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Some Common Fixed Point Theorems for Four Mapping in Generalized Metric Spaces

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Abstract In this paper, we investigate the existence of a common fixed point for four mappings that are the pairs of weakly compatible mappings. Also, about the existence of an answer a class of integral equations, an application is presented to show the main results.

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1. Introduction

In 1964, the principle of Banach contraction was described for contraction mappings in spaces equipped with vector-valued metrics. Later, in [3] the results of Perov were generalized by Filip et al. and they studied in generalized metric space $(X, \mho)$ the FPP (fixed point property) of a self-mapping. In the present paper, the results are a generalization of Theorem 2.1 given in [3] and in the generalized metric space $(X, \mho)$, we consider the local FPP for four mappings. We also study on the generalized metric space $(X, \mho)$, the common FPP for four mappings.

In this article, $\mathbb{R}, \mathbb{N}$ and $\mathbb{C}$ are the sets of all real, natural and complex numbers, respectively.

Let $(\mathcal{U}, \preceq)$ be an ordered Banach space, then the following usual properties for cone $\mathcal{U}_+ = \{u \in \mathcal{U} : \theta \preceq u\}$, where θ is the zero-vector of $\mathcal{U}$, are holds:

(1) $\mathcal{U}_+ \cap -\mathcal{U}_+ = \{\theta\}$;
(2) $\mathcal{U}_+ + \mathcal{U}_+ \subset \mathcal{U}_+$;
(3) $\varsigma \mathcal{U}_+ \subset \mathcal{U}_+$, for $\varsigma \geq 0$.

Suppose the mapping $\mho : X^2 \longrightarrow \mathcal{U}$ satisfies:

(1) $\mho(x^1, x^2) \geq \theta$ for all $x^1, x^2 \in X$. $\mho(x^1, x^2) = \theta$, if and only if $x^1 = x^2$;

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(2) $\mho(x^1,x^2)=\mho(x^2,x^1)$ for each $x^1,x^2\in X$;
(3) $\mho(x^1,x^2)\preceq\mho(x^1,x^3)+\mho(x^3,x^2)$ for each $x^1,x^2,x^3\in X$.

Then $\mho$ is called a *vector-valued metric* on nonempty set X and $(X,\mho)$ is called a *vector-valued metric space*.

It was shown in [2, Theorem 2] that for lower semi-continuous function F from complete vector-valued metric space $(X,\mho)$ on an order continuous and order complete Banach lattice $\mathcal{U}$, if function $T:X\longrightarrow X$ satisfied in the following condition:

$$\mho(x^1,T(x^1))\leq F(x^1)-F(T(x^1)),\quad \forall x^1\in X,$$

then $Fix(T)\neq\emptyset$ (here $Fix(T)$ is the set of fixed points of a maping T).

Definition 1.1. [5]. Let the mapping $\mho:X^2\longrightarrow\mathcal{R}^n$ satisfies:

(1) $\mho(x^1,x^2)\geq 0$ for all $x^1,x^2\in X$. $\mho(x^1,x^2)=0$ if and only if $x^1=x^2$;
(2) $\mho(x^1,x^2)=\mho(x^2,x^1)$ for each $x^1,x^2\in X$;
(3) $\mho(x^1,x^2)\leq\mho(x^1,x^3)+\mho(x^3,x^2)$ for each $x^1,x^2,x^3\in X$.

Then, the set X equipped with vector-valued metric $\mho$ is called a *generalized metric space* and denoted by $(X,\mho)$.

Let x_1^1 be an element of generalized metric space X and $r=(r_i)_{i=1}^n\in\mathcal{R}^n$, with $r_i>0$ for each $1\leq i\leq n$ then $B(x_1^1,r)=\{x^1\in X:\mho(x_1^1,x^1)<r\}$ is the open ball to center x_1^1 and radius r, also $\widetilde{B}(x_1^1,r)=\{x^1\in X:\mho(x_1^1,x^1)\leq r\}$ is the closed ball to center x_1^1 and radius r.

Let $f:X\longrightarrow X$ be a single-valued map. $Fix(f)=\{x^1\in X:f(x^1)=x^1\}$ is the set of all fixed points of f.

In this paper, $M_{p,p}(\mathcal{R}_+)$ represents the set of all $p\times p$ matrices with components in $\mathcal{R}_+$, Θ is the zero matrix and I is the identity $p\times p$ matrix. Let $A\in M_{p,p}(\mathcal{R}_+)$, then A is called convergent to zero, if and only if $A^n\longrightarrow 0$ as $n\longrightarrow\infty$ (see [4–6, 12] for more details).

Let $\alpha,\beta\in\mathcal{R}^n$, where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$ and $c\in\mathcal{R}$. Note that $\alpha\leq\beta$ (resp. $\alpha<\beta$), that is, $\alpha_i\leq\beta_i$ (resp. $\alpha_i<\beta_i$) for each $1\leq i\leq n$ and also $\alpha\leq c$ (resp. $\alpha<c$), that is, $\alpha_i\leq c$ (resp. $\alpha_i<c$) for $1\leq i\leq n$, respectively. We define

$$\alpha+\beta:=(\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots,\alpha_n+\beta_n),$$

and

$$\alpha\cdot\beta:=(\alpha_1\cdot\beta_1,\alpha_2\cdot\beta_2,\ldots,\alpha_n\cdot\beta_n).$$

That are addition and multiplication on $\mathbb{R}^n$ (see [3]).

Now, we have the following equivalent statements that the proof them is the classic results in matrix analysis (see [1, 6, 8, 11]).

