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Convergence Results on Splitting Operator for Convex Minimization Problem and Its Applications

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Abstract The purpose of this paper is to introduce a new iterative method that is a combination of the modified Mann type forward-backward splitting with the viscosity approximation method and the alternating resolvent method for finding the zero of sum of accretive operators in uniformly convex real Banach spaces which are also uniformly smooth spaces. Our result is new and complements many recent and important results in this direction in the literature. Moreover, we also applied our algorithm to solving the convex minimization problem for solving image restoration.

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1. Introduction

Image restoration is the solution of this ill-posed inverse problem which has been extensively studied in various applications such as MRI, medical imaging, astronomical imaging, remote sensing, video coding and image coding. Image restoration is a specific branch in image processing, using the prior knowledge about convex optimization. As mentioned earlier, the problem is about reconstructing an approximate original image from the degraded one. We assume that the degradation process consists of only blurring and the noise addition. These degradation ingredients are observed from statistical data, and then is approximated by the operator R and noise v. Here, $R : H \to H$ is the blurring operator, which is spatial invariant and linear.

Suppose that $f, g : H \times H \to H$ are the spatial data for the original and the degraded images, respectively. Also assume that $R : H \to H$ is linear and spatial invariant and

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$v : H \times H \to H$ is additive noise. Then, the following relation:

$$g(x, y) = R(f(x, y)) + v(x, y)$$

is the model of the degradation process. The image restoration problem is to approximatively find the original image f that minimizes the least-square error of the degradation related to the additive noise:

$$\text{Min } \|v\|^2 = \|g - Rf\|^2.$$

In various cases, R might be calculated as a convolution in an integral equation. But we shall not specify such restrictions so that we can work in a more general form of degradation process.

Let E be a real Banach space. Suppose $A : E \to 2^E$ is a set-valued operator and $B : E \to E$ is an operator. In this paper, we will consider the following inclusion problem (assuming that the solution exists): find $x \in E$ such that

$$0 \in Ax + Bx. \tag{1.1}$$

This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modelled mathematically as this form.

This paper is devoted to designing and discussing an efficient algorithmic framework for minimizing the sum of two proper lower semi-continuous convex functions, i.e.

$$x^* = \arg\min_{x \in \mathbb{R}^n} (f_1 \circ B)(x) + f_2(x), \tag{1.2}$$

where $\Gamma_0(X)$ is the set of proper lower semi-continuous convex function from X to $(-\infty, +\infty]$, $f_1 \in \Gamma_0(\mathbb{R}^m), f_2 \in \Gamma_0(\mathbb{R}^n)$ and f_2 is differentiable on $\mathbb{R}^n$ with a $\frac{1}{\beta}$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$ and $B : \mathbb{R}^n \to \mathbb{R}^m$ a linear transform. Many problems in image processing can be formulated in the form of (1.2). For instance, the following variational sparse recovery models are often considered in image restoration and medical image reconstruction:

$$x^* = \arg\min_{x \in \mathbb{R}^n} \mu\|Bx\|_1 + \frac{1}{2}\|Ax - b\|_2^2, \tag{1.3}$$

where $\|\cdot\|_2$ denotes the usual Euclidean norm for a vector, A is a $p \times n$ matrix representing a linear transform, $b \in \mathbb{R}^p$ and $\mu > 0$ is the regularization parameter. The term $\|Bx\|_1$ is the usual l^1-based regularization in order to promote sparsity under the transform B. Problem (1.3) can be expressed in the form of (1.2) by setting $f_1 = \mu\|\cdot\|_1$ and $f_2(x) = \frac{1}{2}\|Ax - b\|_2^2$. One of the main difficulties in solving it is that f_1 is non-differentiable. The case often occurs in many problems we are interested in.

Another general problem often considered in the literature takes the following form:

$$x^* = \arg\min_{x \in \mathrm{X}} f(x) + h(x), \tag{1.4}$$

where $f, h \in \Gamma_0(\mathrm{X})$ and h is differentiable on X with a $\frac{1}{\beta}$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$. Problem (1.2), which we are interested in this paper, can be viewed as a special case of problem (1.4) for $\mathrm{X} = \mathbb{R}^n$ and $f = f_1 \circ B, h = f_2$. On the other hand, we can also consider that problem (1.4) is a special case of problem (1.2) for $\mathrm{X} = \mathbb{R}^n, f_2 = h, f_1 = f$ and $B = I$, where I denotes the usual identity operator. For problem (1.4), Combettes and Wajs proposed in [1] a proximal forward-backward splitting algorithm, i.e.

$$x_{n+1} = \mathrm{prox}_{\gamma f}(x_n - \gamma \nabla h(x_n)), \quad \forall n \geq 1, \tag{1.5}$$

where $0 < \gamma < 2\beta$ is a stepsize parameter, and the operator prox_f is defined by

$$\begin{aligned} \mathrm{prox}_f &: \mathrm{X} \to \mathrm{X} \\ &x \mapsto \mathrm{argmin}_{y \in \mathrm{X}} f(y) + \frac{1}{2}\|x - y\|_2^2, \end{aligned}$$

called the proximity operator of f.

It is well known that a forward-backward splitting method (please see, e.g., [1–6]) is a classical method for solving problem (1.1). This forward-backward splitting method is given by $x_1 \in E$ and

$$x_{n+1} = (Id + r_n A)^{-1}(Id - r_n B)x_n, \quad \forall n \geq 1, \tag{1.6}$$

where $r_n > 0$. This method generalizes the proximal point algorithm (please see [7–11]) and the gradient method (see, e.g., [12]). There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces). For more details, please, see [13–17] and the references contained therein.

In 2012, López et al. [18] proposed the following modification:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n}^A (x_n - r_n(Bx_n + a_n)) + b_n), \quad \forall n \geq 1, \tag{1.7}$$

where $J_{r_n}^A = (Id + r_n A)^{-1}$ and $\{a_n\}, \{b_n\} \subset E$ stand for the computation, $\{\alpha_n\} \subset (0, 1]$, and $\{r_n\} \subset (0, +\infty)$.

Recently, Shehu and Cai [19] expanded algorithm (1.6) by combining it with the viscosity approximation method

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n}^A (x_n - r_n B x_n), \quad \forall n \geq 1, \tag{1.8}$$

where $f : E \to E$ is contraction with coefficient $k \in (0, 1)$, and $\alpha_n \in (0, 1)$. They studied this algorithm in uniformly smooth Banach spaces, and proved that the sequence $\{x_n\}$ generated by algorithm (1.8) converges strongly to a solution of problem (1.1).

Very recently, Shehu [20] introduced the following modification:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n) x_n \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{r_n}^A (y_n - r_n B y_n), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where $f : E \to E$ is contraction with coefficient $k \in (0,1)$, and $\{\alpha_n\}, \{\beta_n\} \in (0,1)$. They studied this algorithm in a real uniformly convex Banach space which is also uniformly smooth, and proved that the sequence $\{x_n\}$ generated by algorithm (1.9) converges strongly to a solution of problem (1.1).

In this paper, motivated and inspired by above literatures, we are going to consider a problem (1.1) and suggest the following algorithm: for an arbitrary initial $x_1 \in E$,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J^A_{r_n}(Id - r_nB)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J^A_{r_n}(Id - r_nB)y_n, \quad \forall n \geq 1. \end{cases} \tag{1.10}$$

The strong convergence theorems of this iterative algorithms are obtain in Banach spaces. Our algorithm combine between the modified Mann type forward-backward splitting with the viscosity approximation method and the alternating resolvent method complements many recent and important results in this direction in the literature. Moreover, we also applied our algorithm to solving the convex minimization problem for application to image restoration in the last section.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Let $S_E = \{x \in E : \|x\| = 1\}$ and $\mathrm{Fix}(T)$ be the set of fixed point of the mapping $T : E \to E$. A Banach space E is said to be *strictly convex* if for $x, y \in S_E$ with $x \neq y$ and $t \in (0,1)$,

$$\|(1-t)x + ty\| < 1.$$

A Banach space E is said to be *uniformly convex* if for any $\epsilon \in (0,2]$ the inequalities $\|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon$ imply that there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

A Banach space E is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| + \|x\|}{t}$$

exists for each $x, y \in S_E$.

