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Admissible Classes of Multivalent Functions with Higher Order Derivatives

Tamer M. Seoudy

Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt
Jamoum University College,Umm Al-Qura University, Makkah, Saudi Arabia
e-mail : tms00@fayoum.edu.eg; tmsaman@uqu.edu.sa

Abstract In this paper we investigate some applications of the differential subordination and superordination of certain admissible classes of multivalent functions with higher order derivatives in the open unit disk $\mathbb{U}$. Several differential sandwich-type results are also obtained.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{H}[a, n]$ the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with $\mathcal{H} = \mathcal{H}[1, 1]$. Also let $\mathcal{A}(p)$ be the class of all analytic and $p-$valent functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\};\ z \in \mathbb{U}). \tag{1.1}$$

Upon differentiating $j-$times both sides of (1.1) we obtain

$$f^{(j)}(z) = \delta(p, j) z^{p-j} + \sum_{n=p+1}^{\infty} \delta(n, q) a_n z^{n-j} \qquad (z \in \mathbb{U}),$$

where

$$\delta(p, j) = \frac{p!}{(p-j)!} \quad (p \in \mathbb{N},\ j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ p \geq j).$$

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Several research has investigated higher order derivatives of multivalent functions, see (for example [2], [5], [6] and [17]). For $f, F \in \mathcal{H}(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $F(z)$, or the function $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1\,(z \in \mathbb{U})$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent in $\mathbb{U}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (see [12], [19] and [20]).

We denote by $\mathcal{F}$ the set of all functions q that are analytic and injective on $\bar{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0\,(\zeta \in \partial\mathbb{U} \setminus E(q))$. We further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$ and write $\mathcal{F}(1) \equiv \mathcal{F}_1$.

In order to prove our results, we shall make use of The following classes of admissible functions.

Definition 1.1. (see [19, p. 27, Definition 2.3a]) Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{F}$ and $n \in \mathbb{N}$. The class $\Psi_n[\Omega, q]$ of admissible functions consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\zeta), \quad s = k\zeta q'(\zeta) \text{ and } \Re\left\{\frac{t}{s} + 1\right\} \geqq k\Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq n$. We write $\Psi_1[\Omega, q]$ simply as $\Psi[\Omega, q]$.

In particular, if $q(z) = M\frac{Mz+a}{M+\bar{a}z}\,(M > 0, |a| < M)$, then $q(\mathbb{U}) = \mathbb{U}_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \varnothing$ and $q \in \mathcal{F}(a)$. In this case, we set $\Psi_n[\Omega, q] = \Psi_n[\Omega, M, a]$. Moreover, in the special case when we set $\Omega = \mathbb{U}_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.2. (see [20, p. 817, Definition 3]) Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class $\Psi'_n[\Omega, q]$ of admissible functions consists of those functions $\psi : \mathbb{C}^3 \times \bar{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t; \zeta) \in \Omega$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \text{ and } \Re\left\{\frac{t}{s} + 1\right\} \leqq \frac{1}{m}\Re\left(1 + \frac{zq''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq n \geqq 1$. In particular, we write $\Psi'_1[\Omega, q]$ simply as $\Psi'[\Omega, q]$.

In our investigation we need the following lemmas which are proved by Miller and Mocanu (see [19] and [20]).

Lemma 1.3. *(see [19, p. 28, Theorem 2.3b]) Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $g(z)$ given by*

$$g(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

satisfies the following inclusion relationship:

$$\psi\left(g(z), zg'(z), z^2 g''(z); z\right) \in \Omega,$$

then $g \prec q$.

Lemma 1.4. *(see [20, p. 818, Theorem 1]) Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If*

$$\psi\left(g(z), zg'(z), z^2 g''(z); z\right)$$

is univalent function in $\mathbb{U}$ *such that* $g \in \mathcal{F}(a)$, *then*

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2 g''(z); z\right) : z \in \mathbb{U}\right\},$$

implies that $q \prec g$.

In this paper, we determine the sufficient conditions for certain admissible classes of multivalent functions so that

$$q_1(z) \prec \frac{f^{(j)}(z)}{\delta(p,j) z^{p-j}} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$. In addition, we derive sandwich-type results. A similar problem for analytic functions involving certain operators was studied by Aghalary et al. [1], Ali et al. [3], Kim and Srivastava [18], and other authors (see also, [4, 7–10, 13–16, 21])

2. Subordination Results

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}$, $j \in \mathbb{N}_0$, $p > j + 1$, and $z \in \mathbb{U}$.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $g \in \mathcal{F}_1 \cap \mathcal{H}$. The class $\Phi[\Omega, q, p, j]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (p-j) r}{p-j}$$

and

$$\Re\left(\frac{(p-j-1)(w-2v+u)}{v-u} - 1\right) \geqq k\Re\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq 1$. For simplicity, we write $\Phi[\Omega, q, p, 0] = \Phi[\Omega, q, p]$.