(1) $A\longrightarrow 0$;
(2) $A^n\longrightarrow 0$ as $n\longrightarrow\infty$;
(3) for each $\lambda\in\mathcal{C}$ with $det(A-\lambda I)=0$, $|\lambda|<1$, in other words, the eigenvalues of A are in the open unit disc;
(4) the matrix $I-A$ is nonsingular and

$$(I-A)^{-1}=I+A+\cdots+A^n+\cdots;$$

(5) $A^nq\longrightarrow 0$ and $qA^n\longrightarrow 0$ as $n\longrightarrow\infty$, for each $q\in\mathcal{R}^n$.

Definition 1.2. Let $(X,\mho)$ be a generalized metric space, and $\{x_n^1\}$ be a sequence in X, then

(1) for any $\varepsilon > 0$, there is a positive integer N and $x^1 \in X$ such that $\mho(x_n^1, x^1) < \varepsilon$ for all $n > N$, then the sequence $\{x_n^1\}$ is said *convergent*.
(2) for any $\varepsilon > 0$, there is N such that $\mho(x_n^1, x_m^1) < \varepsilon$ for all $m, n > N$, then the sequence $\{x_n^1\}$ is called a *Cauchy sequence*.

A sequence $\{x_n^1\}$ converges to a point $x^1 \in X$ if and only if $\mho(x_n^1, x^1) \to 0$ as $n \to \infty$.

Definition 1.3. [3] Let $f_1 : X \to X$ and $f_2 : X \to X$ are self-mappings. If $x = f_1 x^1 = f_2 x^1$ for some $x^1 \in X$ then x^1 is said a *coincidence point* of f_1 and f_2, where x is said a point of the coincidence of f_1 and f_2.

Definition 1.4. [3] Let $f_1 : X \to X$ and $f_2 : X \to X$ are self-mappings. Then f_1 and f_2 are called to be *w-compatible* if commute at coincidence points.

2. Main Results

Let $(X, \mho)$ be a complete generalized metric space and f_1, f_2, f_3, f_4 be four self-mappings in $(X, \mho)$. To start, first, we have the following lemma.

Lemma 2.1. *Let f_1, f_2, f_3 and f_4 be self-mappings on a complete generalized metric space $(X, \mho)$, satisfying $f_1(X) \subset f_4(X)$ and $f_2(X) \subset f_3(X)$. We define the sequences $\{x_n^1\}$ and $\{x_n^2\}$ in X by*

$$\begin{cases} x_{2n+1}^2 = f_1 x_{2n}^1 = f_4 x_{2n+1}^1 \\ x_{2n+2}^2 = f_2 x_{2n+1}^1 = f_3 x_{2n+2}^1, \qquad \forall n \geq 0. \end{cases} \tag{2.1}$$

Assuming that there is $A \in M_{p,p}(\mathcal{R}_+)$ such that $A \to 0$ and

$$\mho(x_n^2, x_{n+1}^2) \leq A\mho(x_{n-1}^2, x_n^2), \qquad \forall n \geq 1. \tag{2.2}$$

Then

(*a*) *$\{x_n^2\}$ is converges to a point in X and $\{f_1, f_3\}$, $\{f_2, f_4\}$ have coincidence points or*
(*b*) *$\{x_n^2\}$ is a Cauchy sequence in X.*

Further, if X is complete then $x_n^2 \to x^3$ in X and

$$\mho(x_n^2, x^3) \leq A^n (I - A)^{-1} \mho(x_0^2, x_1^2), \qquad \forall n \geq 1. \tag{2.3}$$

Proof. We have

$$\begin{aligned} \mho(x_n^2, x_{n+1}^2) &\leq A\mho(x_{n-1}^2, x_n^2) \\ &\leq A^2 \mho(x_{n-2}^2, x_{n-1}^2) \\ &\leq \cdots \\ &\leq A^n \mho(x_0^2, x_1^2) \longrightarrow 0 \qquad \forall n \geq 1, \qquad as \quad n \longrightarrow \infty. \end{aligned}$$

(a) Let there is a positive integer n such that $x_{2n}^2 = x_{2n+1}^2$. Then, from the definition of $\{x_n^2\}$ we get

$$f_2 x_{2n-1}^1 = f_3 x_{2n}^1 = f_1 x_{2n}^1 = f_4 x_{2n+1}^1,$$

that f_1 and f_3 have a coincidence point x_{2n}^1. Furthermore, by (2.2), one has

$$\mho(x_{2n+1}^2, x_{2n+2}^2) \leq A\mho(x_{2n}^2, x_{2n+1}^2) \to 0,$$

and also $x^2_{2n+1} = x^2_{2n+2}$, that is, $f_1x^1_{2n} = f_4x^1_{2n+1} = f_2x^1_{2n+1} = f_3x^1_{2n+2}$, so f_2 and f_4 have a coincidence point x^1_{2n+1}. Furthermore, (2.2) yields that $x^2_{2n} = x^2_m$ for every $2n < m$, so $\{x^2_n\}$ is converges to a point in X.