In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit attained uniformly for $x \in S_E$. It is well known that every uniformly smooth Banach space has uniformly Gâteaux differentiable norm. A closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty closed convex subset D of C into itself has a fixed point in D.

A subset C of Banach space E is called a retract of E if there is a continuous mapping Q from E onto C such that $Qx = x$ for all $x \in C$. We call such Q a retraction of E onto C. It follows that if a mapping Q is a retraction, then $Qy = y$ for all y in the range of Q. A retraction Q is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. If a sunny retraction Q is also nonexpansive, then C is said to be a sunny non expansive retract of E [21]. In a smooth Banach space E, it is known (cf. [21], p. 48) that $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Q(x), J(z - Q(x))\rangle \leq 0, \ x \in C, \ z \in D.$$

A set-valued operator $A : E \to 2^E$, with domain $D(A)$ and range $R(A)$, is said to be *accretive* if, for all $t > 0$ and every $x, y \in D(A)$,

$$\|x - y\| \leq \|x - y + t(u - v)\|, \quad u \in Ax, v \in Ay.$$

Furthermore, an accretive operator A is said to be m-accretive if the range $R(I + \lambda A) = E$ for all $\lambda > 0$. Given $\alpha > 0$ and $q \in (1, \infty)$, we say that an accretive operator A is α-inverse strongly accretive (α-isa) of order q if, for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q, \quad u \in Ax, v \in Ay.$$

When $q = 2$, we simply say α-isa, instead of α-isa of order 2; that is, A is α-isa if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq \alpha\|u - v\|^2, \quad u \in Ax, v \in Ay.$$

Definition 2.1. A nonlinear operator T whose domain $D(T) \subset H$ and range $R(T) \subset H$ is said to be:

(i) *monotone* if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T);$$

(ii) *β-strongly monotone* if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in D(T);$$

(iii) *ν-inverse strongly monotone*(for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

In other words, T is monotone if its graph, $G(T) = \{(x, y) \in H \times H : x \in D(T), y \in Tx\}$, is a monotone subset of $H \times H$. A monotone operator T is called maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator. Note that a $\nu-$inverse strongly monotone operator is monotone. We know that T is maximal monotone if and only if the range of $Id + T$ is equal to H (i.e., $R(Id+T) = H$). If T is maximal monotone and μ is a positive number, then the resolvent of T is defined by $J^T_\mu(x) = (Id + \mu T)^{-1}(x)$ that a single-valued operator $J^T_\mu : H \to H$.

Definition 2.2. Let D be a nonempty subset of a Hilbert space H and let $T : D \to H$. Then,

(i) firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(Id - T)x - (Id - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in D;$$

(ii) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D;$$

(iii) quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in D, p \in \mathrm{Fix}(T);$$

(iv) stricty quasi-nonexpansive if

$$\|Tx - p\| < \|x - p\|, \quad \forall x \in D, p \in \mathrm{Fix}(T).$$

Definition 2.3. [22] Let $T : E \to E$ be nonexpansive, and let $\alpha \in (0, 1)$. Then, T is said to be α-averaged, if there exists a nonexpansive operator $R : E \to E$ such that

$$T = (1 - \alpha)Id + \alpha R.$$

Remark 2.4. The firmly nonexpansive mappings are $\frac{1}{2}$-averaged.
Suppose that $T : E \to E$ is firmly nonexpansive. So, there exists a mapping defined by $R = 2T - Id : E \to E$ such that

$$\begin{aligned}
\|(2T - Id)x - (2T - Id)y\|^2 &= \|2(Tx - Ty) + (1 - 2)(x - y)\|^2 \\
&= 2\|Tx - Ty\|^2 + (1 - 2)\|x - y\|^2 \\
&\quad - 2(1 - 2)\|Tx - Ty - x + y\|^2 \\
&\leq 2\|x - y\|^2 - \|x - y\|^2 \\
&= \|x - y\|^2.
\end{aligned}$$

Hence, R is a nonexpansive mapping. By a simple transformation, we have

$$\begin{aligned}
T &= Id - Id + T \\
&= (1 - \frac{1}{2})Id + \frac{1}{2}(2T - Id) \\
&= (1 - \frac{1}{2})Id + \frac{1}{2}R.
\end{aligned}$$

Consequently, T is $\frac{1}{2}$-averaged.

Proposition 2.5. [23, 24] *Let D be a nonempty subset of a Hilbert space H, let $(\alpha_1, \alpha_2) \in (0, 1)^2$, let $T_1 : D \to D$ be α_1-averaged, and let $T_2 : D \to D$ be α_2-averaged. Set*

$$T = T_1T_2 \quad \text{and} \quad \alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

Then, $\alpha \in (0, 1)$ and T is α-averaged.

We know that the sunny nonexpansive retract mapping form a Hilbert space H onto a closed convex subset $C \subset H$ is metric projection and we use P_C to denote the metric projection from H onto C.

We now state some known results which will be used in the sequel.

Lemma 2.6. [25] *A Banach space E is uniformly smooth if and only if the duality map J is the single-valued and norm-to-norm uniformly continuous on bounded sets of E.*

Lemma 2.7. [26] *Let E be a Banach space. Then, for every $x, y \in E$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,$$

for all $j(x+y) \in J(x+y)$.

Lemma 2.8. [27] *Let C be a nonempty closed convex subset of a uniformly smooth Banach space E and $T : C \to C$ be a nonexpansive mapping with a fixed point. Let $f : E \to E$ be a fixed contraction with coefficient $k \in (0,1)$. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, and $p = \lim_{t\to 0} z_t$ exists, where $\{z_t\}$ is defined by $z_t = tf(z_t) + (1-t)Tz_t$. Then,*

$$\limsup_{n\to\infty} \langle f(p) - p, j(x_n - p)\rangle \leq 0.$$

Lemma 2.9. [20] *Let E be a real uniformly convex with Fréchet differentiable norm. Assume that B is a single-valued α-inverse strongly accretive mapping on E. Then, given $s > 0$, there exists a continuous, strictly increasing and convex function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\theta(0) = 0$ such that for all $x, y \in E$,*

$$\begin{aligned}\|T_s x - T_s y\|^2 \leq \|x-y\|^2 - s(2\alpha - sc)\|Bx - By\|^2 \\ - \theta(\|(Id - J_s^A)(Id - sB)x - (Id - J_s^A)(Id - sB)y\|),\end{aligned}$$

where $T_s = J_s^A(Id - sB) = (Id + sA)^{-1}(Id - sB)$.

Lemma 2.10. [18] *Let E be a real Banach space. Let $A : E \to 2^E$ be an m-accretive operator and $B : E \to E$ be an α-inverse strongly accretive mapping on E. Then, we have*

- (i) *for $r > 0$, $\mathrm{Fix}(T_r) = (A+B)^{-1}(0)$;*
- (ii) *for $0 < s \leq r$ and $x \in E$, $\|x - T_s x\| \leq \|x - T_r x\|$.*

Theorem 2.11. [28] *Let X be a real Hilbert space and let $S : X \to X$ be quasi-nonexpansive, $T : X \to X$ be strictly quasi-nonexpansive and $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. Then, $\mathrm{Fix}(ST) = \mathrm{Fix}(TS) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.*

Remark 2.12. Let $T_n = (1-\theta_n)Id + \theta_n S_n$ with S_n nonexpansive on $E, \mathrm{Fix}(S_n) \neq \varnothing$ and $\theta_n \in (0,1)$. Then, the following statements are reached:

- (i) $\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$;
- (ii) T_n is nonexpansive.

Lemma 2.13. [29] *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n, \quad \forall\, n \geq 1, \tag{2.1}$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < 0$. Then, the following results hold:

- (i) *If $b_n \leq \delta_n M$ for some $M > 0$, then $\{a_n\}$ is a bounded sequence;*
- (ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

By employing the technique of Mainge [29], He and Yang [30] proved the following lemma.

Lemma 2.14. [30] *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\gamma)s_n + \gamma_n \tau_n, \quad \forall\, n \geq 1,$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad \forall\, n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\tau_n\}$, and $\{\rho_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\lim_{n\to\infty} \rho_n = 0$;
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$, *for any subsequence* $\{n_k\} \subset \{n\}$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.15. [31] *Let $q > 1$. Then, the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q} b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a and b.

