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On $N(k)$-Quasi Einstein Manifolds Admitting a Conharmonic Curvature Tensor

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Abstract In this paper, quasi Einstein manifolds whose characteristic vector field ξ belongs to k-nullity distribution are called $N(k)$-quasi Einstein manifolds. Firstly, we have shown that a conharmonically flat quasi Einstein manifold is an $N\left(\frac{2a+b}{n-2}\right)$-quasi Einstein manifold. Later, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $\tilde{H}(\xi, X)\cdot\tilde{P}=0$ and $\tilde{H}(\xi, X)\cdot\tilde{Z}=0$. Moreover, we also studied conharmonically pseudo-symmetric and conharmonically conservative $N(k)$-quasi Einstein manifolds.

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1. Introduction

An n-dimensional Riemannian or pseudo-Riemannian manifolds whose Ricci tensor S satisfies the relation $S=\mu g$, where μ is a non-zero constant are called Einstein manifolds which are very essential tools for the differential geometry and mathematical physics especially in the field of general theory of relativity. As a generalization of Einstein manifolds, authors Chaki and Maity [8] have developed a type of manifold called quasi Einstein manifold whose Ricci tensor S is defined by

$$S(U,X)=ag(U,X)+b\eta(U)\eta(X), \tag{1.1}$$

for all U, X in TM^n, where a and b are non-zero smooth functions and η is a non zero 1-form such that

$$g(V,\xi)=\eta(V), \;\; g(\xi,\xi)=\eta(\xi)=1, \tag{1.2}$$

for all V in TM^n, ξ is the associated unit vector field. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. Now

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it can be easily seen that Ricci tensor of an n-dimensional $N(k)$-quasi Einstein manifolds has precisely two distinct eigenvalues a and $a+b$, where a is of multiplicity $(n-1)$ and $a+b$ is simple. In particular, if ξ is a parallel vector field then the quasi Einstein manifold is locally isometric to a product manifold of one dimensional distribution U and $(n-1)$ dimensional distribution $U^{\perp}$ [13]. The hypothesis of quasi Einstein manifolds have been weakened by Chaki [19], Guha [19], De and Ghosh [10], Deszcz et. al., [12] and many others with different geometrical properties.

Let M^n be an n-dimensional Riemannian manifold, then the k-nullity distribution [23] is defined by

$$N(k) : p \to N_p(k) = \{X \in T_pM \mid R(U,V)X = k[g(V,X)U - g(U,X)V]\},$$

for all $U,\ V\ \in TM^n$, where k is some smooth function. If the generator ξ belongs to some k-nullity distribution $N(k)$, then the quasi Einstein manifold is called an $N(k)$-quasi Einstein manifold [24]. An n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold has also been demonstrate in [24]. Now in 2007, Ozgur and Tripathi [20] have indicated that in an n-dimensional $N(k)$-quasi Einstein manifold $k = \frac{a+b}{n-1}$ and can not satisfy the conditions $Z(\xi,U)\cdot Z = 0$, $Z(\xi,U)\cdot R = 0$ and $Z(\xi,U)\cdot S = 0$, where Z is a concircular curvature tensor. Again in 2008, Ozgur [19] shown that a conformally flat perfect fluid spacetime (M^4,g) satisfying Einstein equation with or without cosmological constant is an $N(k)$-quasi Einstein manifold. Later, De and Mallick [11] have proved that a special para-Sasakian manifold with vanishing D-concircular curvature tensor and a 4-dimensional Lorentzian space endowed with the Lorentzian metric are $N(k)$-quasi Einstein manifolds. Recently, the notion of $N(k)$-quasi Einstein manifolds with different geometrical and physical properties have been studied by several authors viz., [1, 9, 17, 18, 21] etc.,

Motivated from the above studies, we continues the study of $N(k)$-quasi Einstein manifold endowed with a conharmonic curvature tensor. After giving preliminaries in the Section 2, in Section 3 we have shown that a conharmonically flat quasi Einstein manifold M^n is an $N\left(\frac{2a+b}{n-2}\right)$-quasi Einstein manifold. In section 4, we consider $N(k)$-quasi Einstein manifold satisfying the conditions $\tilde{H}(\xi,X)\cdot\tilde{P} = 0$ and $\tilde{H}(\xi,X)\cdot\tilde{Z} = 0$. In fact Section 5 is devoted to the study of conharmonically pseudo-symmetric $N(k)$-quasi Einstein manifold. Finally, in the last section we have describe conharmonically conservative $N(k)$-quasi Einstein manifold.