Also, a similar result is established, if $x^2_{2n+1} = x^2_{2n+2}$ for positive integer n.
(b) suppose that $x^2_{2n} \neq x^2_{2n+1}$ for all $n \geq 1$. Hence, from (2.2), we have

$$\mho(x^2_n, x^2_{n+1}) \leq A^n \mho(x^2_0, x^2_1), \qquad n \geq 1.$$

For each $n, m \geq 1$ with $n < m$, as a result

$$\begin{aligned} \mho(x^2_n, x^2_m) &\leq \sum_{i=n}^{m-1} \mho(x^2_i, x^2_{i+1}) \leq \sum_{i=n}^{m-1} A^i \mho(x^2_0, x^2_1) \\ &= A^n \mho(x^2_0, x^2_1) \sum_{j=0}^{m-n-1} A^j \qquad (2.4) \\ &\leq A^n (I - A)^{-1} \mho(x^2_0, x^2_1) \longrightarrow 0, \quad as \quad n \longrightarrow \infty. \end{aligned}$$

Therefore, $\{x^2_n\}$ is a Cauchy sequence in X. If X is complete, there exists a point $x^3 \in X$ such that the sequence $x^2_m \to x^3$ as $m \to \infty$. Thus

$$\begin{aligned} \mho(x^2_n, x^3) &\leq \mho(x^2_n, x^2_m) + \mho(x^2_m, x^3) \\ &\leq A^n (I - A)^{-1} \mho(x^2_0, x^2_1) + \mho(x^2_m, x^3), \end{aligned}$$

which yields (2.3). So the proof is complete. ■

Theorem 2.2. *Let f_1, f_2, f_3 and f_4 be self-mappings of a complete generalized metric space $(X, \mho)$, satisfying $f_1(X) \subset f_4(X)$, $f_2(X) \subset f_3(X)$ and there exists a $I \neq A \in M_{p,p}(\mathcal{R}_+)$ such that $A \to 0$ and*

$$\mho(f_1x^1, f_2x^2) \leq A u^1_{x^1,x^2}(f_1, f_2, f_3, f_4), \qquad (2.5)$$

where

$$u^1_{x^1,x^2}(f_1, f_2, f_3, f_4) \in \Big\{\mho(f_3x^1, f_4x^2), \mho(f_1x^1, f_3x^1), \mho(f_2x^2, f_4x^2), \frac{\mho(f_1x^1, f_4x^2) + \mho(f_2x^2, f_3x^1)}{2}\Big\}, \qquad \forall x^1, x^2 \in X.$$

If one of $f_1(X) \cup f_2(X)$ and $f_3(X) \cup f_4(X)$ is complete, then $\{f_1, f_3\}$ and $\{f_2, f_4\}$ have a unique coincidence point in X. Furthermore, if $\{f_1, f_3\}$ and $\{f_2, f_4\}$ are w-compatible then f_1, f_2, f_3 and f_4 have a unique common fixed point in X.

Proof. For each arbitrary point $x^1_0 \in X$, We make the sequences $\{x^1_n\}$ and $\{x^2_n\}$ in X such that

$$\begin{cases} f_1x^1_{2n} = f_4x^1_{2n+1} = x^2_{2n+1}, \\ f_2x^1_{2n+1} = f_3x^1_{2n+2} = x^2_{2n+2}, \qquad \forall n \geq 0. \end{cases}$$

First show that

$$\mho(x^2_{2n+1}, x^2_{2n+2}) \leq A\mho(x^2_{2n}, x^2_{2n+1}).$$

By (2.5), we have

$$\mho(x^2_{2n+1}, x^2_{2n+2}) = \mho(f_1x^1_{2n}, f_2x^1_{2n+1}) \leq A u^1_{x^1{}_{2n}, x^1_{2n+1}}(f_1, f_2, f_3, f_4), \qquad n \geq 1,$$

where

$$u^1_{x^1_{2n},x^1_{2n+1}}(f_1,f_2,f_3,f_4) \in \Big\{\mho(f_3x^1_{2n}, f_4x^1_{2n+1}), \mho(f_1x^1_{2n}, f_3x^1_{2n}), \mho(f_2x^1_{2n+1}, f_4x^1_{2n+1}), \frac{[\mho(f_1x^1_{2n}, f_4x^1_{2n+1}) + \mho(f_2x^1_{2n+1}, f_3x^1_{2n})]}{2}\Big\}$$
$$= \Big\{\mho(x^2_{2n}, x^2_{2n+1}), \mho(x^2_{2n+1}, x^2_{2n}), \mho(x^2_{2n+2}, x^2_{2n+1}), \frac{[\mho(x^2_{2n+1}, x^2_{2n+1}) + \mho(x^2_{2n+2}, x^2_{2n})]}{2}\Big\}$$
$$= \Big\{\mho(x^2_{2n}, x^2_{2n+1}), \mho(x^2_{2n+1}, x^2_{2n+2}), \frac{[\mho(x^2_{2n}, x^2_{2n+1}) + \mho(x^2_{2n+1}, x^2_{2n+2})]}{2}\Big\}.$$

Now, if $u^1_{x^1_{2n},x^1_{2n+1}}(f_1,f_2,f_3,f_4) = \mho(x^2_{2n}, x^2_{2n+1})$ then obviously $\mho(x^2_{2n+1}, x^2_{2n+2}) \leq A\mho(x^2_{2n}, x^2_{2n+1})$. If $u^1_{x^1{}_{2n},x^1_{2n+1}}(f_1,f_2,f_3,f_4) = \mho(x^2_{2n+1}, x^2_{2n+2})$ then $\mho(x^2_{2n+1}, x^2_{2n+2}) \leq A\mho(x^2_{2n+1}, x^2_{2n+2})$, that implies $\mho(x^2_{2n+1}, x^2_{2n+2}) = 0$ and so $x^2_{2n+1} = x^2_{2n+2}$.
If

$$u^1_{x^1_{2n},x^1_{2n+1}}(f_1,f_2,f_3,f_4) = \frac{[\mho(x^2_{2n}, x^2_{2n+1}) + \mho(x^2_{2n+1}, x^2_{2n+2})]}{2},$$

then we get

$$\begin{aligned}\mho(x^2_{2n+1}, x^2_{2n+2}) &\leq \frac{A}{2}[\mho(x^2_{2n}, x^2_{2n+1}) + \mho(x^2_{2n+1}, x^2_{2n+2})]\\ &\leq \frac{A}{2}\mho(x^2_{2n}, x^2_{2n+1}) + \frac{1}{2}\mho(x^2_{2n+1}, x^2_{2n+2}),\end{aligned}$$

that implies

$$\mho(x^2_{2n+1}, x^2_{2n+2}) \leq A\mho(x^2_{2n}, x^2_{2n+1}), \qquad \forall n \geq 0.$$

Thus, condition (2.2) of Lemma 2.1 holds.

