$

where

$$0 < \lambda < \frac{\alpha}{(\|A\|^2 + \alpha)^2}. \tag{2.8}$$

Remark 2.12. ([14]) The mapping $T = P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive.

Lemma 2.13. ([19]) *Assume that the* SFP(1.1) *is consistent. Define a sequence* $\{x_n\}$ *by the iterative algorithm*

$$x_{n+1} = P_C(I - \gamma_n\nabla f_{\alpha_n})x_n = P_C((1 - \gamma_n\alpha_n)x_n - \gamma_n A^*(I - P_Q)Ax_n), \tag{2.9}$$

where $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(i) $0 < \gamma_n \leq \frac{\alpha_n}{\|A\|^2+\alpha_n}$ *for all* n*;*
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \gamma_n = 0$*;*
(iii) $\sum_{n=1}^{\infty} \alpha_n\gamma_n = \infty$*;*
(iv) $\lim_{n\to\infty} = \frac{|\gamma_{n+1}-\gamma_n| - \gamma_n|\alpha_{n+1}-\alpha_n|}{(\alpha_{n+1}\gamma_{n+1})^2} = 0.$

Then $\{x_n\}$ *converges in norm to the minimum-norm solution of the* SFP(1.1).

Lemma 2.14. ([20]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.15. ([15]) *Let* H *be a Hilbert space,* C *a closed convex subset of* H*, and* $S : C \to C$ *a nonexpansive mapping with* $\mathrm{Fix}(S) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence in* C *weakly converging to* $x \in C$ *and if* $\{(I - S)x_n\}$ *converges strongly to* y*, then* $(I - S)x = y$*; inparticular, if* $y = 0$*, then* $x \in\mathrm{Fix}(S)$.

Lemma 2.16. ([10]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n b_n,$$

where $\{\sigma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{b_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \sigma_n = \infty$*;*
(2) $\limsup_{n\to\infty} \sigma_n b_n \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we introduce two algorithms for solving the mixed equilibrium problem (1.3). Namely, we want to find a solution x^* of a mixed equilibrium problem (1.3) and x^* also solves the following variational inequality:

$$x^* \in \Gamma\langle(\gamma g - \mu B)x^*, x - x^*\rangle \leq 0,\ x \in \Gamma, \tag{3.1}$$

where B is a k-Lipschitz and η-strongly monotone operator on H with $k > 0, \eta > 0$ and $0 < \mu < 2\eta/k^2$, and $g : C \to H$ is a β-contraction mapping, $\beta \in (0, 1)$. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions. In order to find a particular solution of the variational inequality (3.1), we construct the following implicit algorithm.

Algorithm 3.1. *For an arbitrary initial point x_0, we defind a sequence $\{x_n\}_{n\geq 0}$ iteratively*

$$x_n = (I - t\mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})x_n,\ \forall t \in (0,1), \tag{3.2}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a real in $[0,1]$, T_r is defined by Lemma 2.7 and ∇f_{α_n} is introduced in Remark 2.11.

We show that the sequence x_n defined by (3.2) converges to a particular solution of the variational inequality (3.1). As a matter of fact, in this paper, we study a general algorithm for solving the variational inequality (3.1).

Let $g : C \to H$ be a β-contraction mapping. For each $t \in (0,1)$, we consider the following mapping S_t given by:

$$S_t x = [t\gamma g + (I - t\mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x,\ x \in C, \tag{3.3}$$

Lemma 3.2. ([14]) *S_t is a contraction. Indeed,*

$$\|S_t x - S_t y\| \leq [1 - (\tau - \gamma\beta)t]\|x - y\|,\ \forall x, y \in H,$$

where $t \in (0, \frac{1}{\tau - \gamma\beta})$, and the sequence of $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the conditions (i)-(iv) *in Lemma 2.13.*

From Lemma 3.2 and using the Banach contraction principle, there exists a unique fixed point x_t of S_t in C, i.e., we obtain the following algorithm.