3. Algorithm and convergence analysis

In this section, we start a new algorithm for solving (1.1), we know that a common zero of sum of two operators i.e., $A^{-1}0 \cap B^{-1}0$ which is a partial of a zero sum of two operators that is $(A+B)^{-1}0$ (i.e., $A^{-1}0 \cap B^{-1}0 \subset (A+B)^{-1}0$). Hence our problem different of [32] which is extended and improved more general.

Next, we will start our main result as follows:

Theorem 3.1. *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $A : E \to 2^E$ be an m-accretive operator and let $B : E \to E$ be an α-inverse strongly accretive mapping. Assume that $\Omega := (A+B)^{-1}(0) \neq \varnothing$. Let $f : E \to E$ be a fixed contraction with coefficient $k \in (0,1)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E$ and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J_{r_n}^A (Id - r_n B)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n}^A (Id - r_n B) y_n, \quad \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $J_{r_n}^A = (Id + r_n A)^{-1}$ and $\{r_n\}_{n=1}^{\infty}$ is a sequence of positive real number and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequence in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\liminf_{n\to\infty} \beta_n > 0$;
(iii) $0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \frac{2\alpha}{c}$

for some constant c. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z = Q_\Omega f(z)$, where Q_Ω is the unique sunny nonexpansive retraction of E onto Ω; that is, z solves the variational inequality

$$\langle (Id - f)z, j(z-x) \rangle \leq 0, \quad \forall x \in \Omega. \tag{3.2}$$

Proof. Denote by $S_n := J_{r_n}^A(Id - r_nB), T_n = (1-\beta_n)Id + \beta_n S_n$ and $U_n = S_nT_n$. Then, (3.1) can be rewritten as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)U_nx_n, \quad \forall n \geq 1, \tag{3.3}$$

Step 1. We prove that the sequence $\{x_n\}$ dened by (3.1) is well dened
For each $n \geq 1$, and let $S_n := J_{r_n}^A(Id - r_nB)$. Then, for all $x, y \in E$, we have

$$\begin{aligned}
\|S_nx - S_ny\|^2 &= \|J_{r_n}^A(Id - r_nB)x - J_{r_n}^A(Id - r_nB)y\|^2 \\
&\leq \|x - y - r_n(Bx - By)\|^2 \\
&\leq \|x-y\|^2 - 2r_n\langle Bx - By, j(x-y)\rangle + cr_n^2\|Bx - By\|^2 \\
&\leq \|x-y\|^2 - 2r_n\alpha\|Bx - By\|^2 + cr_n^2\|Bx - By\|^2 \\
&\leq \|x-y\|^2 - (2\alpha - cr_n)r_n\|Bx - By\|^2 \\
&\leq \|x-y\|^2.
\end{aligned} \tag{3.4}$$

Thus, S_n is nonexpansive for all $n \geq 1$.
Set $R_n = (Id - r_nB)$. Since B is α-inverse strongly accretive mapping, we have

$$\begin{aligned}
\|R_nx - R_ny\|^2 &= \|(Id - 2\alpha B)x - (Id - 2\alpha B)y\|^2 \\
&= \|(x-y) - 2\alpha(Bx - By)\|^2 \\
&= \|x-y\|^2 + 4\alpha^2\|Bx - By\|^2 - 4\alpha\langle x - y, Bx - By\rangle \\
&\leq \|x-y\|^2 + 4\alpha^2\|Bx - By\|^2 - 4\alpha^2\|Bx - By\|^2 \\
&= \|x-y\|^2.
\end{aligned} \tag{3.5}$$

Hence, R_n is nonexpansive. By a simple transformation, we have

$$\begin{aligned}
Id - r_nB &= (1 - \frac{r_n}{2\alpha})Id + \frac{r_n}{2\alpha}Id - r_nB \\
&= (1 - \frac{r_n}{2\alpha})Id + \frac{r_n}{2\alpha}(Id - 2\alpha B) \\
&= (1 - \frac{r_n}{2\alpha})Id + \frac{r_n}{2\alpha}R_n.
\end{aligned} \tag{3.6}$$

On the other hand, in view of $r_n \in (0, 2\alpha)$, $\frac{r_n}{2\alpha} \in (0,1)$. Consequently, $Id - r_nB$ is $\frac{r_n}{2\alpha}$-averaged. Set $L_n = 2J_{r_n}^A - Id$. Since A is m-accretive operator and $J_{r_n}^A$ is nonexpansive mapping, we have

$$\begin{aligned}
\|L_nx - L_ny\|^2 &= \|(2J_{r_n}^A - Id)x - (2J_{r_n}^A - Id)y\|^2 \\
&= \|(y-x) - 2(J_{r_n}^Ax - J_{r_n}^Ay)\|^2 \\
&= \|x-y\|^2 + 4\|J_{r_n}^Ax - J_{r_n}^Ay\|^2 - 4\langle y - x, J_{r_n}^Ax - J_{r_n}^Ay\rangle \\
&\leq \|x-y\|^2 + 4\|x-y\|^2 - 4\|x-y\|^2 \\
&= \|x-y\|^2.
\end{aligned} \tag{3.7}$$

Hence, L_n is nonexpansive. By a simple transformation, we have

$$\begin{aligned} J^A_{r_n} &= (1-\frac{1}{2})Id + \frac{1}{2}Id + J^A_{r_n} - Id \\ &= (1-\frac{1}{2})Id + \frac{1}{2}(2J^A_{r_n} - Id) \\ &= (1-\frac{1}{2})Id + \frac{1}{2}L_n. \end{aligned} \tag{3.8}$$

Consequently, $J^A_{r_n}$ is $\frac{1}{2}$-averaged. By Proposition 2.5, we have $S_n = J^A_{r_n}(Id - r_nB)$ is an $\frac{2\alpha}{4\alpha - r_n}$-averaged. For each $n \geq 1$, and let $T_n := (1-\beta_n)Id + \beta_n S_n$. Since S_n is a nonexpansive mapping, T_n is β_n-averaged. For each $n \geq 1$, and let $U_n := S_nT_n$. Then, for all $x, y \in E$, we have

$$\begin{aligned} \|U_nx - U_ny\| &= \|S_nT_nx - S_nT_ny\| \\ &\leq \|S_nx - S_ny\| \\ &\leq \|x-y\|. \end{aligned} \tag{3.9}$$

Thus, U_n is nonexpansive for all $n \geq 1$. By Proposition 2.5, we have $\theta_n = \frac{2\alpha - r_n\beta_n}{4\alpha - r_n - 2\alpha\beta_n} \in (0,1)$ such that $r_n \in (0, 2\alpha)$ and $\beta_n \in (0,1)$ and so, $U_n = S_nT_n$ is an θ_n-averaged. Let us define a mapping $W_n : E \to E$ as follows

$$W_n(x) := \alpha_n f(x) + (1-\alpha_n)U_nx. \tag{3.10}$$

For any $x, y \in E$, we have

$$\begin{aligned} \|W_nx - W_ny\| &= \|\alpha_n f(x) + (1-\alpha_n)U_nx - \alpha_n f(y) + (1-\alpha_n)U_ny\| \\ &\leq \alpha_n\|f(x) - f(y)\| + (1-\alpha_n)\|U_nx - U_ny\| \\ &\leq \alpha_n k\|x-y\| + (1-\alpha_n)\|x-y\| \\ &= (1-\alpha_n(1-k))\|x-y\| \\ &= (1-\epsilon)\|x-y\| \end{aligned} \tag{3.11}$$

for some $\epsilon \in (0,1)$. This implies that W_n is a contraction mapping. So, there exists a unique fixed point z_n of W_n. Note that $x_{n+1} := z_n$ satisfies (3.1).