To show that $\{f_1, f_3\}$ and $\{f_2, f_4\}$ have coincidence points in X. Without loss of generality, we assume that $x^2_n \neq x^2_{n+1}$ for each $n \geq 1$. If there is equality for some n, in this case from assertion (a) of Lemma 2.1, $\{f_1, f_3\}$ and $\{f_2, f_4\}$ have coincidence points in X. Therefore, from assertion (b) of Lemma 2.1, the sequence $\{x^2_n\}$ is a Cauchy sequence.

(1) Suppose that $f_3(X)\cup f_4(X)$ is complete. Then there is $u^1 \in f_3(X)\cup f_4(X)$ such that $x^2_n \to u^1$ as $n \to \infty$. Furthermore, the subsequences $\{f_3x^1_{2n+2}\} = \{f_2x^1_{2n+1}\} = \{x^2_{2n+2}\}$ and $\{f_4x^1_{2n+1}\} = \{f_1x^1_{2n}\} = \{x^2_{2n+1}\}$ of $\{x^2_n\}$, converge to the point u^1.
Since $u^1 \in f_3(X) \cup f_4(X)$, we have $u^1 \in f_3(X)$ or $u^1 \in f_4(X)$.

If $u^1 \in f_3(X)$, then we can find $u^2 \in X$ such that $f_3u^2 = u^1$ and assertion that $f_1u^2 = u^1$. To show this, consider

$$\begin{aligned}\mho(f_1u^2, u^1) &\leq \mho(f_1u^2, f_2x^1_{2n+1}) + \mho(f_2x^1_{2n+1}, u^1)\\ &\leq Au^1_{u^2,x^1_{2n+1}}(f_1,f_2,f_3,f_4) + \mho(f_2x^1_{2n+1}, u^1),\end{aligned}$$

where

$$u^1_{u^2,x^1_{2n+1}}(f_1,f_2,f_3,f_4) \in \Big\{\mho(f_3u^2, f_4x^1_{2n+1}), \mho(f_1u^2, f_3u^2), \mho(f_2x^1_{2n+1}, f_4x^1_{2n+1}), \frac{\mho(f_1u^2, f_4x^1_{2n+1}) + \mho(f_2x^1_{2n+1}, f_3u^2)}{2}\Big\}, \tag{2.6}$$

for each $n \geq 1$. Then, by (2.6), We have the following conditions:
(i) If $u^1_{u^2, x^1_{2n_k+1}}(f_1, f_2, f_3, f_4) = \mho(f_3u^2, f_4x^1_{2n_k+1})$ for all $k \geq 1$, then we have

$$\mho(f_1u^2, u^1) \leq A\mho(f_3u^2, f_4x^1_{2n_k+1}) + \mho(f_2x^1_{2n_k+1}, u^1),$$

hence, $\mho(f_1u^2, u^1) \to 0$, as $k \to \infty$.
(ii) If $u^1_{u^2, x^1_{2n_k}+1}(f_1, f_2, f_3, f_4) = \mho(f_1u^2, f_3u^2)$, then we have

$$\mho(f_1u^2, u^1) \leq A\mho(f_1u^2, f_3u^2) + \mho(f_2x^1_{2n_k+1}, u^1),$$

hence, $\mho(f_1u^2, u^1) \to 0$, as $k \to \infty$.
(iii) If $u^1_{u^2, x^1_{2n_k+1}}(f_1, f_2, f_3, f_4) = \mho(f_2x^1_{2n_k+1}, f_4x^1_{2n_k+1})$, then we have

$$\mho(f_1u^2, u^1) \leq A\mho(f_2x^1_{2n_k+1}, f_4x^1_{2n_k+1}) + \mho(f_2x^1_{2n_k+1}, u^1),$$

hence, $\mho(f_1u^2, u^1) \to 0$, as $k \to \infty$.
(iv) If $u^1_{u^2, x^1_{2n_k+1}}(f_1, f_2, f_3, f_4) = \dfrac{\mho(f_1u^2, f_4x^1_{2n_k+1}) + \mho(f_2x^1_{2n_k+1}, sv)}{2}$, then we have

$$\begin{aligned}\mho(f_1u^2, u^1) &\leq A\frac{\mho(f_1u^2, f_4x^1_{2n_k+1}) + \mho(f_2x^1_{2n_k+1}, sv)}{2} + \mho(f_2x^1_{2n_k+1}, u^1)\\ &\leq \frac{A}{2}\mho(f_1u^2, f_4x^1_{2n_k+1}) + \frac{1}{2}\mho(f_2x^1_{2n_k+1}, sv) + \mho(f_2x^1_{2n_k+1}, u^1),\end{aligned}$$

hence, $\mho(f_1u^2, u^1) \to 0$, as $k \to \infty$.
Therefore, from (i)-(iv), we have $\mho(f_1u^2, u^1) = 0$. As a result, we have $f_1u^2 = f_3u^2 = u^1$ and since $u^1 \in f_1(X) \subset f_4(X)$, there exists $u^3 \in X$ such that $f_4u^3 = u^1$.
Now, we show that $f_2u^3 = u^1$. Consider

$$\begin{aligned}\mho(f_2u^3, u^1) &\leq \mho(f_2u^3, f_1x^1_{2n}) + \mho(f_1x^1_{2n}, u^1)\\ &= \mho(f_1x^1_{2n}, f_2u^3) + \mho(f_1x^1_{2n}, u^1)\\ &\leq Au^1_{x^1_{2n}, u^3}(f_1, f_2, f_3, f_4) + \mho(f_1x^1_{2n}, u^1),\end{aligned}$$