Algorithm 3.3. *For an arbitrary initial point x_0, we defind a sequence $\{x_n\}_{n\geq 0}$ iteratively*

$$x_n = [\varepsilon_n \gamma g + (I - \varepsilon_n \mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x_n,\ x \in C, \tag{3.4}$$

for all $n \geq 0$, where $\{\alpha_n\}$ and $\{\varepsilon_n\}$ are two real sequences in $[0,1]$, T_r is defined by Lemma 2.7 and ∇f_{α_n} is introduced in Remark 2.11.

At this point, we would like to point out that Algorithm 3.3 includes Algorithm 3.1 as a special case due to the fact that the contraction g is a possible nonself-mapping.

Theorem 3.4. *Let C be a closed convex subset of a real Hilbert space H. Let B be a k-Lipschitz and η-strongly monotone operator on H with $k > 0, \eta > 0$ and $0 < \mu < 2\eta/k^2$, and the sequence of $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the conditions* (i)-(iv) *in Lemma 2.13. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions which satisfy the conditions $(f_1) - (f_4), (h_1) - (h_3)$ and (H) in Lemma 2.7. Let $g : C \to H$ be a β-contraction. Let S be a nonexpansive mapping of C into itself. Assume that $\Omega =$Fix$(S) \cap \Gamma \cap$ MEP$(F_1, F_2) \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$ generated by*

$$x_{n+1} = \delta_n x_n + (1 - \delta_n)S[\varepsilon_n \gamma g + (I - \varepsilon_n \mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x_n,\ n \geq 0, \tag{3.5}$$

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are two sequence in $[0,1)$, satisfy the following conditions:

(C1) $\lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=1}^{\infty} \varepsilon_n = \infty$;
(C2) $\lim_{n\to\infty} \delta_n = 0$ and $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$;
(C3) $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* which is the unique solution of the variational inequality (3.1). In particular, if $g = 0$, then the sequence $\{x_n\}$ generated by

$$x_{n+1} = \delta_n x_n + (1 - \delta_n)S[(I - \varepsilon_n \mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x_n,\ n \geq 0,$$

converges strongly to a solution of the following variational inequality:

$$\langle \mu Bx^*, x - x^* \rangle \geq 0, \ \ \forall x \in \Omega.$$

Proof. We devide the proof into five steps as follows.

Step 1. We first show that the sequences $\{x_n\}$ is bounded. Indeed, pick $p \in \Omega$.

Let $p = P_C(I - \lambda_n \nabla f_{\alpha_n})p$. Set $u_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n$ for all $n \geq 0$. Since $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive mapping, then from (3.5), we have

$$\|u_n - p\| = \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \leq \|x_n - p\|,$$

and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\delta_n x_n + (1-\delta_n)S[\varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n] - p\| \\
&= \|\delta_n(x_n - p) + (1-\delta_n)\big(S[\varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n] - Sp\big)\| \\
&\leq \|\delta_n(x_n - p)\| + \|(1-\delta_n)\big(S[\varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n] - Sp\big)\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\|[\varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n] - p\| \\
&= \delta_n\|x_n - p\| + (1-\delta_n)\|(I - \varepsilon_n \mu B)(T_r u_n - p) + \varepsilon_n(\gamma g(x_n) - \mu Bp)\| \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\big((1-\varepsilon_n\tau)\|u_n - p\| + \varepsilon_n\gamma\beta\|x_n - p\| \\
&\quad +\varepsilon_n\|\gamma g(p) - \mu Bp\|\big) \\
&\leq \delta_n\|x_n - p\| + (1-\delta_n)\big((1-\varepsilon_n\tau)\|x_n - p\| + \varepsilon_n\gamma\beta\|x_n - p\| \\
&\quad +\varepsilon_n\|\gamma g(p) - \mu Bp\|\big) \\
&= \big(1 - (\tau - \gamma\beta)(1-\delta_n)\varepsilon_n\big)\|x_n - p\| \\
&\quad +\varepsilon_n(1-\delta_n)(\tau - \gamma\beta)\frac{\|\gamma g(p) - \mu Bp\|}{\tau - \gamma\beta} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma g(p) - \mu Bp\|}{\tau - \gamma\beta}\right\}.
\end{aligned} \tag{3.6}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma g(p) - \mu Bp\|}{\tau - \gamma\beta}\right\}, \ \forall n \geq 0. \tag{3.7}$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}, \{g(x_n)\}$ and $\{\nabla f(x_n)\}$ are all bounded.