2. Preliminaries

For a quasi Einstein manifold, the following relations holds true:

$$Q = aI + b\eta\otimes\xi, \tag{2.1}$$

$$S(U,\xi) = (a+b)\eta(U), \tag{2.2}$$

$$r = na + b. \tag{2.3}$$

Now, it can be noticed that an $N(K)$-quasi Einstein manifold satisfies (See [20]):

$$R(U,V)\xi = \frac{a+b}{n-1}\{\eta(V)U - \eta(U)V\}, \tag{2.4}$$

$$R(U,\xi)V = \frac{a+b}{n-1}\{\eta(V)U - g(U,V)\xi\} = R(\xi,U)V. \tag{2.5}$$

Definition 2.1. A Riemannian manifold (M^n, g) is said to possesses a quasi-constant curvature if the curvature tensor R is not identically zero and satisfies the relation

$$\begin{aligned} R(X,Y,W,U) &= \alpha[g(U,Y)g(X,W) - g(Y,W)g(U,X)] \\ &+\beta[g(X,U)\eta(Y)\eta(W) + g(Y,W)\eta(X)\eta(U) \\ &-g(Y,U)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(U)], \end{aligned} \tag{2.6}$$

for every arbitrary vector fields X, Y, W and U, where α and β are smooth functions not identically zero.

3. Conharmonic Curvature Tensor on $N(k)$-quasi Einstein Manifold

The notion of conharmonic curvature tensor $\tilde{H}$ in an n-dimensional Riemannian manifold (M^n, g) is defined by ([14]):

$$\begin{aligned} \tilde{H}(U,V)W &= R(U,V)W - \frac{1}{n-2}\{S(V,W)U - S(U,W)V \\ &+g(V,W)QU - g(U,W)QV\}, \end{aligned} \tag{3.1}$$

for all vector fields U, V, W on M^n, where R is a Riemannian curvature tensor Q is a Ricci operator.

Proposition 3.1. *In an $N(k)$-quasi Einstein manifold, the conharmonic curvature tensor $\tilde{H}$ satisfies*

$$\tilde{H}(U,V)\xi = \frac{na+b}{(n-1)(n-2)}\{\eta(U)V - \eta(V)U\}, \tag{3.2}$$

$$\tilde{H}(\xi,U)V = \frac{na+b}{(n-1)(n-2)}\{\eta(V)U - g(U,V)\xi\} = -H(U,\xi)V, \tag{3.3}$$

$$\eta(\tilde{H}(U,V)W) = \frac{na+b}{(n-1)(n-2)}\{g(U,W)\eta(V) - g(V,W)\eta(U)\}. \tag{3.4}$$

Proof. From (2.2), (2.4), (2.5) and (3.1), the equations (3.2)-(3.4) follow easily. ■

Let us consider an n-dimensional conharmonically flat quasi Einstein manifold, that is $\tilde{H}(U,V)W = 0$.
Now by virtue of (1.1), (2.1) and (2.4), we have

$$\begin{aligned} R(U,V)W &= \frac{2a}{n-2}\{g(V,W)U - g(U,W)V\} \\ &+\frac{b}{n-2}\{g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi \\ &+\eta(V)\eta(W)U - \eta(U)\eta(W)V\}. \end{aligned} \tag{3.5}$$

On plugging $W = \xi$ in the above equation, we obtain

$$R(U,V)\xi = \frac{2a+b}{n-2}\{\eta(V)U - \eta(U)V\}, \tag{3.6}$$

that is, in an n-dimensional conharmonically flat quasi Einstein manifold, the generator ξ belongs to the $\left(\frac{2a+b}{n-2}\right)$-nullity distribution. Hence this leads us to state the following result:

Theorem 3.2. *An n-dimensional conharmonically flat quasi Einstein manifold M^n is an $N\left(\frac{2a+b}{n-2}\right)$-quasi Einstein manifold.*

4. $N(k)$-quasi Einstein Manifold Satisfying $\tilde{H}(\xi,U)\cdot\tilde{P}=0$ and $\tilde{H}(\xi,U)\cdot\tilde{Z}=0$

The notion of projective curvature tensor $\tilde{P}$ in an n-dimensional Riemannian manifold is given by [26]:

$$\tilde{P}(U,V)W = R(U,V)W - \frac{1}{n-1}[S(V,W)U - S(U,W)V], \tag{4.1}$$

for all vector fields U, V and W on TM^n.

Proposition 4.1. *In an n-dimensional $N(k)$-quasi Einstein manifold, the projective curvature tensor $\tilde{P}$ satisfies the following:*

$$\tilde{P}(U,V)\xi = 0, \tag{4.2}$$

$$\tilde{P}(\xi,U)V = \frac{b}{n-1}\{g(U,V)\xi - \eta(U)\eta(V)\}\xi = -\tilde{P}(U,\xi)V, \tag{4.3}$$

$$\eta(\tilde{P}(U,V)W) = \frac{b}{n-1}\{g(V,W)\eta(U) - g(U,W)\eta(V)\}. \tag{4.4}$$

Proof. By the virtue of (2.2), (2.4), (2.5) and (4.1), it can be easily seen that the equations (4.2)-(4.4) holds true. ∎

Now we have prove the following theorem:

Theorem 4.2. *An n-dimensional $N(k)$-quasi Einstein manifold satisfies the condition $\tilde{H}(\xi,U)\cdot\tilde{P}=0$ if and only if $na+b=0$.*

Proof. Let M^n be an n-dimensional $N(k)$-quasi Einstein manifold satisfying the condition $\tilde{H}(\xi,U)\cdot\tilde{P}=0$, we can write

$$\begin{aligned} 0 &= \tilde{H}(\xi,U)\tilde{P}(V,W)X - \tilde{P}(\tilde{H}(\xi,U)V,W)X \\ &\quad -\tilde{P}(V,\tilde{H}(\xi,U)V)X - \tilde{P}(V,W)\tilde{H}(\xi,U)X. \end{aligned} \tag{4.5}$$

Now by using (2.5) in (4.5), we found that

$$\begin{aligned} 0 &= \frac{na+b}{(n-1)(n-2)}[\tilde{P}(V,W,X,U)\xi - \eta(\tilde{P}(V,W)X)U \\ &\quad -g(U,V)\tilde{P}(\xi,W)X + \eta(V)\tilde{P}(U,W)X - g(U,W)\tilde{P}(V,\xi)X \\ &\quad +\eta(W)\tilde{P}(V,U)X - g(U,X)\tilde{P}(V,W)\xi + \eta(X)P(V,W)U], \end{aligned} \tag{4.6}$$

which implies that either $na+b=0$ or

$$\begin{aligned} 0 &= \tilde{P}(V,W,X,U)\xi - \eta(\tilde{P}(V,W)X)U - g(U,V)\tilde{P}(\xi,W)X \\ &\quad +\eta(V)\tilde{P}(U,W)X - g(U,W)\tilde{P}(V,\xi)X + \eta(W)\tilde{P}(V,U)X \\ &\quad -g(U,X)\tilde{P}(V,W)\xi + \eta(X)P(V,W)U. \end{aligned} \tag{4.7}$$

Taking inner product of above equation with respect to ξ and then by using (4.2)-(4.4) gives

$$\begin{aligned} 0 &= R(V,W,X,U) + \frac{a+b}{n-1}\{g(V,X)g(U,W) - g(W,X)g(U,V)\} \\ &\quad + \frac{b}{n-1}\{g(U,W)\eta(V)\eta(X) - g(U,V)\eta(W)\eta(X)\}. \end{aligned} \tag{4.8}$$

So by a suitable contraction of (4.8), we get

$$bg(Z,W) = 0, \tag{4.9}$$

which turns into $b = 0$. This contradicts to the assumption that M is an $N(k)$-quasi Einstein manifold. The converse statement is trivial. Hence the proof. ∎

Finally, we have described the concircular curvature tensor $\tilde{Z}$ ([25]) in an n-dimensional Riemannian manifold by

$$\tilde{Z}(U,V)W = R(U,V)W - \frac{r}{n(n-1)}V, \tag{4.10}$$

for all vector fields U, V, W on TM^n, where r is a scalar curvature.