Step 2. We prove that the sequence $\{x_n\}$ is bounded.
For each $n \in \mathbb{N}$, we put $U_nx = S_nT_nx$ where $S_n = J^A_{r_n}(Id - r_nB)$ and $T_n = (1-\beta_n)Id + \beta_nS_n$ and let $\{y_n\}$ be defined by

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)U_ny_n. \tag{3.12}$$

Firstly, we compute the following:

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|\alpha_n f(x_n) + (1-\alpha_n)U_nx_n - \alpha_n f(y_n) - (1-\alpha_n)U_ny_n\| \\ &\leq \alpha_n\|f(x_n) - f(y_n)\| + (1-\alpha_n)\|U_nx_n - U_ny_n\| \\ &\leq \alpha_n k\|x_n - y_n\| + (1-\alpha_n)\|x_n - y_n\| \\ &= (1-\alpha_n(1-k))\|x_n - y_n\|. \end{aligned} \tag{3.13}$$

By the assumption and Lemma 2.13 (ii), we conclude that $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Let $p \in E$, we have

$$\begin{aligned} p = S_n p &\Longleftrightarrow p = (Id + r_n A)^{-1}(Id - r_n B)p \\ &\Longleftrightarrow p - r_n Bp \in p + r_n Ap \\ &\Longleftrightarrow 0 \in Ap + Bp. \end{aligned} \tag{3.14}$$

Thus, $\mathrm{Fix}(S_n) = (A+B)^{-1}(0)$ for all $n \geq 1$.
Let $p \in E$, we have

$$\begin{aligned} p = T_n p &\Longleftrightarrow p = ((1-\beta_n)Id + \beta_n S_n)p \\ &\Longleftrightarrow p = (1-\beta_n)p + \beta_n S_n p \\ &\Longleftrightarrow p = S_n p. \end{aligned} \tag{3.15}$$

From (3.14), we have, $\mathrm{Fix}(T_n) = (A+B)^{-1}(0)$ for all $n \geq 1$. By Theorem 2.11, $\mathrm{Fix}(U_n) = \mathrm{Fix}(S_n T_n) = \mathrm{Fix}(S_n) \cap \mathrm{Fix}(T_n) = (A+B)^{-1}(0)$. Taking $p \in \Omega = (A+B)^{-1}(0) = \mathrm{Fix}(U_n)$, we next show that $\{y_n\}$ is bounded. Indeed,

$$\begin{aligned} \|y_{n+1} - p\| &= \|\alpha_n f(y_n) + (1-\alpha_n)U_n y_n - p\| \\ &\leq \alpha_n \|f(y_n) - p\| + (1-\alpha_n)\|U_n y_n - p\| \\ &\leq \alpha_n k\|y_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leq (1 - \alpha_n(1-k))\|y_n - p\| + \alpha_n \frac{\|f(p) - p\|}{1-k} \end{aligned} \tag{3.16}$$

for every $n \in \mathbb{N}$. Thus, by induction on n, we have

$$\|y_n - p\| \leq \max\{\|x_1 - p\|, \|f(p) - p\|/(1-k)\}. \tag{3.17}$$

This shows that $\{y_n\}$ bounded by Lemma 2.13 (i) and $\{x_n\}$ is also bounded.

Step 3. We prove that $x_n \to p = Qf(p) \in \Omega = (A+B)^{-1}(0) = \mathrm{Fix}(U_n)$.
Let $p \in \mathrm{Fix}(U_n) = \mathrm{Fix}(S_n T_n)$, we have $p \in \mathrm{Fix}(S_n)$ and $p \in \mathrm{Fix}(T_n)$. From Lemma 2.9 $(q = 2)$ and Lemma 2.15 and setting $w_n = T_n y_n$, we consider the following

$$\begin{aligned} \|w_n - p\|^2 &= \|T_n y_n - p\|^2 \\ &= \|(1-\beta_n)y_n + \beta_n S_n y_n - p\|^2 \\ &= \|(1-\beta_n)(y_n - p) + \beta_n(S_n y_n - p)\|^2 \\ &\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n \|S_n y_n - S_n p\|^2 \\ &\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|y_n - p\|^2 - \beta_n r_n(2\alpha - r_n c)\|By_n - Bp\|^2 \\ &\quad - \beta_n \phi(\|y_n - r_n Ay_n - S_n y_n + r_n Ap\|) \\ &\leq \|y_n - p\|^2 - \beta_n r_n(2\alpha - r_n c)\|By_n - Bp\|^2 \\ &\quad - \beta_n \phi(\|y_n - r_n Ay_n - S_n y_n + r_n Ap\|) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \|S_n w_n - p\|^2 \leq \|w_n - p\|^2 &- r_n(2\alpha - r_n c)\|Bw_n - Bp\|^2 \\ &- \phi(\|w_n - r_n Aw_n - S_n w_n + r_n Ap\|). \end{aligned} \tag{3.19}$$

By the definition of y_{n+1}, we have

$$\begin{aligned}
\|y_{n+1}-p\|^2 &= \|\alpha_n(f(y_n)-p)+(1-\alpha_n)(U_ny_n-p)\|^2 \\
&= \|(1-\alpha_n)(S_nw_n-p)+\alpha_n(f(y_n)-p)\|^2 \\
&\le (1-\alpha_n)\|S_nw_n-p\|^2+2\alpha_n\langle f(y_n)-p, j(y_{n+1}-p)\rangle \\
&\le (1-\alpha_n)\|S_nw_n-p\|^2+2\alpha_n\Big(\langle f(y_n)-f(p), j(y_{n+1}-p)\rangle \\
&\quad +\langle f(p)-p, j(y_{n+1}-p)\rangle\Big) \\
&\le (1-\alpha_n)\|S_nw_n-p\|^2+2\alpha_nk\|y_n-p\|\|y_{n+1}-p\| \\
&\quad +2\alpha_n\langle f(p)-p, j(y_{n+1}-p)\rangle \\
&\le (1-\alpha_n)\|S_nw_n-p\|^2+2\alpha_n\Big(\frac{1}{2}\|y_n-p\|^2+\frac{1}{2}\|y_{n+1}-p\|^2\Big) \\
&\quad +2\alpha_n\langle f(p)-p, j(y_{n+1}-p)\rangle.
\end{aligned} \tag{3.20}$$

After simplifying, it follows that

$$\begin{aligned}
\|y_{n+1}-p\|^2 \le{}& 1-\frac{\alpha_n(1-2k)}{1-\alpha_nk}\|y_n-p\|^2+\frac{2\alpha_n}{1-\alpha_nk}\langle f(p)-p, j(y_{n+1}-p)\rangle \\
&-(1-\alpha_n)\beta_nr_n(2\alpha-r_nc)\|By_n-Bp\|^2 \\
&-(1-\alpha_n)r_n(2\alpha-r_nc)\|Bw_n-Bp\|^2 \\
&-(1-\alpha_n)\beta_n\phi(\|y_n-r_nBy_n-S_ny_n+r_nBp\|) \\
&-(1-\alpha_n)\phi(\|w_n-r_nBw_n-S_nw_n+r_nBp\|).
\end{aligned} \tag{3.21}$$

We can check that $\alpha_n\Big(\dfrac{1-2k}{1-k\alpha_n}\Big)$ is in $(0,1)$, since $\{\alpha_n\}\subset(0,1)$ and $\lim_{n\to\infty}\alpha_n=0$. Moreover, by condition (ii), $\dfrac{2\alpha_n}{1-k\alpha_n}$ is a nonnegative real numbers. For each $n\ge 1$, we set

$$\begin{aligned}
& s_n := \|y_n-p\|^2; \\
& \gamma_n := \frac{\alpha_n(1-2k)}{1-k\alpha_n}; \\
& \tau_n := \frac{2}{1-2k}\langle f(p)-p, j(y_{n+1}-p)\rangle; \\
& \eta_n := (1-\alpha_n)\beta_nr_n(2\alpha-r_nc)\|By_n-Bp\|^2 \\
&\quad +(1-\alpha_n)\beta_n\phi(\|y_n-r_nBy_n-S_ny_n+r_nBp\|) \\
&\quad +(1-\alpha_n)r_n(2\alpha-r_nc)\|Bw_n-Bp\|^2 \\
&\quad +(1-\alpha_n)\phi(\|w_n-r_nBw_n-S_nw_n+r_nBp\|); \\
& \rho_n := \frac{2\alpha_n}{1-k\alpha_n}\langle f(p)-p, j(y_{n+1}-p)\rangle.
\end{aligned} \tag{3.22}$$

From (3.21), we have

$$s_{n+1}\le(1-\gamma_n)s_n+\gamma_n\tau_n, \quad \forall n\ge 1, \tag{3.23}$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad \forall n \geq 1. \tag{3.24}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. By the boundedness of $\{y_n\}$ and $\lim_{n\to\infty} \alpha_n = 0$, we see that $\lim_{n\to\infty} \rho_n = 0$. In order to complete proof, by using Lemma 2.13, it remains to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$, for any subsequence $\{n_k\} \subset \{n\}$.