Theorem 2.2. *Let* $\phi \in \Phi[\Omega, q, p, j]$. *If* $f \in \mathcal{A}(p)$ *satisfies the following condition:*

$$\left\{\phi\left(\frac{f^{(j)}(z)}{\delta(p,j) z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1) z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2) z^{p-j-2}}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \quad (2.1)$$

then

$$\frac{f^{(j)}(z)}{\delta(p,j) z^{p-j}} \prec q(z).$$

Proof. We begin by defining the analytic function g in $\mathbb{U}$ by

$$g(z)=\frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}}\qquad(z\in\mathbb{U}).\tag{2.2}$$

Then, in view of (2.2), we get

$$\frac{f^{(j+1)}(z)}{\delta(p,j+1)\,z^{p-j-1}}=g(z)+\frac{zg'(z)}{p-j}.\tag{2.3}$$

and

$$\frac{f^{(j+2)}(z)}{\delta(p,j+2)\,z^{p-j-2}}=g(z)+\frac{2zg'(z)}{p-j-1}+\frac{z^2g''(z)}{(p-j)(p-j-1)}.\tag{2.4}$$

Now, we define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ \ v=\frac{s+(p-j)\,r}{p-j}\ \text{and}\ w=\frac{t+2(p-j)\,s+(p-j)(p-j-1)\,r}{(p-j)(p-j-1)}.\tag{2.5}$$

Suppose that

$$\psi(r,s,t;z)=\phi(u,v,w;z)=\phi\left(r,\tfrac{s+(p-j)r}{p-j},\tfrac{t+2(p-j)s+(p-j)(p-j-1)r}{(p-j)(p-j-1)};z\right).\tag{2.6}$$

The proof shall make use of Lemma 1.3. Indeed, by using (2.2) to (2.6), we obtain

$$\psi\left(g(z),zg'(z),z^2g''(z);z\right)=\phi\left(\tfrac{f^{(j)}(z)}{\delta(p,j)z^{p-j}},\tfrac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}},\tfrac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}};z\right).\tag{2.7}$$

Hence (2.1) becomes

$$\psi\left(g(z),zg'(z),z^2g''(z);z\right)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi[\Omega,q,p,j]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. We note that

$$\frac{t}{s}+1=\frac{(p-j-1)(w-2v+u)}{v-u}-1,$$

and hence $\psi\in\Psi_1[\Omega,q]$. By Lemma 1.3, we thus obtain

$$g(z)\prec q(z)\qquad\text{or}\qquad\frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}}\prec q(z).$$

which evidently proves Theorem 2.2. ∎

If $\Omega\neq\mathbb{C}$ is a simply-connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi[h(\mathbb{U}),q,p,j]$ is written, for convenience, as $\Phi[h,q,p,j]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3. *Let $\phi\in\Phi[h,q,p,j]$. If $f\in\mathcal{A}(p)$ satisfies the following condition:*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}},\frac{f^{(j+1)}(z)}{\delta(p,j+1)\,z^{p-j-1}},\frac{f^{(j+2)}(z)}{\delta(p,j+2)\,z^{p-j-2}};z\right)\prec h(z),\tag{2.8}$$

then

$$\frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}}\prec q(z).$$

Putting $j = 0$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let $\phi \in \Phi[h, q, p]$ with $p > 1$. If $f \in \mathcal{A}(p)$ satisfies the following condition*:

$$\phi\left(\frac{f(z)}{z^p}, \frac{f'(z)}{pz^{p-1}}, \frac{f''(z)}{p(p-1)z^{p-2}}; z\right) \prec h(z),$$

then

$$\frac{f(z)}{z^p} \prec q(z).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ and suppose that the function q is univalent in $\mathbb{U}$ with $q(0) = 1$. Also let $\phi \in \Phi[\Omega, q_\rho, p, j]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}(p)$ and*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}}; z\right) \in \Omega,$$

then

$$\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}} \prec q(z).$$

Proof. Theorem 2.2 readily yields

$$\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}} \prec q_\rho(z).$$

The asserted result is now deduced from the fact that $q_\rho(z) \prec q(z)$. ∎

Theorem 2.6. *Let the functions h and q be univalent in $\mathbb{U}$, with $q(0) = 1$, and set*

$$q_\rho(z) = q(\rho z) \quad \text{and} \quad h_\rho(z) = h(\rho z).$$

Also let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfy one of the following conditions:

(1) *$\phi \in \Phi[h, q_\rho, p, j]$ for some $\rho \in (0,1)$; or*
(2) *There exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi[h_\rho, q_\rho, p, j]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}(p)$ satisfies the condition (2.8), then

$$\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}} \prec q(z).$$

Proof. The proof of Theorem 2.6 is similar to the proof of a known result [19, p. 30, Theorem 2.3d] and is, therefore, omitted. ∎

The next theorem yields the best dominant of the differential subordination (2.8).

Theorem 2.7. *Let the function h be univalent in $\mathbb{U}$. Also let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Suppose that the following differential equation:*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{2.9}$$

has a solution q with $q(0) = 1$ and satisfying one of the following conditions:

(1) $q \in \mathcal{F}_1$ and $\phi \in \Phi[h, q, p, j]$;
(2) The function q is univalent in $\mathbb{U}$ and $\phi \in \Phi[h, q_\rho, p, j]$ for some $\rho \in (0,1)$; or

(3) The function q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi[h_\rho, q_\rho, p, j]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}(p)$ satisfies (2.8), then

$$\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}} \prec q(z)$$

and q is the best dominant.

Proof. Following the same arguments in [19, p. 31, Theorem 2.3e], we deduce that q is a dominant from Theorems 2.3 and 2.6. Since q satisfies (2.9), it is also a solution of (2.8) and, therefore, q will be dominated by all dominants. Hence q is the best dominant. ■

In the particular case when $q(z) = 1 + Mz (M > 0)$, in view of the Definition 2.1, the class $\Phi[\Omega, q, p, j]$ of admissible functions, denoted by $\Phi[\Omega, M, p, j]$, is described below.

Definition 2.8. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\Phi[\Omega, M, p, j]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k+p-j}{p-j} Me^{i\theta}, 1 + \frac{L + (p-j)(2k+p-j-1) Me^{i\theta}}{(p-j)(p-j-1)}; z\right) \notin \Omega \tag{2.10}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\left(Le^{-i\theta}\right) \geqq (k-1) kM$ for all real θ and $k \geqq 1$.

Corollary 2.9. *Let $\phi \in \Phi[\Omega, M, p, j]$. If $f \in \mathcal{A}(p)$ satisfies the following condition:*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\, z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\, z^{p-j-2}}; z\right) \in \Omega,$$

then

$$\left|\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}} - 1\right| < M \qquad (z \in \mathbb{U}).$$

In the special case when $\Omega = q(\mathbb{U}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi[\Omega, M, p, j]$ is simply denoted by $\Phi[M, p, j]$.

Corollary 2.10. *Let $\phi \in \Phi[M, p, j]$. If $f \in \mathcal{A}(p)$ satisfies the following condition:*

$$\left|\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\, z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\, z^{p-j-2}}; z\right) - 1\right| < M,$$

then

$$\left|\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}} - 1\right| < M.$$

3. Superordination and Sandwich-Type Results

In this section we investigate the differential superordination of multivalent functions with higher order derivatives. For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class $\Phi'[\Omega, q, p, j]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \bar{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(p-j)q(z)}{m(p-j)}$$

and

$$\Re\left(\frac{(p-j-1)(w-2v+u)}{v-u} - 1\right) \geqq \frac{1}{m}\Re\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}$ and $m \geqq 1$. For convenience, we write $\Phi'[\Omega, q, p, 0] = \Phi'[\Omega, q, p]$.

Theorem 3.2. *Let $\phi \in \Phi'[\Omega, q, p, j]$. If $f \in \mathcal{A}(p)$, $\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}} \in \mathcal{F}_1$ and*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}}; z\right) : z \in \mathbb{U}\right\} \tag{3.1}$$

implies that

$$q(z) \prec \frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}.$$

Proof. From (2.7) and (3.1), we find that

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2g''(z); z\right) : z \in \mathbb{U}\right\}.$$

We also see from (2.5) that the admissibility condition for the function class $\phi \in \Phi'[\Omega, q, p, j]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'_1[\Omega, q]$. Thus, by Lemma 1.4, we have

$$q(z) \prec g(z) \qquad \text{or} \qquad q(z) \prec \frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}},$$

which evidently completes the proof of Theorem 3.2. ∎

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'[h(\mathbb{U}), q, p, j]$ is written simply as $\Phi'[h, q, p, j]$.