Proposition 4.3. *In an n-dimensional $N(k)$-quasi Einstein manifold M^n, the concircular curvature tensor $\widetilde{Z}$ satisfies*

$$\tilde{Z}(U,V)\xi = \frac{b}{n}\{\eta(V)U - \eta(U)V\}, \tag{4.11}$$

$$\tilde{Z}(\xi,U)V = \frac{b}{n}\{g(U,V)\xi - \eta(V)U\} = -\tilde{Z}(U,\xi)V, \tag{4.12}$$

$$\eta(\tilde{Z}(U,V)W) = \frac{b}{n}\{g(V,W)\eta(U) - g(U,W)\eta(V)\}. \tag{4.13}$$

Proof. By using (2.2), (2.4), (2.5) and (4.10), the equations (4.11)-(4.13) follow easily. ∎

Let us consider an $N(k)$-quasi Einstein manifold satisfying $\tilde{H}(\xi,U)\cdot\tilde{Z} = 0$, from which it follows that

$$\begin{aligned} 0 &= \tilde{H}(\xi,U)\tilde{Z}(V,W)X - \tilde{Z}(\tilde{H}(\xi,U)V,W)X \\ &\quad -\tilde{Z}(V,\tilde{H}(\xi,U)W)X - \tilde{Z}(V,W)\tilde{H}(\xi,U)X. \end{aligned} \tag{4.14}$$

By virtue of (2.5) in (4.14), we find

$$\begin{aligned} 0 &= \frac{na+b}{(n-1)(n-2)}[\tilde{Z}(V,W,X,U)\xi - \eta(\tilde{Z}(V,W)X)U \\ &\quad -g(U,V)\tilde{Z}(\xi,W)X + \eta(V)\tilde{Z}(U,W)X - g(U,W)\tilde{Z}(V,\xi)X \\ &\quad +\eta(W)\tilde{Z}(V,U)X - g(U,X)\tilde{Z}(V,W)\xi + \eta(X)\tilde{Z}(V,W)U], \end{aligned} \tag{4.15}$$

which implies that either $na+b=0$ or

$$\begin{aligned} 0 &= \tilde{Z}(V,W,X,U)\xi - \eta(\tilde{Z}(V,W)X)U - g(U,V)\tilde{Z}(\xi,W)X \\ &\quad +\eta(V)\tilde{Z}(U,W)X - g(U,W)\tilde{Z}(V,\xi)X + \eta(W)\tilde{Z}(V,U)X \\ &\quad -g(U,X)\tilde{Z}(V,W)\xi + \eta(X)\tilde{Z}(V,W)U. \end{aligned} \tag{4.16}$$

Taking inner product of (4.16) with respect to ξ and then by using (4.11)-(4.13), turns into

$$
\begin{aligned}
0 &= R(V,W,X,U) + \frac{na+b}{n(n-1)}\{g(U,W)g(V,X) - g(U,V)g(W,X)\} \qquad (4.17)\\
&\quad + \frac{b}{n}\{g(U,W)g(V,X) - g(U,V)g(W,X)\}.
\end{aligned}
$$

On contracting (4.17), we get

$$b(g(Z,W) - \eta(Z)\eta(W)) = 0, \tag{4.18}$$

which gives $b = 0$. This contradicts to the assumption that M^n is an $N(k)$-quasi Einstein manifold. The converse statement is trivial.
Thus we can state the following result:

Theorem 4.4. *Let M^n be a n-dimensional $N(k)$-quasi Einstein manifold. Then M^n satisfies the condition $\tilde{H}(\xi,U)\cdot\tilde{Z} = 0$ if and only if $na+b=0$.*

Hence in view of theorem 3.2 in [22], theorem 4.2 and theorem 4.4, we can state the following corollary:

Corollary 4.5. *Let M^n be an n-dimensional $N(k)$-quasi Einstein manifold. Then the following statements are equivalent:*

$$
\begin{aligned}
&(i) && \tilde{H}(\xi,V)\cdot S = 0,\\
&(ii) && \tilde{H}(\xi,V)\cdot\tilde{P} = 0,\\
&(iii) && \tilde{H}(\xi,V)\cdot\tilde{Z} = 0,\\
&(iv) && na+b = 0,
\end{aligned}
$$

for every vector field V on TM^n.

5. Conharmonically Pseudo-symmetric $N(k)$-quasi Einstein Manifold

In 1992, Deszcz [15] was first to introduced and studied the idea of pseudo-symmetric manifolds which is defined by

$$(R(U,V)\cdot R)(X,Y)W = L_R[((U\wedge V)\cdot R)(X,Y)W],$$

where L_R is some smooth function on M^n and $U\wedge V$ is an endomorphism defined by

$$(U\wedge V)W = g(V,W)U - g(U,W)V.$$

An n-dimensional $N(k)$-quasi Einstein manifold M^n is said to be conharmonically pseudo-symmetric if the condition

$$(R(U,V)\cdot\tilde{H})(X,Y)W = L_{\tilde{H}}[((U\wedge V)\cdot\tilde{H})(X,Y)W], \tag{5.1}$$

holds on the set $U_{\tilde{H}} = \{x\in M : \tilde{H}\neq 0 \ at\ x\}$, where $L_{\tilde{H}}$ is some function on $U_{\tilde{H}}$ and $\tilde{H}$ is the conharmonic curvature tensor. In particular, if $L_{\tilde{H}} = 0$, then M^n reduces to conharmonically semi-symmetric manifold.

Let us consider an n-dimensional conharmonically pseudo-symmetric $N(k)$-quasi Einstein manifold. Now on plugging $Y = \xi$ in (5.1), we obtain

$$\begin{aligned}(R(U,\xi)\cdot\tilde{H})(X,Y)W &= L_{\tilde{H}}[(U\wedge\xi)(\tilde{H}(X,Y)W) - \tilde{H}((U\wedge\xi)X,Y)W \\ &\quad -\tilde{H}(X,(U\wedge\xi))W - \tilde{H}(X,Y)(U\wedge\xi)W].\end{aligned} \tag{5.2}$$

Now the left hand side of (5.2), yields

$$\begin{aligned}&\frac{a+b}{n-1}[\eta(\tilde{H}(X,Y)W)U - \tilde{H}(X,Y,W,U)\xi - \eta(U)\tilde{H}(U,Y)W \\ &+g(U,X)\tilde{H}(\xi,Y)W - \eta(Y)\tilde{H}(X,U)W + g(U,Y)\tilde{H}(X,\xi)W \\ &-\eta(W)\tilde{H}(X,Y)U + g(U,W)\tilde{H}(X,Y)\xi].\end{aligned} \tag{5.3}$$

Similarly the right hand side of (5.2), gives

$$\begin{aligned}&L_{\tilde{H}}[\eta(\tilde{H}(X,Y)W)U - \tilde{H}(X,Y,W,U)\xi - \eta(U)\tilde{H}(U,Y)W \\ &+g(U,X)\tilde{H}(\xi,Y)W - \eta(Y)\tilde{H}(X,U)W + g(U,Y)\tilde{H}(X,\xi)W \\ &-\eta(W)\tilde{H}(X,Y)U + g(U,W)\tilde{H}(X,Y)\xi].\end{aligned} \tag{5.4}$$