Now, we can choose a constant $M > 0$ such that

$$\sup_n \big\{\|x_n - u_n\|, \|\mu BT_r u_n\| + \|\gamma g(x_n)\|, \|\mu BT_r u_n - \gamma g(x_n)\|^2\big\} \leq M.$$

Step 2. We show that $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Let $d_n = \varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n$ for all $n > 0$, then from (3.5) we get $x_{n+1} = \delta_n x_n + (1-\delta_n)Sd_n$ for all $n > 0$ and

$$\begin{aligned}
\|d_{n+1} - d_n\| &= \|[\varepsilon_{n+1}\gamma g(x_{n+1}) + (I - \varepsilon_{n+1}\mu B)T_r u_{n+1}] - [\varepsilon_n \gamma g(x_n) + (I - \varepsilon_n \mu B)T_r u_n]\| \\
&\leq \|u_{n+1} - u_n\| + \varepsilon_{n+1}\big(\|\mu B(T_r u_{n+1}\| + \|\gamma g(x_{n+1})\|\big) \\
&\quad + \varepsilon_n\big(\|\mu B)(T_r u_n\| + \|\gamma g(x_n)\|\big) \\
&\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_{n+1} - P_C(I - \lambda_n \nabla f_{\alpha_n})x_n\| + M(\varepsilon_{n+1} + \varepsilon_n) \\
&\leq \|x_{n+1} - x_n\| + M(\varepsilon_{n+1} + \varepsilon_n).
\end{aligned}$$

This together with (i) implies that

$$\limsup_{n\to\infty}\left(\|d_{n+1}-d_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

Hence, by Lemma 2.14 we get $\lim_{n\to\infty}\|d_n-x_n\|=0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\delta_n)\|d_n-x_n\|=0. \tag{3.8}$$

By the convexity of the norm $\|\cdot\|$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\delta_n x_n+(1-\delta_n)Sd_n-p\|^2\\
&\le \delta_n\|x_n-p\|^2+(1-\delta_n)\|Sd_n-p\|^2\\
&\le \delta_n\|x_n-p\|^2+(1-\delta_n)\|T_ru_n-p-\varepsilon_n(\mu BT_ru_n-\gamma g(x_n))\|^2\\
&= \delta_n\|x_n-p\|^2+(1-\delta_n)\big[\|u_n-p\|^2\\
&\quad -2\varepsilon_n\langle\mu BT_ru_n-\gamma g(x_n),T_ru_n-p\rangle+\varepsilon_n^2\|\mu BT_ru_n-\gamma g(x_n)\|^2\big]\\
&\le \delta_n\|x_n-p\|^2+(1-\delta_n)\|u_n-p\|^2+\varepsilon_n M.
\end{aligned} \tag{3.9}$$

Let $y_n=T_ru_n$ and by $u_n=P_C(I-\lambda_n\nabla f_{\alpha_n})x_n$, we obtain

$$\begin{aligned}
\|y_n-p\|^2 &= \|T_ru_n-T_rp\|^2\\
&\le \|u_n-p\|^2\\
&= \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n-P_C(I-\lambda_n\nabla f_{\alpha_n})p\|^2\\
&\le \langle(I-\lambda_n\nabla f_{\alpha_n})x_n-(I-\lambda_n\nabla f_{\alpha_n})p,u_n-p\rangle\\
&= \frac{1}{2}\big(\|(I-\lambda_n\nabla f_{\alpha_n})x_n-(I-\lambda_n\nabla f_{\alpha_n})p\|^2+\|u_n-p\|^2\\
&\quad -\|(x_n-p)-\lambda_n(\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p))-(u_n-p)\|^2\big)\\
&\le \frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-u_n)-\lambda_n(\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p))\|^2\big)\\
&\le \frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\\
&\quad +2\lambda_n\langle x_n-u_n,\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\rangle\\
&\quad -\lambda_n^2\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|^2\big).
\end{aligned} \tag{3.10}$$