where

$$\begin{aligned}u^1_{x^1_{2n}, u^3}(f_1, f_2, f_3, f_4) \in \Big\{&\mho(f_3x^1_{2n}, f_4u^3), \mho(f_1x^1_{2n}, f_3x^1_{2n}), \mho(f_2u^3, f_4u^3),\\ &\frac{\mho(f_1x^1_{2n}, f_4u^3) + \mho(f_2u^3, f_3x^1_{2n})}{2}\Big\},\end{aligned} \tag{2.7}$$

for each $n \geq 1$. Then, from (2.7), we have the following four:
(v) If $u^1_{x^1_{2n_k}, u^3}(f_1, f_2, f_3, f_4) = \mho(f_3x^1_{2n_k}, f_4u^3)$ for each $k \geq 1$, then

$$\mho(f_2u^3, u^1) \leq A\mho(f_3x^1_{2n_k}, f_4u^3) + \mho(f_1x^1_{2n_k}, u^1),$$

hence, $\mho(f_2u^3, u^1) \to 0$, as $k \to \infty$.
(vi) If $u^1_{x^1_{2n_k}, u^3}(f_1, f_2, f_3, f_4) = \mho(f_1x^1_{2n_k}, f_3x^1_{2n_k})$, then

$$\mho(f_2u^3, u^1) \leq A\mho(f_1x^1_{2n_k}, f_3x^1_{2n_k}) + \mho(f_1x^1_{2n_k}, u^1),$$

hence, $\mho(f_2u^3, u^1) \to 0$, as $k \to \infty$.

(vii) If $u^1_{x^1_{2n_k},u^3}(f_1,f_2,f_3,f_4) = \mho(f_2u^3, f_4u^3)$, then

$$\begin{aligned}\mho(f_2u^3,u^1) &\le A\mho(f_2u^3,f_4u^3) + \mho(f_1x^1_{2n_k},u^1)\\ &= A\mho(f_2u^3,u^1) + \mho(f_1x^1_{2n_k},u^1),\end{aligned}$$

hence, $\mho(f_2u^3,u^1) \to 0$, as $k \to \infty$.

(ix) If $u^1_{x^1_{2n_k},u^3}(f_1,f_2,f_3,f_4) = \dfrac{\mho(f_1x^1_{2n_k}, f_4u^3) + \mho(f_2u^3, f_3x^1_{2n_k})}{2}$, then

$$\begin{aligned}\mho(f_2u^3,u^1) &\le A\frac{\mho(f_1x^1_{2n_k}, f_4u^3) + \mho(f_2u^3, f_3x^1_{2n_k})}{2} + \mho(f_1x^1_{2n_k},u^1)\\ &\le \frac{A}{2}\mho(f_1x^1_{2n_k},u^1) + \frac{1}{2}\mho(f_2u^3, f_3x^1_{2n_k}) + \mho(f_1x^1_{2n_k},u^1),\end{aligned}$$

hence, $\mho(f_2u^3,u^1) \to 0$, as $k \to \infty$.

Therefore, from (v)-(ix), $\mho(f_2u^3,u^1) = 0$ and following the same arguments as above, we get $f_2u^3 = f_4u^3 = u^1$. Hence $\{f_1,f_3\}$ and $\{f_2,f_4\}$ have a common coincidence point in X.

Now, if $\{f_1,f_3\}$ and $\{f_2,f_4\}$ are w-compatible, $f_1u^1 = f_1f_3v = f_3f_1v = f_3u^1 := u^3_1$ and $f_2u^1 = f_2f_4w = f_4f_2w = f_4u^1 := u^3_2$. Then

$$\mho(u^3_1,u^3_2) = \mho(f_1u^1, f_2u^1) \le Au_{u^1,u^1}(f_1,f_2,f_3,f_4),$$

where

$$\begin{aligned}u^1_{u^1,u^1}(f_1,f_2,f_3,f_4)) \in \Big\{&\mho(f_3u^1,f_4u^1), \mho(f_1u^1,f_3u^1), \mho(f_2u^1,f_4u^1),\\ &\frac{\mho(f_1u^1,f_4u^1)+\mho(f_2u^1,f_3u^1)}{2}\Big\}\\ =\ &\mho(u^3_1,u^3_2).\end{aligned} \tag{2.8}$$

Therefore, $\mho(u^3_1,u^3_2) \le A\mho(u^3_1,u^3_2)$, which implies that $u^3_1 = u^3_2$ and thus $f_1u^1 = f_2u^1 = f_3u^1 = f_4u^1$, that is, the point u^1 is a coincidence point of $\{f_1,f_3\}$ and $\{f_2,f_4\}$. Now, we show that $u^1 = f_2u^1$. Indeed, we have

$$\mho(u^1, f_2u^1) = \mho(f_1u^2, f_2u^1) \le Au_{u^2,u^1}(f_1,f_2,f_3,f_4),$$

where

$$\begin{aligned}u^1_{u^2,u^1}(f_1,f_2,f_3,f_4)) \in \Big\{&\mho(f_3u^2,f_4u^1), \mho(f_1u^2,f_3u^2), \mho(f_2u^1,f_4u^1),\\ &\frac{\mho(f_1u^2,f_4u^1)+\mho(f_2u^1,f_3u^2)}{2}\Big\}\\ =\ &\{\mho(u^1,f_2u^1)\}.\end{aligned}$$

So $\mho(u^1,f_2u^1) \le A\mho(u^1,f_2u^1)$, which implies that $f_2u^1 = u^1$ and thus u^1 is a common fixed point of f_1,f_2,f_3 and f_4.