The following result can be derived as an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let the function h be analytic in $\mathbb{U}$ and $\phi \in \Phi'[h, q, p, j]$. If $f \in \mathcal{A}(p)$, $\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}} \in \mathcal{F}_1$ and*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{f^{(j)}(z)}{\delta(p,j)z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)z^{p-j-2}}; z\right) \tag{3.2}$$

implies that

$$q(z) \prec \frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}.$$

Putting $j=0$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let the function h be analytic in $\mathbb{U}$ and $\phi \in \Phi'[h,q,p]$. If $f \in \mathcal{A}(p)$, $\frac{f(z)}{z^p} \in \mathcal{F}_1$ and*

$$\phi\left(\frac{f(z)}{z^p}, \frac{f'(z)}{pz^{p-1}}, \frac{f''(z)}{p(p-1)\, z^{p-2}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{f(z)}{z^p}, \frac{f'(z)}{pz^{p-1}}, \frac{f''(z)}{p(p-1)\, z^{p-2}}; z\right)$$

implies that

$$q(z) \prec \frac{f(z)}{z^p}.$$

Theorems 3.2 and 3.3 can only be used to obtain subordinants of the differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for a specified ϕ.

Theorem 3.5. *Let the function h be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \bar{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation:*

$$\phi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z)$$

has a solution $q \in \mathcal{F}_1$. If $\phi \in \Phi'[h,q,p,j]$, $f \in \mathcal{A}(p)$, $\frac{f^{(j)}(z)}{\delta(p,j) z^{p-j}} \in \mathcal{F}_1$ and

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\, z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\, z^{p-j-2}}; z\right) \tag{3.3}$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\, z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\, z^{p-j-2}}; z\right)$$

implies that

$$q(z) \prec \frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}$$

and q is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.7. We, therefore, omit the details involved. ∎

Combining Theorems 2.3 and 3.3, we obtain the following sandwich-type theorem.

Theorem 3.6. *Let the functions h_1 and q_1 be analytic in $\mathbb{U}$, the function h_2 be univalent in $\mathbb{U}$, $q_2 \in \mathcal{F}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi[h_2,q_2,p,j] \cap \Phi'[h_1,q_1,p,j]$. If $f \in \mathcal{A}(p)$, $\frac{f^{(j)}(z)}{\delta(p,j) z^{p-j}} \in \mathcal{H} \cap \mathcal{F}_1$ and*

$$\phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\, z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\, z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\, z^{p-j-2}}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec \phi\left(\frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}}, \frac{f^{(j+1)}(z)}{\delta(p,j+1)\,z^{p-j-1}}, \frac{f^{(j+2)}(z)}{\delta(p,j+2)\,z^{p-j-2}}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{f^{(j)}(z)}{\delta(p,j)\,z^{p-j}} \prec q_2(z).$$

Upon setting $j=0$ in Theorem 3.6, we get the following result.

Corollary 3.7. *Let the functions h_1 and q_1 be analytic in $\mathbb{U}$, the function h_2 be univalent in $\mathbb{U}$, $q_2 \in \mathcal{F}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi[h_2, q_2, p] \cap \Phi'[h_1, q_1, p]$. If $f \in \mathcal{A}(p)$, $\frac{f(z)}{z^p} \in \mathcal{H} \cap \mathcal{F}_1$ and*

$$\phi\left(\frac{f(z)}{z^p}, \frac{f'(z)}{pz^{p-1}}, \frac{f''(z)}{p(p-1)\,z^{p-2}}\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec \phi\left(\frac{f(z)}{z^p}, \frac{f'(z)}{pz^{p-1}}, \frac{f''(z)}{p(p-1)\,z^{p-2}}\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{f(z)}{z^p} \prec q_2(z).$$

Remark 3.8. Our results of Corollaries 2.4, 3.4 and 3.7, respectively, are an improvement of the results obtained by Aouf et al. [11, Corollaries 8, 19 and 22, respectively].

Concluding Remark

In our present investigation, we have derived several second order differential subordination and superordination results for multivalent functions with higher order derivatives in the open unit disk $\mathbb{U}$. Our results have been obtained by considering suitable classes of admissible functions. Furthermore, we have obtained some differential sandwich-type results.

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