Indeed, if $\{n_k\}$ is a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$, then by the assumptions that $r_n(2\alpha - r_n c) > 0$ and the property of ϕ, we can deduce that

$$\begin{cases} \lim_{k\to\infty} \|By_{n_k} - Bp\| = 0; \\ \lim_{k\to\infty} \|Bw_{n_k} - Bp\| = 0; \\ \lim_{k\to\infty} \|y_{n_k} - r_n By_{n_k} - S_{n_k} y_{n_k} + r_n Bp\| = 0; \\ \lim_{k\to\infty} \|w_{n_k} - r_n By_{n_k} - S_{n_k} w_{n_k} + r_n Bp\| = 0. \end{cases} \tag{3.25}$$

This implies, by the triangle inequality, that

$$\begin{cases} \lim_{k\to\infty} \|S_{n_k} y_{n_k} - y_{n_k}\| = 0; \\ \lim_{k\to\infty} \|S_{n_k} w_{n_k} - w_{n_k}\| = 0. \end{cases} \tag{3.26}$$

Since $w_n = (1-\beta_n)y_n + \beta_n S_n y_n$, we have

$$\begin{aligned} \|w_{n_k} - S_{n_k} y_{n_k}\| &= \|(1-\beta_{n_k})y_{n_k} + \beta_{n_k} S_{n_k} y_{n_k} - S_{n_k}\| \\ &\leq (1-\beta_{n_k})\|S_{n_k} y_{n_k} - y_{n_k}\|. \end{aligned} \tag{3.27}$$

This together with (3.26) and (3.27) implies that

$$\begin{aligned} \|U_{n_k} y_{n_k} - y_{n_k}\| &= \|S_{n_k} w_{n_k} - y_{n_k}\| \\ &\leq \|S_{n_k} w_{n_k} - w_{n_k}\| + \|w_{n_k} - S_{n_k} y_{n_k}\| + \|S_{n_k} y_{n_k} - y_{n_k}\| \\ &\leq \|S_{n_k} w_{n_k} - w_{n_k}\| + (1-\beta_{n_k})\|S_{n_k} y_{n_k} - y_{n_k}\| + \|S_{n_k} y_{n_k} - y_{n_k}\| \\ &\to 0 \quad (\text{as } k \to \infty). \end{aligned} \tag{3.28}$$

Since $\liminf_{k\to\infty} r_n > 0$, there exists $r > 0$ such that $r_n \geq r$, for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$.

$$\begin{aligned} \|w_{n_k} - y_{n_k}\| &= \|(1-\beta_{n_k})y_{n_k} + \beta_{n_k} S_{n_k} y_{n_k} - y_{n_k}\| \\ &\leq \beta_{n_k}\|S_{n_k} y_{n_k} - y_{n_k}\|. \end{aligned} \tag{3.29}$$

Lemma 2.10 (ii) yields that

$$\begin{aligned} \|S_r^{A,B} w_{n_k} - y_{n_k}\| &\leq \|S_r^{A,B} w_{n_k} - w_{n_k}\| + \|w_{n_k} - y_{n_k}\| \\ &\leq 2\|S_{n_k} w_{n_k} - y_{n_k}\| + \beta_{n_k}\|S_{n_k} y_{n_k} - y_{n_k}\|. \end{aligned} \tag{3.30}$$

Then, by (3.26) and (3.28), we obtain

$$\limsup_{k\to\infty} \|S_r^{A,B} w_{n_k} - y_{n_k}\| \leq 2 \lim_{k\to\infty} \|S_{n_k} w_{n_k} - y_{n_k}\| + \lim_{k\to\infty} \beta_{n_k}\|S_{n_k} y_{n_k} - y_{n_k}\|. \tag{3.31}$$

It follows that

$$\lim_{k\to\infty} \|S_r^{A,B} w_{n_k} - y_{n_k}\| = \lim_{k\to\infty} \|U_r^{A,B} y_{n_k} - y_{n_k}\| = 0. \tag{3.32}$$

Let

$$z_t = tf(z_t) + (1-t)U_r^{A,B} z_t, \quad t \in (0,1). \tag{3.33}$$

By Lemma 2.8, $\{z_t\}$ converges strongly as $t \to 0$ to the unique fixed point $z = Qf(z) \in \mathrm{Fix}(S_n) = (A+B)^{-1}(0)$, where $Q : E \to \mathrm{Fix}(S_n)$ is the unique sunny nonexpansive retraction from E onto $\mathrm{Fix}(S_n) = (A+B)^{-1}(0)$. So we obtain

$$\begin{aligned}
\|z_t - y_{n_k}\|^2 &= \|t(f(z_t) - y_{n_k}) + (1-t)(U_r^{A,B} z_t - y_{n_k})\|^2 \\
&\leq (1-t)^2 \|U_r^{A,B} z_t - y_{n_k}\|^2 + 2t\langle f(z_t) - z_t, j(z_t - y_{n_k})\rangle \\
&\quad + 2t\langle z_t - x_{n_k}, j(z_t - x_{n_k})\rangle \\
&\leq (1-t)^2 (\|U_r^{A,B} z_t - U_r^{A,B} y_{n_k}\| + \|U_r^{A,B} y_{n_k} - t_{n_k}\|)^2 \\
&\quad + 2t\langle f(z_t) - z_t, j(z_t - y_{n_k})\rangle + 2t\|z_t - y_{n_k}\|^2 \\
&\leq (1-t)^2 (\|z_t - y_{n_k}\| + \|U_r^{A,B} y_{n_k} - t_{n_k}\|)^2 \\
&\quad + 2t\langle f(z_t) - z_t, j(z_t - y_{n_k})\rangle + 2t\|z_t - y_{n_k}\|^2.
\end{aligned} \tag{3.34}$$

After simplifying we have

$$\begin{aligned}
&\langle z_t - f(z_t), j(z_t - y_{n_k})\rangle \\
&\quad \leq \frac{(1-t)^2}{2t} (\|z_t - y_{n_k}\| + \|U_r^{A,B} y_{n_k} - y_{n_k}\|)^2 + \frac{(2t-1)}{2t}\|z_t - x_{n_k}\|^2.
\end{aligned} \tag{3.35}$$

It follows from (3.34) and (3.28) that

$$\limsup_{k\to\infty} \langle z_t - f(z_t), j(z_t - y_{n_k})\rangle \leq \frac{1}{2t}[(1-t)^2 + (2t-1)]M^2, \tag{3.36}$$

where $M = \sup_{k\geq 0, t\in(0,1)} \|z_t - y_{n_k}\|$. Since $\lim_{t\to 0} \frac{1}{2t}[(1-t)^2 + (2t-1)] = 0$, $z_t \to z = Qf(z)$ as $t \to 0$ and by Lemma 2.6 (ii), we know that j is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\|j(z_t - y_{n_k}) - j(z - y_{n_k})\| \to 0 \quad (\text{as } t \to 0). \tag{3.37}$$

Observe that

$$\begin{aligned}
&|\langle z_t - f(z_t), j(z_t - y_{n_k})\rangle - \langle z - f(z_t), j(z - y_{n_k})\rangle| \\
&\quad = |\langle z_t - z + z - f(z_t), j(z_t - y_{n_k})\rangle - \langle z - f(z_t), j(z - y_{n_k})\rangle| \\
&\quad \leq |\langle z_t - z, j(z_t - y_{n_k})\rangle| + |\langle z - f(z_t), j(z_t - y_{n_k}) - j(z - y_{n_k})\rangle| \\
&\quad \leq \|z_t - z\|\|z_t - y_{n_k}\| + \|z - f(z_t)\|\|j(z_t - y_{n_k}) - j(z - y_{n_k})\|.
\end{aligned} \tag{3.38}$$

This together with (3.36) and (3.37) shows that

$$\begin{aligned}
\limsup_{k\to\infty} \langle z - f(z), j(z - y_{n_k})\rangle &= \limsup_{t\to 0} \limsup_{k\to\infty} \langle z - f(z_t), j(z_t - y_{n_k})\rangle \\
&= \limsup_{t\to 0} \limsup_{k\to\infty} \langle z_t - f(z_t), j(z_t - y_{n_k})\rangle \\
&= 0.
\end{aligned} \tag{3.39}$$