Therefore, we deduce

$$\begin{aligned}
\|u_n-p\|^2 &\le \|x_n-p\|^2-\|x_n-u_n\|^2+2\lambda_n\|x_n-u_n\|\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|\\
&\le \|x_n-p\|^2-\|x_n-u_n\|^2+\lambda_n M\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|.
\end{aligned} \tag{3.11}$$

By (3.9) and (3.11), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \delta_n\|x_n-p\|^2+(1-\delta_n)\|u_n-p\|^2+\varepsilon_n M\\
&\le \delta_n\|x_n-p\|^2\\
&\quad +(1-\delta_n)\big[\|x_n-p\|^2-\|x_n-u_n\|^2+\lambda_n M\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|\big]\\
&\quad +\varepsilon_n M\\
&= \|x_n-p\|^2-(1-\delta_n)\|x_n-u_n\|^2+(\lambda_n\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|+\varepsilon_n)M.
\end{aligned}$$

It follows that

$$\begin{aligned}(1-\delta_n)\|x_n-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\lambda_n\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|+\varepsilon_n)M\\ &\leq -\|x_{n+1}-x_n\|^2+(\lambda_n\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(p)\|+\varepsilon_n)M.\end{aligned}$$

Wherewith $\liminf_{n\to\infty}(1-\delta_n)>0, \lim_{n\to\infty}\varepsilon_n=0, \lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\{\nabla f(x_n)\}$ is bounded and $\lim_{n\to\infty}\lambda_n=0$, we derive that

$$\|x_n-u_n\|\to 0 \text{ as } n\to\infty. \tag{3.12}$$

Step 3. We show that $\|y_n-u_n\|\to 0$ and $\|Sy_n-y_n\|\to 0$ as $n\to\infty$, where $y_n=T_ru_n$. From (3.5), we have

$$\begin{aligned}\|x_n-y_n\| &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-y_n\|\\ &\leq \|x_{n+1}-x_n\|+\|\delta_nx_n+(1-\delta_n)S[\varepsilon_n\gamma g(x_n)+(I-\varepsilon_n\mu B)y_n]-y_n\|\\ &\leq \|x_{n+1}-x_n\|+\delta_n\|x_n-y_n\|+(1-\delta_n)\|\varepsilon_n\gamma g(x_n)-\varepsilon_n\mu BT_ru_n\|\\ &\leq \|x_{n+1}-x_n\|+\delta_n\|x_n-y_n\|\\ &\quad +(1-\delta_n)\big(\varepsilon_n\gamma\beta\|x_n-u_n\|+\varepsilon_n\|\gamma g(u_n)-\mu BT_ru_n\|\big).\end{aligned}$$

Therefore,

$$\|x_n-y_n\|\leq \frac{1}{1-\delta_n}\|x_{n+1}-x_n\|+\varepsilon_n\gamma\beta\|x_n-u_n\|+\varepsilon_n\|\gamma g(u_n)-\mu BT_ru_n\|.$$

From $\lim_{n\to\infty}\varepsilon_n=0$ and $\{u_n\}$ is bounded, we obtain

$$\|x_n-y_n\|\to 0 \text{ as } n\to\infty. \tag{3.13}$$

Since $\|y_n-u_n\|\leq\|y_n-x_n\|+\|x_n-u_n\|$, and by (3.12), (3.13) it follows that

$$\|y_n-u_n\|\to 0 \text{ as } n\to\infty.$$

Next, we will show that $\|Sy_n-y_n\|\to 0$ as $n\to\infty$. We have

$$\begin{aligned}\|y_n-d_n\| &= \|y_n-[\varepsilon_n\gamma g(x_n)+(I-\varepsilon_n\mu B)T_ru_n]\|\\ &= \|-\varepsilon_n\gamma g(x_n)+\varepsilon_n\mu BT_ru_n\|\\ &= \|\varepsilon_n\gamma g(u_n)-\varepsilon_n\gamma g(x_n)+\varepsilon_n\mu BT_ru_n-\varepsilon_n\gamma g(u_n)\|\\ &\leq \|\varepsilon_n\gamma(g(u_n)-g(x_n))\|+\|\varepsilon_n(\mu BT_ru_n-\gamma g(u_n))\|\\ &\leq \varepsilon_n\gamma\beta\|u_n-x_n\|+\varepsilon_n\|\mu BT_ru_n-\gamma g(u_n)\|.\end{aligned}$$

