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Some k-Fibonacci and k-Lucas Identities by a Matrix Approach with Applications

Piyanut Puangjumpa

Department of Mathematics and Statistics, Faculty of Science and Technology, Surindra Rajabhat University, Surin 32000, Thailand
e-mail : piyanut@math.sut.ac.th

Abstract In this research, we study and find some identities involving k-Fibonacci and k-Lucas numbers by using a matrix approach. As an application of these identities we then obtain the solutions of some Diophantine equations.

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1. Introduction

There are many research articles about sequence of integers and applications thereof (see [1–4]). In 2016, Srisawat and Sriprad[5] studied some identities involving Pell and Pell-Lucas numbers by using matrix methods and they presented the solutions of some Diophantine equations by employing these identities. The above articles were the motivation to apply matrix methods to k-Fibonacci and k-Lucas numbers, and in this work we are thus interested in discovering k-Fibonacci and k-Lucas identities, together with some applications.

The k-Fibonacci sequence $\{F_{k,n}\}$ is an additive sequence similar to the Fibonacci sequence, defined by the recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for all $n \geq 2$ with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$. The first few terms of $\{F_{k,n}\}$ are $0, 1, k, k^2 + 1, k^3 + 2k, \ldots$. A number in the sequence is called a k-Fibonacci number and we denote the n^{th} k-Fibonacci number by $F_{k,n}$. The k-Fibonacci numbers for negative subscripts are defined as $F_{k,-n} = (-1)^{-n+1}F_{k,n}$, similarly, the k-Lucas sequence $\{L_{k,n}\}$ is defined by the same recurrence relation as the k-Fibonacci sequence, but with different initial values: $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$, for all $n \geq 2$ while the initial values are $L_{k,0} = 2$ and $L_{k,1} = k$. The first few terms of $\{L_{k,n}\}$ are $2, k, k^2+2, k^3+3k, k^4+4k^2+2, \ldots$. The numbers in this sequence are called k-Lucas numbers and we denote the n^{th} k-Lucas number by $L_{k,n}$. The k-Lucas numbers for negative subscripts are defined as $L_{k,-n} = (-1)^{-n}L_{k,n}$. It can be

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seen that $L_{k,n} = F_{k,n+1} + F_{k,n-1}$ for all $n \in \mathbb{Z}$. For $\{F_{k,n}\}$ and $\{L_{k,n}\}$, the characteristic equation is $x^2 - kx - 1 = 0$ with roots $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$, while the Binet formulae are $F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_{k,n} = \alpha^n + \beta^n$, respectively, for all $n \geq 0$. (see [4, 6, 7]).

2. Main Results

In this section, we will derive some identities for the k-Fibonacci and k-Lucas numbers by using the matrix approach. We begin with the following lemma:

Lemma 2.1. *Let X be a square matrix satisfying $X^2 = kX + I$. Then $X^n = F_{k,n}X + F_{k,n-1}I$ for all $n \in \mathbb{Z}$.*

Proof. If $n = 0$, then the assertion is obvious. Next, we will use mathematical induction to show that $X^n = F_{k,n}X + F_{k,n-1}I$ for all $n \in \mathbb{N}$.

When $n = 1$, we have $X = (1)X + (0)I = F_{k,1}X + F_{k,0}I$, so the assertion is seen to be true. Now assume that it is true for some positive $n = m$. We will show that it is true for $n = m + 1$ as follows:

$$\begin{aligned}
X^{m+1} &= X^m X \\
&= (F_{k,m}X + F_{k,m-1}I)\, X \\
&= F_{k,m}X^2 + F_{k,m-1}X \\
&= F_{k,m}\,(kX + I) + F_{k,m-1}X \\
&= F_{k,m+1}X + F_{k.m}I.
\end{aligned}$$

Hence, $X^n = F_{k,n}X + F_{k,n-1}I$ for all $n \in \mathbb{N}$. Finally, we will show that $X^{-n} = F_{k,-n}X + F_{k,-n-1}I$ for all $n \in \mathbb{N}$. Let us consider

$$\begin{aligned}
&(F_{k,n}X + F_{k,n-1}I)\left(F_{k,-n}X + F_{k,-(n+1)}I\right) \\
&= F_{k,n}F_{k,-n}X^2 + F_{k,n-1}F_{k,-n}X + F_{k,n}F_{k,-(n+1)}X + F_{k,n-1}F_{k,-(n+1)}I \\
&= (-1)^{-n+1}F_{k,n}^2\,(kX + I) + F_{k,n-1}(-1)^{-n+1}F_{k,n}X \\
&\qquad + F_{k,n}(-1)^{-n}F_{k,n+1}X + F_{k,n-1}(-1)^{-n}F_{k,n+1}I \\
&= (-1)^{-n+1}kF_{k,n}^2X + (-1)^{-n+1}F_{k,n}^2I + (-1)^{-n+1}F_{k,n-1}F_{k,n}X \\
&\qquad + (-1)^{-n}F_{k,n}F_{k,n+1}X + (-1)^{-n}F_{k,n-1}F_{k,n+1}I \\
&= (-1)^{-n}\left(-kF_{k,n}^2 - F_{k,n-1}F_{k,n} + F_{k,n}F_{k,n+1}\right)X \\
&\qquad + (-1)^{-n}\left(-F_{k,n}^2 + F_{k,n-1}F_{k,n+1}\right)I \\
&= (-1)^{-n}\left(-F_{k,n}\,(kF_{k,n} + F_{k,n-1}) + F_{k,n}F_{k,n+1}\right)X \\
&\qquad + (-1)^{-n}\left(-\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 + \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)\right)I \\
&= (-1)^{-n}\left(-F_{k,n}F_{k,n+1} + F_{k,n}F_{k,n+1}\right)X \\
&\qquad + (-1)^{-n}\left(\frac{2(\alpha\beta)^n - \alpha^{n-1}\beta^{n+1} - \alpha^{n+1}\beta^{n-1}}{(\alpha - \beta)^2}\right)I
\end{aligned}$$