From $\{y_n\}$ is bounded, and so is $\{f(y_n)\}$, by condition (i), (3.12) and (3.28), we have

$$\begin{aligned} \|y_{n_k+1} - y_{n_k}\| &= \|\alpha_{n_k} f(y_{n_k}) + (1-\alpha_{n_k})U_{n_k} y_{n_k} - y_{n_k}\| \\ &\leq \alpha_{n_k}\|f(y_{n_k}) - y_{n_k}\| + (1-\alpha_{n_k})\|U_{n_k} y_{n_k} - y_{n_k}\| \\ &\to 0, \end{aligned} \tag{3.40}$$

as $k \to \infty$. By combining (3.39) and (3.40), we get that

$$\limsup_{k\to\infty}\langle z - f(z), j(z - y_{n_k+1})\rangle \leq 0. \tag{3.41}$$

This implies that $\limsup_{k\to\infty} \tau_{n_k} \leq 0$. We conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.14, $y_n \to z$ (as $n \to \infty$), by the boundedness of $\{y_n\}$, we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$, so $\lim_{n\to\infty} x_n = z \in \Omega$. This complete the proof of Theorem 3.1. ∎

If the mapping f maps every point in E to a fixed element, then we have the following result.

Corollary 3.2. *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $A : E \to 2^E$ be an m-accretive operator and let $B : E \to E$ be an α-inverse strongly accretive mapping. Assume that $\Omega := (A+B)^{-1}(0) \neq \varnothing$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive real number and suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequence in $(0,1)$ satisfying the following conditions:*

- (i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
- (ii) $\liminf_{n\to\infty}\beta_n > 0$;
- (iii) $0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < \frac{2\alpha}{c}$

for some constant c. For a fixed element $u \in E$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $u, x_1 \in E$,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J_{r_n}^A (Id - r_n B)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n}^A (Id - r_n B) y_n, \quad \forall n \geq 1, \end{cases} \tag{3.42}$$

where $J_{r_n}^A = (Id + r_n A)^{-1}$. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $z = Q_\Omega u$, where Q_Ω is the unique sunny nonexpansive retraction of E onto Ω; that is, p solves the variational inequality

$$\langle z - u, j(z - x)\rangle \leq 0, \quad \forall x \in \Omega. \tag{3.43}$$

As well known, if H is a real Hilbert space, then it is a uniformly convex and 2-uniformly smooth Banach space, the 2-uniform smoothness coefficient $c = 1$. And the monotonicity coincides with the accretivity. Hence from Theorem (3.1) we can obtain the following result.

Theorem 3.3. *Let H be a real Hilbert space, $A : H \to 2^H$ be a maximal monotone operator and let $B : H \to H$ be an α-inverse strongly monotone operator. Assume that $\Omega := (A+B)^{-1}(0) \neq \varnothing$. Let $f : H \to H$ be a fixed contraction with coefficient $k \in (0,1)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J_{r_n}^A (Id - r_n B)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n}^A (Id - r_n B) y_n, \quad \forall n \geq 1, \end{cases} \tag{3.44}$$

where $J_{r_n}^A = (Id + r_n A)^{-1}$ and $\{r_n\}_{n=1}^{\infty}$ is a sequence of positive real number and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequence in $(0,1)$ satisfying the following conditions:

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(*ii*) $\liminf_{n\to\infty} \beta_n > 0$;
(*iii*) $0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\alpha$.

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $z = P_\Omega f(z)$, *where* P_Ω *is the metric projection of* H *onto* Ω; *that is,* z *solves the variational inequality*

$$\langle (Id - f)z, z - x\rangle \leq 0, \quad \forall x \in \Omega. \tag{3.45}$$

Corollary 3.4. *Let* H *be a real Hilbert space,* $A : H \to 2^H$ *be a maximal monotone operator and let* $B : H \to H$ *be an* α*-inverse strongly monotone operator. Assume that* $\Omega := (A+B)^{-1}(0) \neq \varnothing$. *Let* $\{x_n\}$ *be the sequence generated by* $u, x_1 \in H$ *and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J_{r_n}^A (Id - r_n B)x_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n}^A (Id - r_n B) y_n, \quad \forall n \geq 1, \end{cases} \tag{3.46}$$

where $J_{r_n}^A = (Id + r_n A)^{-1}$ *and* $\{r_n\}_{n=1}^{\infty}$ *is a sequence of positive real number and* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequence in* $(0,1)$ *satisfying the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(*ii*) $\liminf_{n\to\infty} \beta_n > 0$;
(*iii*) $0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\alpha$.

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $z = P_\Omega u$, *where* P_Ω *is the metric projection of* H *onto* Ω; *that is,* z *solves the variational inequality*

$$\langle z - u, z - x\rangle \leq 0, \quad \forall x \in \Omega. \tag{3.47}$$

4. Computational experiments

Example 4.1. Let $A : l_3 \to l_3$ be defined by $Ax = 10x$ and let $B : l_3 \to l_3$ be defined by $Bx = 4x + (1,1,1,0,0,0,...)$, where $x = (x_1, x_2, x_3, ...) \in l_3$. Find $x \in l_3$ such that $0 \in Ax + Bx$.

We see that A is an m-accretive and B is a $1/4$-isa operator. Indeed, let $x, y \in l_3$, then

$$\langle Ax - Ay, j(x-y)\rangle = 10\|x-y\|_{l_3}^2 \geq 0. \tag{4.1}$$

We also have

$$\begin{aligned} \langle Bx - By, j(x-y)\rangle &= \langle 4x - 4y, j(x-y)\rangle \\ &= 4\|x-y\|_{l_3}^2 \\ &= \frac{1}{4}\|Ax - Ay\|_{l_3}^2 \end{aligned} \tag{4.2}$$

and $R(Id + rA) = l_3$ for all $r > 0$. By a direct calculation, we have for $r > 0$

$$\begin{aligned} J_r^A(x - rBx) &= (Id + rA)^{-1}(x - rBx) \\ &= \frac{1-4r}{1+10r}x - \frac{r}{1+10r}(1,1,1,0,0,0,...), \end{aligned} \tag{4.3}$$

where $x = (x_1, x_2, x_3, 0, 0, 0, ...) \in l_3$. Since, in l_3 and $\alpha = 1/4$, we can choose $r_n = 0.1$ for all $n \in \mathbb{N}$. Let $\beta_n = 0.01$ and $\alpha_n = 0.001$. Starting $x_1 = (4, 10, 7, 0, 0, 0, ...)$ and computing iteratively algorithm (3.1) in Theorem 3.1 and algorithm (1.6), we obtain the following numerical results. From Table 1, the solution is $(-0.07143, -0.07143, -0.07143, 0, 0, 0, ...)$.

TABLE 1. Numerical results of Example 4.1 for iteration process (3.1) and (1.6)

n	x_n in algorithm (3.1)	error in (3.1)	error in (1.6)
1	(4.00000, 10.0000, 7.00000, 0, 0, 0, ...)	-	-
2	(0.30355, 0.85615, 0.57985, 0, 0, 0, ...)	1.17683E+01	9.07345E+00
3	(-0.03689, 0.01400, -0.01144, 0, 0, 0, ...)	1.08386E+00	2.72204E+00
4	(-0.06825, -0.06356, -0.06590, 0, 0, 0, ...)	9.98239E-02	8.16611E-01
5	(-0.07113, -0.07070, -0.07092, 0, 0, 0, ...)	9.19382E-03	2.44983E-01
6	(-0.07140, -0.07136, -0.07138, 0, 0, 0, ...)	8.46753E-04	7.34949E-02
7	(-0.07143, -0.07142, -0.07142, 0, 0, 0, ...)	7.79863E-05	2.20485E-02
8	(-0.07143, -0.07143, -0.07143, 0, 0, 0, ...)	7.18256E-06	6.61455E-03
9	(-0.07143, -0.07143, -0.07143, 0, 0, 0, ...)	6.61516E-07	1.98436E-03
10	(-0.07143, -0.07143, -0.07143, 0, 0, 0, ...)	6.09259E-08	5.95309E-04
11	(-0.07143, -0.07143, -0.07143, 0, 0, 0, ...)	5.61129E-09	1.78593E-04