By considering (5.3) and (5.4) in (5.2), we get

$$\begin{aligned}0 &= \left(L_{\tilde{H}} - \frac{a+b}{n-1}\right)[\eta(\tilde{H}(X,Y)W)U - \tilde{H}(X,Y,W,U)\xi \\ &\quad -\eta(U)\tilde{H}(U,Y)W + g(U,X)\tilde{H}(\xi,Y)W - \eta(Y)\tilde{H}(X,U)W \\ &\quad +g(U,Y)\tilde{H}(X,\xi)W - \eta(W)\tilde{H}(X,Y)U + g(U,W)\tilde{H}(X,Y)\xi].\end{aligned} \tag{5.5}$$

From (5.5), it is found that either $L_{\tilde{H}} = \frac{a+b}{n-1}$ or

$$\begin{aligned}0 &= \eta(\tilde{H}(X,Y)W)U - \tilde{H}(X,Y,W,U)\xi - \eta(U)\tilde{H}(U,Y)W \\ &\quad +g(U,X)\tilde{H}(\xi,Y)W - \eta(Y)\tilde{H}(X,U)W + g(U,Y)\tilde{H}(X,\xi)W \\ &\quad -\eta(W)\tilde{H}(X,Y)U + g(U,W)\tilde{H}(X,Y)\xi,\end{aligned} \tag{5.6}$$

which is equivalent to

$$\begin{aligned}0 &= \eta(\tilde{H}(X,Y)W)\eta(U) - \tilde{H}(X,Y,W,U) - \eta(U)\eta(\tilde{H}(U,Y)W) \\ &\quad +g(U,X)\eta(\tilde{H}(\xi,Y)W) - \eta(Y)\eta(\tilde{H}(X,U)W) - \eta(W)\eta(\tilde{H}(X,Y)U) \\ &\quad +g(U,Y)\eta(\tilde{H}(X,\xi)W) + g(U,W)\eta(\tilde{H}(X,Y)\xi).\end{aligned} \tag{5.7}$$

In consequence of (4.2)-(4.4) in (5.7) and then with the help of (1.1) and (3.1), we obtain

$$\begin{aligned}R(X,Y,W,U) &= \alpha[g(U,Y)g(X,W) - g(Y,W)g(U,X)] \\ &\quad +\beta[g(X,U)\eta(Y)\eta(W) + g(Y,W)\eta(X)\eta(U) \\ &\quad -g(Y,U)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(U)],\end{aligned} \tag{5.8}$$

where $\alpha = \frac{a(2-n)+b}{(n-1)(n-2)}$ and $\beta = \frac{b}{n-2}$ are smooth functions on M^n and are not identically zero.
Hence from (5.8) and (2.6), we can state the following:

Theorem 5.1. *Let M^n be an n-dimensional conharmonically pseudo-symmetric $N(k)$-quasi Einstein manifold. Then, either $L_{\tilde{H}} = \frac{a+b}{n-1}$ or M^n is a manifold of quasi constant curvature.*

Further, if we consider $L_{\tilde{H}} = 0$ then we have $a + b = 0$. This leads us to the following corollary:

Corollary 5.2. *An n-dimensional conharmonically pseudo-symmetric $N(k)$-quasi Einstein manifold turns into conharmonically semi-symmetric manifold, then the sum of the associated scalars is always zero.*

6. Conharmonically Conservative $N(k)$-quasi Einstein Manifold

Definition 6.1. An n-dimensional $N(k)$-quasi Einstein manifold will be called conharmonically conservative, if $(div\tilde{H})(U,V)W = 0$.

From (3.1), we get

$$(div\tilde{H})(U,V)W = (divR)(U,V)W - \frac{1}{2(n-2)}[(\nabla_U S)(V,W) - (\nabla_V S)(U,W) + dr(U)g(V,W) - dr(V)g(U,W)]. \tag{6.1}$$

Again it is to be noticed that in a Riemannian manifold, we have

$$(divR)(U,V)W = (\nabla_U S)(V,W) - (\nabla_V S)(U,W). \tag{6.2}$$

Hence in view of (6.2), equation (6.1) turns into

$$(div\tilde{H})(U,V)W = \frac{2n-5}{2(n-2)}[(\nabla_U S)(V,W) - (\nabla_V S)(U,W)] - \frac{1}{2(n-2)}[dr(U)g(V,W) - dr(V)g(U,W)]. \tag{6.3}$$