To prove the uniqueness of the point u^1, we assume that u^{1*} is another common fixed point of f_1,f_2,f_3 and f_4. By (2.5), it concludes that

$$\mho(u^1,u^{1*}) = \mho(f_1u^1, f_2u^{1*}) \le Au_{u^1,u^{1*}}(f_1,f_2,f_3,f_4),$$

where

$$u^1_{u^1,u^{1*}}(f_1,f_2,f_3,f_4)) \in \Big\{\mho(f_3u^1,f_4u^{1*}),\mho(f_1u^1,f_3u^1),\mho(f_2u^{1*},f_4u^{1*}),\\ \frac{\mho(f_1u^1,f_4u^{1*})+\mho(f_2u^{1*},f_3u^1)}{2}\Big\}\\ =\mho(u^1,f_2u^{1*}),$$

that implies that $u^1=u^{1*}$.

(2) Let $f_1(X)\cup f_2(X)$ is complete and $u^1\in f_4(X)$. In this case, the proof is similar to the completeness of $f_3(X)\cup f_4(X)$ and $u^1\in f_4(X)$. ■

Corollary 2.3. *Let f_1, f_2, f_3 and f_4 be self-mappings on complete generalized metric space $(X,\mho)$, satisfying $f_1(X)\subset f_4(X)$, $f_2(X)\subset f_3(X)$ and for some $m,n\geq 1$ there is a $A\in M_{p,p}(\mathcal{R}_+)$ such that $A\to 0$ and*

$$\mho(f_1^m x^1,f_2^n x^2)\leq A u^1_{x^1,x^2}(f_1^m,f_2^n,f_3^m,f_4^n), \tag{2.9}$$

where

$$u^1_{x^1,x^2}(f_1^m,f_2^n,f_3^m,f_4^n) \in \Big\{\mho(f_3^m x^1,f_4^n x^2),\mho(f_1^m x^1,f_3^m x^1),\mho(f_2^n x^2,f_4^n x^2),\\ \frac{\mho(f_1^m x^1,f_4^n x^2)+\mho(f_2^n x^2,f_3^m x^1)}{2}\Big\}, \qquad \forall x^1,x^2\in X.$$

If one of $f_1(X)\cup f_2(X)$ and $f_3(X)\cup f_4(X)$ is complete subspace of X then $\{f_1,f_3\}$ and $\{f_2,f_4\}$ have a unique coincidence point in X. Furthermore, if $\{f_1,f_3\}$ and $\{f_2,f_4\}$ are w-compatible then f_1, f_2, f_3 and f_4 have a unique common fixed point in X.

Proof. According to Theorem 2.2, it follows that $\{f_1^m,f_3^m\}$ and $\{f_2^n,f_4^n\}$ have a unique common fixed point $s\in X$. Now, we have

$$\begin{aligned} f_1(s) &= f_1(f_1^m(s))=f_1^{m+1}(s)=f_1^m(f_1(s)),\\ f_3(s) &= f_3(f_3^m(s))=f_3^{m+1}(s)=f_3^m(f_3(s)). \end{aligned}$$

So $f_1(s)$ and $f_3(s)$ are again fixed points for the mappings f_1^m and f_3^m. Thus, $f_1(s)=f_3(s)=s$. Using the same method to prove the Theorem 2.2, we get $f_2(s)=f_4(s)=s$. So the proof is complete. ■

Corollary 2.4. *Let f_1, f_2, f_3 and f_4 be self-mappings on complete generalized metric space $(X,\mho)$, satisfying $f_1(X)\subset f_4(X)$, $f_2(X)\subset f_3(X)$ and there is a $A\in M_{p,p}(\mathcal{R}_+)$ such that $A\to 0$ and*

$$\mho(f_1x^1,f_2x^2)\leq A\mho(f_3x^1,f_4x^2), \qquad \forall x^1,x^2\in X.$$

If one of $f_1(X)\cup f_2(X)$ and $f_3(X)\cup f_4(X)$ is complete subspace of X then $\{f_1,f_3\}$ and $\{f_2,f_4\}$ have a unique coincidence point in X. Furthermore, if $\{f_1,f_3\}$ and $\{f_2,f_4\}$ are w-compatible then f_1, f_2, f_3 and f_4 have a unique common fixed point in X.

Corollary 2.5. *Let f_1, f_2 and f_4 be self-mappings on complete generalized metric space $(X,\mho)$, satisfying $f_1(X)\cup f_2(X)\subset f_4(X)$ and there is a $A\in M_{p,p}(\mathcal{R}_+)$ such that $A\to 0$ and*

$$\mho(f_1x^1,f_2x^2)\leq A u_{x^1,x^2}(f_1,f_2,f_4),$$

where

$$u^{1}_{x^1,x^2}(f_1,f_2,f_4) \in \left\{\mho(f_4x^1,f_4x^2),\mho(f_1x^1,f_4x^1),\mho(f_2x^2,f_4x^2),\frac{\mho(f_1x^1,f_4x^2)+\mho(f_2x^2,f_4x^1)}{2}\right\}, \quad \forall x^1,x^2\in X.$$

If one of $f_1(X)\cup f_2(X)$ or $f_4(X)$ is complete subspace of X then $\{f_1,f_4\}$ and $\{f_2,f_4\}$ have a unique coincidence point in X. Furthermore, if $\{f_1,f_4\}$ and $\{f_2,f_4\}$ are w-compatible then the mappings f_1, f_2 and f_4 have a unique common fixed point in X.