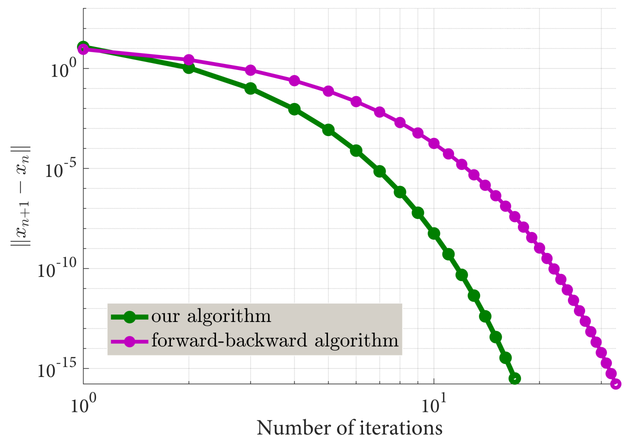


FIGURE 1. Error of $\|x_{n+1} - x_n\|_{l_3}$ of different algorithm shown in Table. 1

5. Application

In this section, we shall utilize the generalized viscosity implicit rules presented in the paper to study the convex minimization problem and convexly constrained linear inverse problem. Throughout this section, let C be a nonempty closed and convex subset of a real Hilbert space H. Note that in this case the concept of monotonicity coincides with the concept of accretivity.

5.1. Application to the convex minimization problem

Let $h : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R}$ be a proper convex and lower-semicontinuous function. We consider the following convex minimization problem of finding $x^* \in H$ such that

$$h(x^*) + g(x^*) = \min_{x \in H} \Big\{ h(x) + g(x) \Big\}. \tag{5.1}$$

This problem (5.1) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \tag{5.2}$$

where ∇h is a gradient of h and ∂g is a subdifferential of g. Set $B = \nabla h$ and $A = \partial g$ in Theorem 3.3. If ∇h is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone. Moreover, ∂g is maximal monotone. Hence from Theorem 3.3 we have the following result.

Theorem 5.1. *Let $h : H \to \mathbb{R}$ be a convex and differentiable function with $(1/L)$-Lipschitz continuous gradient ∇h and $g : H \to \mathbb{R}$ be a proper convex and lower semi-continuous function such that $h+g$ attains a minimizer. Let $f : H \to H$ be a k-contractive mapping with $k \in (0,1), \{\beta_n\}$ and $\{\alpha_n\}$ be sequence in $(0,1)$. Let $\{x_n\}$ be the sequence generated by $x_1 \in H$ and*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n J_{r_n}^{\partial g}(Id - r_n\nabla h)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)J_{r_n}^{\partial g}(Id - r_n\nabla h)y_n, \quad \forall n \geq 1, \end{cases} \tag{5.3}$$

where $J_{r_n}^{\partial g} = (Id + r_n\partial g)^{-1}$ and $\{r_n\}_{n=1}^{\infty}$ is a sequence of positive real number and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequence in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} \beta_n > 0$;

(iii) $0 < \liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2L$.

Then, $\{x_n\}$ strongly converges to a minimizer of $h+g$.

5.2. Application to Signal Processing

For consider some applications of our algorithm to inverse problems occurring from signal processing:

$$b = Ax + v, \tag{5.4}$$

where $x \in \mathbb{R}^n$ is recovered, $b \in \mathbb{R}^k$ is noisy observations, $A : \mathbb{R}^n \to \mathbb{R}^k$ is a bounded linear observation operator. It determines a process with loss of information. For finding solutions of the linear inverse problems (5.4), a successful one of some models is the convex unconstrained minimization problem, see more detail in [34].

5.3. Application to image restoration problems

In this section, we apply our algorithm to image deblurring. General image recovery problem can be formulated by the inversion of the following observation model:

$$b = Ax + v, \tag{5.5}$$

where $x \in \mathbb{R}^n$, v and b are unknown original image, unknown additive random noise and known degraded observation, respectively. A linear operator A depends on the concerned image recovery problem.

This is approximately equivalent to several different formulations available for optimization problems. In the literature, there is a growing interest in using l_1 norm for solving these types of problems. The l_1 regularization can remove noise in the restoration process that it is given by (see [33])

$$\min_x \left\{ \frac{1}{2}\|Ax - b\|_2^2 + \lambda_n\|x\|_1 \right\}, \tag{5.6}$$

Next, an iteration is used to find the solution of the following convex minimization problem:

$$\text{Find } x \in \underset{x\in\mathbb{R}^n}{\arg\min} \left\{ \frac{1}{2}\|Ax-b\|_2^2 + \lambda_n \|x\|_1 \right\}, \tag{5.7}$$

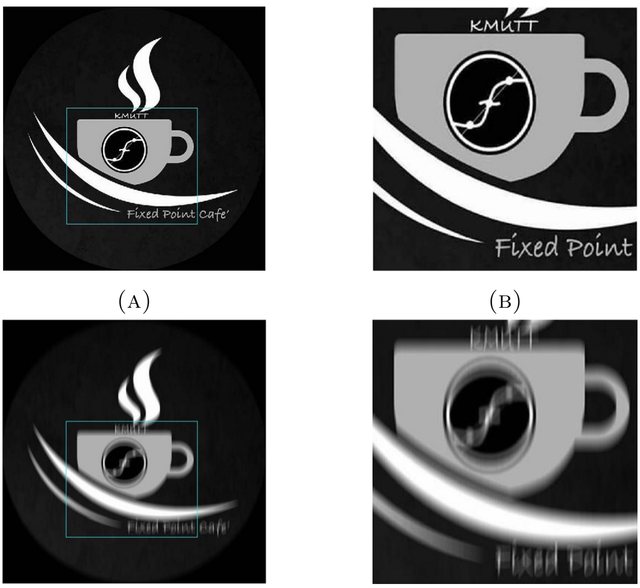
where b is the degraded image, and A an operator representing the mask. Therefore, we use our Theorem 5.1 to solve (5.7). We set $g(x) = \|x\|_1$, $h(x) = \frac{1}{2}\|Ax-b\|_2^2$ and $\lambda_n = 0.023$. We define the gradient as:

$$\nabla h(x) = A^*(Ax-b).$$

The image went through a random blur and random noise. It is followed from Theorem 5.1, we set $f(x) = \frac{x}{8}, \beta_n = 0.1$ and $\alpha_n = 1/(10n+1)$. In algorithm (1.6), we set $\lambda_n = 0.1$. The improvement in signal to noise ratio (ISNR) is used to measure the quality of the restored images. They are defined as follows:

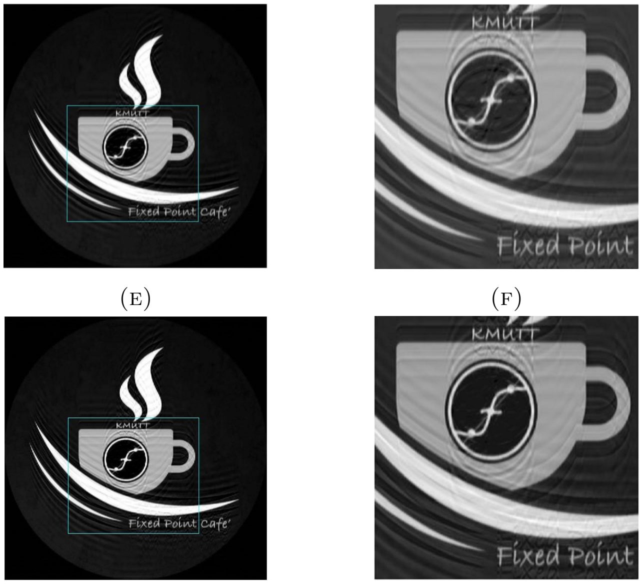
$$\text{ISNR} = 10\log\frac{\|x-b\|_2^2}{\|x-x_n\|_2^2},$$

where x, b and x_n are the original image, observed and estimated image at iteration n, respectively. The original, observed and restored images are given in Figure 2 (grey image) and 4 (color image).



(A) (B)

(C) (D)



(E) (F)

(G) (H)

FIGURE 2. Figure (A) shows the original image, (B) shows the crop original image, figure (C) shows the image degraded by a random blur and random noise, figure (D) shows the crop image degraded, figure (E) shows the restored image by forward-backward (1.6), figure (F) shows the crop restored image by forward-backward, figure (G) shows the restored image by our algorithm (5.3) and figure (H) shows the crop restored image by our algorithm.