From $\varepsilon_n\to 0$ and $\{u_n\}$ is bounded, we obtain

$$\|y_n-d_n\|\to 0 \text{ as } n\to\infty. \tag{3.14}$$

On the other hand, we have

$$\|Sd_n-x_{n+1}\|=\|Sd_n-(\delta_nx_n+(1-\delta_n)Sd_n)\|=\delta_n\|Sd_n-x_n\|.$$

From $\delta_n\to 0$, we obtain

$$\|Sd_n-x_{n+1}\|\to 0 \text{ as } n\to\infty. \tag{3.15}$$

Since $\|Sd_n-x_n\|\leq\|Sd_n-x_{n+1}\|+\|x_{n+1}-x_n\|$, then by (3.15) and (3.8) it follows that

$$\|Sd_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.16}$$

Since $\|Sy_n - x_n\| \leq \|Sy_n - Sd_n\| + \|Sd_n - x_n\| \leq \|y_n - d_n\| + \|Sd_n - x_n\|$, then by (3.14) and (3.16) it follows that

$$\|Sy_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Since $\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|$, then by (3.17) and (3.13) it follows that

$$\|Sy_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

Step 4. We show that

$$\limsup_{n\to\infty} \langle (\gamma g - \mu B)x^*, y_n - x^* \rangle \leq 0, \text{where } x^* \in P_\Omega g(x^*).$$

Since $\{y_n\}$ is bounded, then we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$, such that

$$\limsup_{n\to\infty} \langle (\gamma g - \mu B)x^*, y_n - x^* \rangle \leq \limsup_{k\to\infty} \langle (\gamma g - \mu B)x^*, y_{n_k} - x^* \rangle \leq 0.$$

Without loss of generality, we can assume that $y_{n_k} \rightharpoonup x^* \in C$.

Next, we show that $x^* \in \Omega = \text{Fix}(S) \cap \Gamma \cap \text{MEP}(F_1, F_2)$. Indeed, first we show that $x^* \in\text{Fix}(S)$. Since S is nonexpansive, then by (3.18) and Lemma 2.15, we obtain that $x^* \in\text{Fix}(S)$.

Second, we show that $x^* \in\text{MEP}(F_1, F_2)$.
Since $y_n = T_r u_n$, then we obtain

$$F_1(y_n, y) + F_2(y_n, y) + \frac{1}{r}\langle y - y_n, y_n - u_n \rangle \geq 0, \forall y \in C.$$

From the monotoniccity of F_1 and F_2, we obtain

$$\frac{1}{r}\langle y - y_n, y_n - u_n \rangle \geq F_1(y_n, y) + F_2(y_n, y), \forall y \in C.$$

Hence,

$$\left\langle y - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{r} \right\rangle \geq F_1(y_{n_k}, y) + F_2(y_{n_k}, y), \forall y \in C. \tag{3.19}$$

Since $\frac{y_{n_k} - x_{n_k}}{r} \to 0$ and $y_n \rightharpoonup x^*$, from (A2), it follows $F_1(y_n, y) + F_2(y_n, y) \leq 0$ for all $y \in H$. Put $z_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in H$, then we have $F_1(z_t, x^*) + F_2(z_t, x^*) \leq 0$. So, from (A1) and (A4), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) + F_2(y_t, y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, x^*) + tF_2(y_t, y) + (1-t)F_2(y_t, x^*) \\
&\leq F_1(y_t, y) + F_2(y_t, y)
\end{aligned}$$

and hence $0 \leq F_1(y_t, y) + F_2(y_t, y)$. From (A3), we have $0 \leq F_1(x^*, y) + F_2(x^*, y)$ for all $y \in H$. Thus, $x^* \in\text{MEP}(F_1, F_2)$.