Corollary 2.6. *Let f_1 and f_4 be self-mappings on complete generalized metric space $(X,\mho)$, satisfying $f_1(X)\subset f_4(X)$ and there exists a $A\in M_{p,p}(\mathcal{R}_+)$ such that $A\to 0$ and*

$$\mho(f_1x^1,f_1x^2)\le Au_{x^1,x^2}(f_1,f_4), \tag{2.10}$$

where

$$u^{1}_{x^1,x^2}(f_1,f_4) \in \left\{\mho(f_4x^1,f_4x^2),\mho(f_1x^1,f_4x^1),\mho(f_1x^2,f_4x^2),\frac{\mho(f_1x^1,f_4x^2)+\mho(f_1x^2,f_4x^1)}{2}\right\}, \tag{2.11}$$

$$\forall x^1,x^2\in X.$$

If $f_1(X)$ or $f_4(X)$ is complete subspace of X then $\{f_1,f_4\}$ have a unique coincidence point in X. Furthermore, if $\{f_1,f_4\}$ is w-compatible then the mappings f_1 and f_4 have a unique common fixed point in X.

Example 2.7. Let $X=[0,\infty)$ and $\mho: X^2\to\mathbb{R}^2$ with $\mho(x^1,x^2)=\big(|x^1-x^2|,|x^1-x^2|\big)$. Then $(X,\mho)$ is a complete generalized metric space. Consider four mappings $f_1,f_2,f_3,f_4: X\to X$ defined by

$$f_1x^1=\frac{3x^1}{5},\qquad f_2x^1=\frac{2x^1}{5},\qquad f_4x^1=\frac{5x^1}{3},\qquad f_3x^1=\frac{5x^1}{2},\quad \textit{for all}\ \ x^1\in X.$$

Clearly, $f_1(X)\subseteq f_4(X)$ and $f_2(X)\subseteq f_3(X)$. Also, $\{f_1,f_3\}$ and $\{f_2,f_4\}$ have a unique coincidence point in X. Furthermore, $\{f_1,f_3\}$ and $\{f_2,f_4\}$ are w-compatible, that is,

$$f_1f_3x^1=f_3f_1x^1=x^1 \quad \textit{and}\ f_2f_4x^1=f_4f_2x^1=x^1.$$

Now, for all $x^1,x^2\in X$,

$$\mho(f_1x^1,f_2x^2)=(|\frac{3x^1}{5}-\frac{2x^2}{5}|,|\frac{3x^1}{5}-\frac{2x^2}{5}|)=\frac{1}{5}(|3x^1-2x^2|,|3x^1-2x^2|),$$

$$\mho(f_3x^1,f_4x^2)=(|\frac{5x^1}{2}-\frac{5x^2}{3}|,|\frac{5x^1}{2}-\frac{5x^2}{3}|),$$

$$\mho(f_1x^1,f_3x^1)=(|\frac{3x^1}{5}-\frac{5x^1}{2}|,|\frac{3x^1}{5}-\frac{5x^1}{2}|)=(\frac{19x^1}{10},\frac{19x^1}{10}),$$

$$\mho(f_2x^2,f_4x^2)=(|\frac{2x^2}{5}-\frac{5x^2}{3}|,|\frac{2x^2}{5}-\frac{5x^2}{3}|)=(\frac{19x^2}{15},\frac{19x^2}{15}),$$

$$\mho(f_1x^1,f_4x^2)+\mho(f_2x^2,f_3x^1)=(|\frac{3x^1}{5}-\frac{5x^2}{3}|,|\frac{2x^2}{5}-\frac{5x^1}{2}|+|\frac{2x^2}{5}-\frac{5x^1}{2}|,|\frac{3x^1}{5}-\frac{5x^2}{3}|).$$

Let $A = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$ be a matrix convergent to zero. If $x^1 \geq x^2$ then

$$\begin{aligned}\mho(f_1x^1, f_2x^2) &= \frac{1}{5}(|3x^1 - 2x^2|, |3x^1 - 2x^2|)\\ &\leq (\frac{3x^1}{5}, \frac{3x^1}{5})\\ &\leq A(\frac{19x^1}{10}, \frac{19x^1}{10})\\ &= Ad(f_1x^1, f_3x^1)\\ &= Au_{x^1,x^2}(f_1, f_2, f_3, f_4).\end{aligned}$$

If $x^1 \leq x^2$ then

$$\begin{aligned}\mho(f_1x^1, f_2x^2) &= \frac{1}{5}(|3x^1 - 2x^2|, |3x^1 - 2x^2|)\\ &\leq (\frac{2x^2}{5}, \frac{2x^2}{5})\\ &\leq A(\frac{19x^2}{15}, \frac{19x^2}{15})\\ &= Ad(f_2x^2, f_4x^2)\\ &= Au_{x^1,x^2}(f_1, f_2, f_3, f_4).\end{aligned}$$

Therefore, all the conditions of Theorem 2.2 hold. Then the mappings f_1, f_2, f_3 and f_4 have a unique common fixed point.