Next, we compare algorithms between our algorithm and forward-backward in Table 2 and Figure 3.

TABLE 2. Numerical results of ISNR in Figure 2.

n	The improvement in signal to noise ratio (ISNR)	
	forward-backward algorithm (1.6)	our algorithm (5.3)
1	0.25040	0.12804
10	1.26985	1.55567
20	1.97277	2.59079
30	2.51430	3.32105
40	2.95140	3.86967
50	3.31486	4.30754
60	3.62476	4.67321
70	3.89473	4.98815
80	4.13418	5.26502
90	4.34968	5.51188
100	4.54594	5.73422

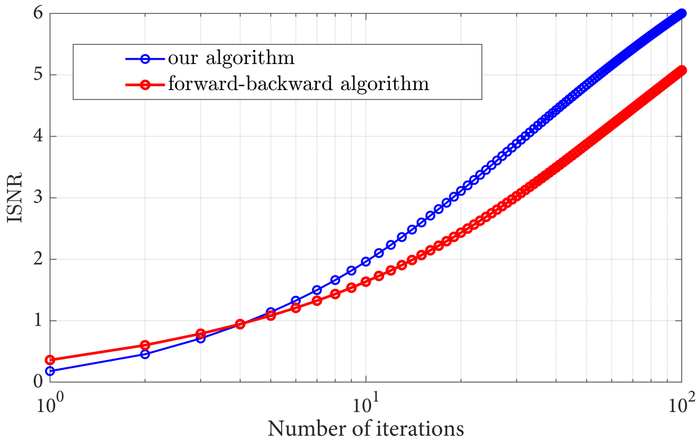
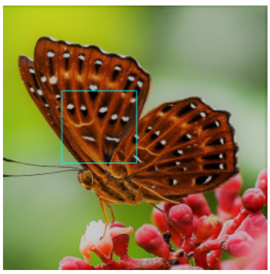
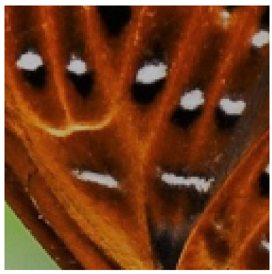


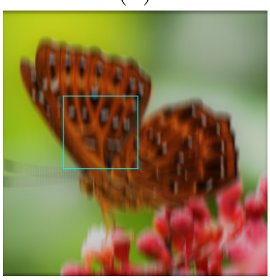
FIGURE 3. ISNR of different algorithm shown in Figure 2



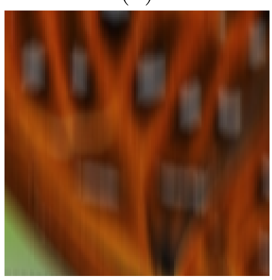
(A)



(B)

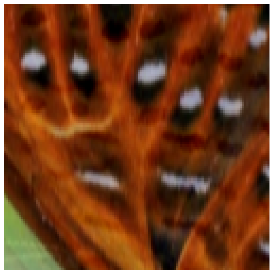


(C)



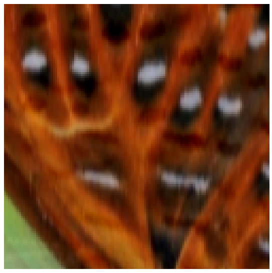
(D)

(E)



(F)

(G)



(H)

FIGURE 4. Figure (A) shows the original image, (B) shows the crop original image, figure (C) shows the image degraded by a random blur and random noise, figure (D) shows the crop image degraded, figure (E) shows the restored image by forward-backward (1.6), figure (F) shows the crop restored image by forward-backward, figure (G) shows the restored image by our algorithm (5.3) and figure (H) shows the crop restored image by our algorithm.

In addition, we use two state-of-the-art metrics for image quality: the structural similarity index (SSIM) defined by

$$\mathrm{SSIM}(x, x_n) = \frac{(2\mu_x \mu_{x_n} + C_1)(2\sigma_{x x_n} + C_2)}{(\mu_x^2 + \mu_{x_n}^2 + C_1)(\sigma_x^2 + \sigma_{x_n}^2 + C_2)},$$

where μ_x and μ_{x_n} are averages of x and x_n respectively, σ_x and σ_{x_n} are the variance of x and x_n respectively and $\sigma_{x x_n}$ is the covariance of x and x_n. The positive constants C_1 and C_2 can be thought of as stabilizing constants for near-zero denominator values. The one important property of SSIM is

$$\lim_{n\to\infty} \mathrm{SSIM}(x, x_n) = 1 \text{ if and only if } \lim_{n\to\infty} x_n = x.$$

All algorithms are implemented under Windows 10 and MATLAB 2017b running on a Dell laptop with Intel(R) Core(TM) i5 CPU and 4 GB of RAM. The stopping criterion of the algorithm is

$$\frac{\|x_{n+1} - x_n\|_2}{\|x_{n+1}\|_2} < 10^{-4}.$$

Finally, we use method SSIM for comparing between our algorithm and forward-backward in Figure 5.

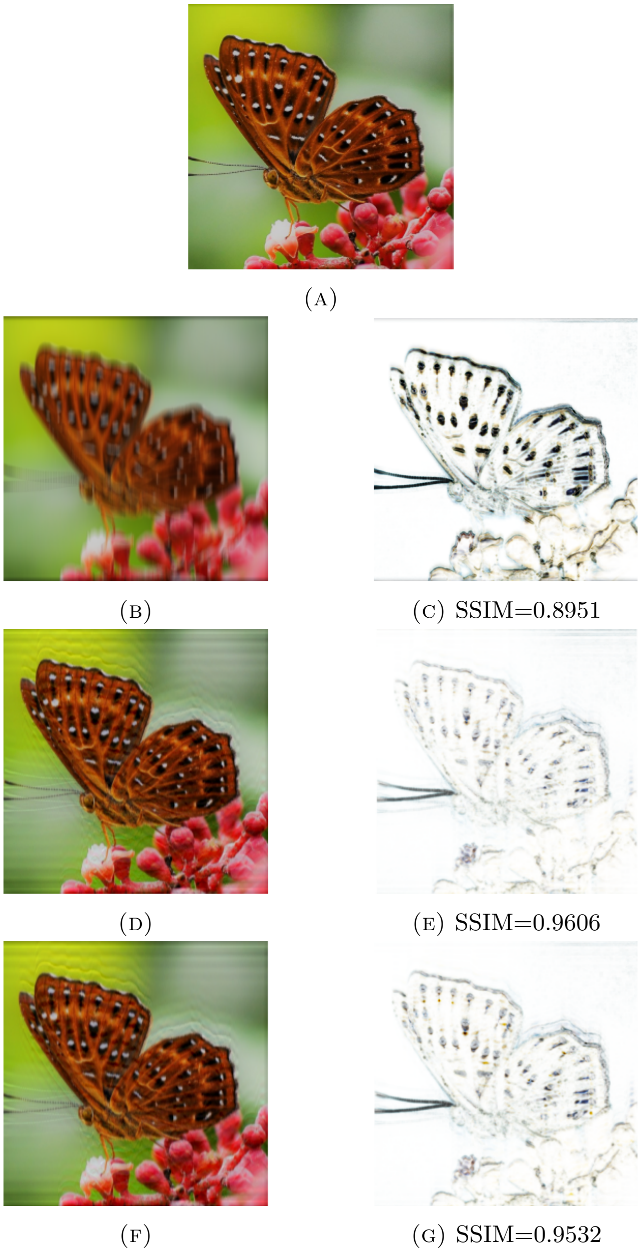


FIGURE 5. Figure (A) shows the original image, figure (B) shows the image degraded by a random blur and random noise, figure (C) shows the SSIM of figure (B) image, figure (D) shows the restored image by forward-backward (1.6), figure (E) shows the SSIM of figure (D) image, figure (F) shows the restored image by our algorithm (5.3) and figure (G) shows the SSIM of figure (F) image.

6. Conclusion

In this work, we introduce a new iterative method that is a combination of the modified Mann type forward-backward splitting with the viscosity approximation method and the alternating resolvent method for finding the zero of sum of m-accretive operators in uniformly convex real Banach spaces. The results obtain in this paper extend many recent ones in the literature.

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