Third, we show that $x^* \in \Gamma$. From Remark (2.12), we know that the mapping $T = P_C(I - \lambda_n \nabla f)$ is nonexpansive, then we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - u_n\| + \|u_n - Tx_n\| \\
&= \|x_n - u_n\| + \|P_C(I - \lambda_n \nabla f_{\alpha_n})x_n - P_C(I - \lambda_n \nabla f)x_n\| \\
&= \|x_n - u_n\| + \|(I - \lambda_n \nabla f_{\alpha_n})x_n - (I - \lambda_n \nabla f)x_n\| \\
&= \|x_n - u_n\| + \lambda_n \alpha_n \|x_n\|.
\end{aligned}$$

So, from $\{x_n\}$ is bounded sequence, $\lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\lambda_n = 0, \sum_{n=1}^{\infty}\alpha_n\lambda_n = \infty$ and Step 2 it follows that

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Thus, taking into account $x_{n_k} \to x^*$ and $u_{n_k} \to x^*$, and from Lemma 2.15, we get $x^* \in \Gamma$. Therefore $x^* \in \Omega = \mathrm{Fix}(S) \cap \Gamma \cap \mathrm{MEP}(F_1, F_2)$. Since $x^* = P_\Omega g(x^*)$. Indeed, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma g - \mu B)x^*, y_n - x^*\rangle &= \lim_{k\to\infty}\langle(\gamma g - \mu B)x^*, y_{n_k} - x^*\rangle \\ &= \langle(\gamma g - \mu B)x^*, w - x^*\rangle \le 0. \end{aligned}\tag{3.21}$$

Step 5. Finally, we show that $\{x_n\}$ strongly converge to $x^* \in \Omega$.
Let $x^* \in \Omega$ and $y_n = T_r u_n$, where $u_n = P_C(I - \lambda_n \nabla f_{\alpha_n})x_n$ for all $n \ge 0$, then from (3.5), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\delta_n x_n + (1-\delta_n)S[\varepsilon_n\gamma g(x_n) + (I - \varepsilon_n\mu B)y_n] - x^*\|^2 \\
&= \big\|\delta_n(x_n - x^*) + (1-\delta_n)\big[(S[y_n + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n]) - x^*\big]\big\|^2 \\
&= \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big\|\big[S[y_n + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n] - x^*\big]\big\|^2 \\
&\quad - \delta_n(1-\delta_n)\big\|(x_n - x^*) - S[y_n + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n] - x^*\big\|^2 \\
&\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big\|\big[S[y_n + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n] - x^*\big]\big\|^2 \\
&= \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big\|\big[S[y_n + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n] - Sx^*\big]\big\|^2 \\
&\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\|y_n - x^* + \varepsilon_n\gamma g(x_n) - \varepsilon_n\mu B y_n\|^2 \\
&\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big(\|y_n - x^*\|^2 + 2\varepsilon_n\gamma\langle g(x_n), y_n - x^*\rangle \\
&\quad - 2\varepsilon_n\langle\mu B y_n, y_n - x^*\rangle + \varepsilon_n^2\|\gamma g(x_n) - \mu B y_n\|^2\big) \\
&= \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big(\|T_r u_n - x^*\|^2 + 2\varepsilon_n\gamma\langle g(x_n) - g(x^*), y_n - x^*\rangle \\
&\quad - 2\varepsilon_n\langle\mu B y_n - \mu B x^*, y_n - x^*\rangle + 2\varepsilon_n\langle\gamma g(x^*) - \mu B x^*, y_n - x^*\rangle \\
&\quad + \varepsilon_n^2\|\gamma g(x_n) - \mu B y_n\|^2\big) \\
&\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big(\|u_n - x^*\|^2 + 2\varepsilon_n\gamma\|g(x_n) - g(x^*)\|\|T_r u_n - x^*\| \\
&\quad - 2\varepsilon_n\|\mu B y_n - \mu B x^*\|\|T_r u_n - x^*\| + 2\varepsilon_n\langle\gamma g(x^*) - \mu B x^*, y_n - x^*\rangle \\
&\quad + \varepsilon_n^2\|\gamma g(x_n) - \mu B T_r u_n\|^2\big) \\
&\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\big(\|x_n - x^*\|^2 + 2\varepsilon_n\gamma\beta\|x_n - x^*\|^2 \\
&\quad - 2\varepsilon_n\tau\|x_n - x^*\|^2 + 2\varepsilon_n\langle\gamma g(x^*) - \mu B x^*, y_n - x^*\rangle + \varepsilon_n^2 M\big) \\
&= (1 - 2\varepsilon_n(\gamma\beta - \tau))\|x_n - x^*\|^2 + 2(1-\delta_n)\varepsilon_n\langle\gamma g(x^*) - \mu B x^*, y_n - x^*\rangle \\
&\quad + \varepsilon_n^2(1-\delta_n)M.\end{aligned}$$