Example 2.8. Let $X = [0, 1] \cup \{2, 3\}$ and $\mho : X^2 \to \mathbb{R}^2$ with $\mho(x^1, x^2) = \big(|x^1 - x^2|, |x^1 - x^2|\big)$. Then $(X, \mho)$ is a complete generalized metric space. Consider four mappings $f_1, f_2, f_3, f_4 : X \to X$ defined by

$$f_1x^1 = \begin{cases} \frac{1-x^1}{2}, & x^1 \in [0, 1] \\ x^1, & x^1 \in \{2, 3\} \end{cases} \quad f_2x^1 = \begin{cases} \frac{2x^1}{5}, & x^1 \in [0, 1] \\ x^1, & x^1 \in \{2, 3\} \end{cases} \quad f_3x^1 = \begin{cases} \frac{x^1}{2}, & x^1 \in [0, 1] \\ x^1, & x^1 \in \{2, 3\} \end{cases}$$

$$f_4x^1 = \begin{cases} \frac{3x^1}{5}, & x^1 \in [0, 1] \\ x^1, & x^1 \in \{2, 3\}. \end{cases}$$

Clearly, $f_1(X) \subseteq f_4(X)$ and $f_2(X) \subseteq f_3(X)$. Also, $\{f_1, f_3\}$ and $\{f_2, f_4\}$ have a unique coincidence point in X. Furthermore, $\{f_1, f_3\}$ and $\{f_2, f_4\}$ are w-compatible, that is,

$$f_1f_3x^1 = f_3f_1x^1 = x^1 \quad \textit{and } f_2f_4x^1 = f_4f_2x^1 = x^1.$$

Since $\mho(f_12, f_23) = (|2-3|, |2-3|) = (1, 1) = \mho(2, 3)$ and $\mho(f_32, f_43) = \mho(f_12, f_32) = \mho(f_23, f_43) = \frac{1}{2}\mho(f_12, f_43) + \mho(f_23, f_32) = (1, 1)$. Then, we have

$$\mho(f_12, f_23) \geq Au_{2,3}(f_1, f_2, f_3, f_4),$$

where $A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ is a matrix convergent to zero. Therefore, Theorem 2.2 cannot be used for this example

3. Application

Let $X = L^2(C)$ be the set of comparable functions on $C = [0, 1]$ whose square is integrable on C. Consider the following integral equations

$$x^1(r) = \int_C g_1(r, s, x^1(s))ds + u^2(r),$$
$$x^2(r) = \int_C g_2(r, s, x^1(s))ds + u^2(r), \tag{3.1}$$

where $g_1, g_2 : C \times C \times \mathcal{R} \to \mathcal{R}^2$ and $u^2 : C \to \mathcal{R}_+$ are given continuous mappings. We will study the sufficient conditions for the existence of a common solution of integral equations in the frame of complete generalized metric spaces. We define $\mho : X^2 \to \mathcal{R}^2$ with

$$\mho(x^1, x^2) = \Big(|x^1(r) - x^2(r)|, |x^1(r) - x^2(r)|\Big).$$

Then $\mho$ is a complete generalized metric on X. Assume that the following conditions hold:

(i) For each $r, s \in C$, we have

$$g_1(r, s, x^1(s)) = u_1^1(r) \quad \le \quad \int_C g_1(r, s, u_1^1(s))ds$$

and

$$g_2(r, s, x^1(s)) = u_2^1(r) \quad \le \quad \int_C g_1(r, s, u_2^1(s))ds.$$

(ii) There is $\rho : C \to M_{2\times 2}(C)$ that the following condition satisfies

$$\int_C |g_1(r, s, u^1(s)) - g_1(r, s, u^2(s))|ds \le \rho(r)|f_4u^1(t) - f_4u^2(r)|,$$

for all $r, s \in C$ with $A \ge \rho(t)$ where $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a matrix that converges to zero.

So the integral equations (3.1) have a common solution in $L^2(C)$.

Proof. Define $(f_1x^1)(r) = \int_C g_1(r, s, x^1(s))ds + u^2(r)$ and $(f_4x^1)(r) = \int_C g_2(r, s, x^1(s))ds + u^2(r)$. From (i), we have

$$\begin{aligned} (f_1x^1)(r) &= \int_C g_1(r, s, x^1(s))ds + u^2(r) \\ &\ge x^1(r) + u^2(r) \\ &\ge x^1(r) \end{aligned}$$

and

$$\begin{aligned} (f_4x^1)(r) = &\int_C g_2(r, s, x^1(s))ds + u^2(r) \\ &\ge x^1(r) + u^2(r) \\ &\ge x^1(r). \end{aligned}$$

Hence f_1 and f_4 are mappings on X. Now, for all comparable $x^1, x^2 \in X$, we have

$$\begin{aligned}
\mho(f_1x^1, f_1x^2) &= \Big(|f_1x^1(r) - f_1x^2(r)|, |f_1x^1(r) - f_1x^2(r)|\Big) \\
&= \Big(\left|\int_C g_1(r,s,x^1(s))ds - \int_C g_1(r,s,x^2(s))ds\right|, \\
&\qquad \left|\int_C g_1(r,s,x^1(s))ds - \int_C g_1(r,s,x^2(s))ds\right|\Big) \\
&\leq \Big(\int_C \left|g_1(r,s,x^1(s))ds - g_1(r,s,x^2(s))\right| ds, \\
&\qquad \int_C \left|g_1(r,s,x^1(s))ds - g_1(r,s,x^2(s))\right| ds\Big) \\
&\leq \Big(\rho(r)|f_4x^1(r) - f_4x^2(r)|, \rho(t)|f_4x^1(r) - f_4x^2(r)|\Big) \\
&\leq A\Big(|f_4x^1(r) - f_4x^2(r)|, |f_4x^1(r) - f_4x^2(r)|\Big) \\
&= A\mho(f_4x^1, f_4x^2) \\
&= Au_{x^1,x^2}(f_1, f_4),
\end{aligned}$$

where

$$u^1_{x^1,x^2}(f_1,f_4) = \mho(f_4x^1, f_4x^2) \in \Big\{\mho(f_4x^1, f_4x^2), \mho(f_1x^1, f_4x^1), \mho(f_1x^2, f_4x^2), \frac{\mho(f_1x^1, f_4x^2) + \mho(f_1x^2, f_4x^1)}{2}\Big\}.$$

Thus equation (2.10) is hold. Now, by apply Corollary 2.6 we can get the answer of common of integral equations (3.1) in $L^2(C)$. ■

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