Now consider conharmonically conservative $N(k)$-quasi Einstein manifold i.e., $(div\tilde{H})(U,V)W = 0$. Then equation (6.3) reduces to

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = \frac{1}{2n-5}[dr(U)g(V,W) - dr(V)g(U,W)]. \tag{6.4}$$

Using (1.1) in (6.4), we get

$$da(U)g(U,W) - da(V)g(U,W) + db(U)\eta(V)\eta(W) - db(V)\eta(U)\eta(W) + b[(\nabla_U\eta)(V)\eta(W) + \eta(V)(\nabla_U\eta)(W) - (\nabla_V\eta)(U)\eta(W) - \eta(U)(\nabla_V\eta)(W)] = \frac{1}{2n-5}[dr(U)g(V,W) - dr(V)g(U,W)]. \tag{6.5}$$

Let us take the associated scalar b is non zero constant. Then $db(X) = 0$ and hence equation (6.5) becomes

$$0 = \frac{n-5}{2n-5}[da(U)g(V,W) - da(V)g(U,W)] + b[(\nabla_U\eta)(V)\eta(W) + \eta(V)(\nabla_U\eta)(W) - (\nabla_V\eta)(U)\eta(W) - \eta(U)(\nabla_V\eta)(W)]. \tag{6.6}$$

Setting $V = W = \xi$ in (6.6), we get

$$b(\nabla_\xi\eta)(U) = \frac{n-5}{2n-5}[da(U) - da(\xi)\eta(U)]. \tag{6.7}$$

On contracting (6.6), we obtain

$$b\{(\nabla_\xi\eta)(U) + \eta(U)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)\} - \frac{(n-5)(n-1)}{2n-5}da(U) = 0. \tag{6.8}$$

Using (6.7) in (6.8), we get

$$b\eta(U)\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) = \frac{(n-5)(n-1)}{2n-5}da(U) - \frac{n-5}{2n-5}[da(U) - da(\xi)\eta(U)]. \tag{6.9}$$

Setting $U = \xi$ in (6.9), we have

$$b\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i) = \frac{(n-5)(n-1)}{2n-5}da(\xi). \tag{6.10}$$

By virtue of (6.7) and (6.10), it follows from (6.8) that

$$da(U) = \eta(U)da(\xi). \tag{6.11}$$

Again setting $W = \xi$ in (6.6) and then by using (6.11), we get

$$b\{(\nabla_U\eta)(V) - (\nabla_V\eta)(U)\} = 0, \tag{6.12}$$

which implies that

$$(\nabla_U\eta)(V) - (\nabla_U\eta)(V) = 0. \tag{6.13}$$

On plugging $U = \xi$ in (6.13), we obtain

$$(\nabla_\xi\eta)(V) = 0, \tag{6.14}$$

which implies that $\nabla_\xi\xi = 0$ and hence we can state the following theorem:

Theorem 6.2. *Let M^n be an conharmonically conservative $N(k)$-quasi Einstein manifold admitting a non-zero constant scalar b, then the associated 1-form η is closed and the integral curves of the generator ξ are geodesics.*

7. Conclusions

The hypothesis of quasi Einstein manifolds emerged in the course of exact solutions of Einstein field equations. In this study, the existence of $N(k)$-quasi Einstein manifold as a conharmonically flat quasi Einstein manifold are given. Next an $N(k)$-quasi Einstein manifold satisfies $\tilde{H}(\xi,U)\cdot\tilde{H} = 0$, $\tilde{H}(\xi,U)\cdot\tilde{Z} = 0$ and $\tilde{H}(\xi,U)\cdot S = 0$, then the associated scalars are relatively dependent to each other. Later we have studied that a conharmonically pseudo-symmetric $N(k)$-quasi Einstein manifold and hence shown that it is a space of quasi constant curvature. Finally we have proved that an $N(k)$-quasi Einstein manifold is conharmonically conservative, then the integral curves of ξ are geodesics.

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