It then follows that

$$\|x_{n+1} - x^*\|^2 \le (1-\sigma_n)\|x_n - x^*\|^2 + \sigma_n b_n, \tag{3.22}$$

where $\sigma_n = 2\varepsilon_n(\gamma\beta - \tau)$ and $b_n = \frac{(1-\delta_n)}{(\gamma\beta-\tau)}\langle\gamma g(x^*) - \mu B x^*, y_n - x^*\rangle + \frac{\varepsilon_n(1-\delta_n)}{2(\gamma\beta-\tau)}M$.
Let $a_n = \|x_n - x^*\|^2$ then, we can write the last inequality as

$$a_{n+1} \le (1-\sigma_n)a_n + \sigma_n b_n.$$

Note that in virtue of condition (C1), $\sum_{n=1}^{\infty}\sigma_n = \infty$. Moreover

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \frac{1}{\bar{\gamma}-\gamma\alpha}\limsup_{n\to\infty} 2\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle.$$

By Step 4, we obtain

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0. \tag{3.23}$$

Now applying Lemma 2.16 to (3.22), we conclude that $x_n \to x^*$ as $n \to \infty$. The proof is now complete. ∎

Putting $S = I$ which is the identity operator in Theorem 3.4, we obtain the following results.

Corollary 3.5. *Let C be a closed convex subset of a real Hilbert space H. Let B be a k-Lipschitz and η-strongly monotone operator on H with $k > 0, \eta > 0$ and $0 < \mu < 2\eta/k^2$, and the sequence of $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the conditions* (i)-(iv) *in Lemma 2.13. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions which satisfy the conditions $(f_1)-(f_4), (h_1)-(h_3)$ and (H) in Lemma 2.7. Let $g : C \to H$ be a β-contraction. Assume that $\Omega = \Gamma \cap MEP(F_1, F_2) \neq \emptyset$. For given $\forall x_0 \in C$, let the sequences $\{x_n\}$ generated by*

$$x_{n+1} = \delta_n x_n + (1-\delta_n)[\varepsilon_n \gamma g + (I - \varepsilon_n \mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x_n,\ n \ge 0, \tag{3.24}$$

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are two sequence in $[0, 1)$, satisfy the following conditions:

(C1) $\lim_{n\to\infty}\varepsilon_n = 0$ and $\sum_{n=1}^{\infty}\varepsilon_n = \infty$;
(C2) $\lim_{n\to\infty}\delta_n = 0$ and $0 < \liminf_{n\to\infty}\delta_n \le \limsup_{n\to\infty}\delta_n < 1$;
(C3) $\lim_{n\to\infty}\lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to x^* which is the unique solution of the variational inequality (3.1). In particular, if $g = 0$, then the sequence $\{x_n\}$ generated by

$$x_{n+1} = \delta_n x_n + (1-\delta_n)[(I - \varepsilon_n \mu B)T_r P_C(I - \lambda_n \nabla f_{\alpha_n})]x_n,\ n \ge 0,$$

converges strongly to a solution of the following variational inequality:

$$\langle \mu B x^*, x - x^*\rangle \ge 0,\ \ \forall x \in \Omega.$$

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