$$= (0)\,X + (-1)^{-n}\left(\frac{-(\alpha\beta)^{n-1}\left(\alpha^2 - 2\alpha\beta + \beta^2\right)}{(\alpha-\beta)^2}\right) I$$
$$= -(-1)^{-n}(-1)^{n-1}\frac{(\alpha-\beta)^2}{(\alpha-\beta)^2} I = I.$$

In a similar way, we have $\left(F_{k,-n}X + F_{k,-(n+1)}I\right)\left(F_{k,n}X + F_{k,n-1}I\right) = I$. Thus, $X^{-n} = F_{k,-n}X + F_{k,-n-1}I$. This completes the proof of the lemma. ∎

From this lemma, we can easily derive Corollary 2.2. More details about the matrix F can be seen in [9].

Corollary 2.2. *Let* $F = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$. *Then* $F^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}$ *for all* $n \in \mathbb{Z}$.

The matrix Z considered in the following lemma will be used to obtain some identities for k-Fibonacci and k-Lucas numbers further below.

Lemma 2.3. *Let* $Z = \begin{bmatrix} \frac{k}{2} & \frac{\sqrt{k^2+4}}{2} \\ \frac{\sqrt{k^2+4}}{2} & \frac{k}{2} \end{bmatrix}$. *Then* $Z^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\sqrt{k^2+4}F_{k,n}}{2} \\ \frac{\sqrt{k^2+4}F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$ *for all* $n \in \mathbb{Z}$.

Proof. Note that $Z^2 = \begin{bmatrix} \frac{k^2}{2}+1 & \frac{k\sqrt{k^2+4}}{2} \\ \frac{k\sqrt{k^2+4}}{2} & \frac{k^2}{2}+1 \end{bmatrix} = kZ + I$. By Lemma 2.1, we have $Z^n = F_{k,n}Z + F_{k,n-1}I$. It follows that

$$Z^n = \begin{bmatrix} \frac{k}{2}F_{k,n} + F_{k,n-1} & \frac{\sqrt{k^2+4}}{2}F_{k,n} \\ \frac{\sqrt{k^2+4}}{2}F_{k,n} & \frac{k}{2}F_{k,n} + F_{k,n-1} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{kF_{k,n}+2F_{k,n-1}}{2} & \frac{\sqrt{k^2+4}}{2}F_{k,n} \\ \frac{\sqrt{k^2+4}}{2}F_{k,n} & \frac{kF_{k,n}+2F_{k,n-1}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{F_{k,n+1}+F_{k,n-1}}{2} & \frac{\sqrt{k^2+4}}{2}F_{k,n} \\ \frac{\sqrt{k^2+4}}{2}F_{k,n} & \frac{F_{k,n+1}+F_{k,n-1}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\sqrt{k^2+4}F_{k,n}}{2} \\ \frac{\sqrt{k^2+4}F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}.$$

This completes the proof of the lemma. ∎

By using the matrix Z, we obtain the next two lemmas.

Lemma 2.4. *For any integer* n, *the following equality holds:*

$$L_{k,n}^2 - \left(k^2+4\right)F_{k,n}^2 = 4(-1)^n.$$

Proof. Since $\det(Z) = -1$ and $\det\left(Z^n\right) = \frac{L_{k,n}^2}{4} - \frac{\left(k^2+4\right)F_{k,n}^2}{4}$, it follows that $L_{k,n}^2 - \left(k^2+4\right)F_n^2 = 4(-1)^n$, and the proof is complete. ∎

Lemma 2.5. *Let* m *and* n *be any integers. Then the following equalities hold:*

(1) $2L_{k,m+n} = L_{k,m}L_{k,n} + \left(k^2+4\right)F_{k,m}F_{k,n}$,
(2) $2F_{k,m+n} = L_{k,n}F_{k,m} + L_{k,m}F_{k,n}$.

Proof. Since $Z^{m+n} = Z^m Z^n$, by Lemma 2.3, we have

$$\begin{bmatrix} \frac{L_{k,m+n}}{2} & \frac{\sqrt{k^2+4}F_{k,m+n}}{2} \\ \frac{\sqrt{k^2+4}F_{k,m+n}}{2} & \frac{L_{k,m+n}}{2} \end{bmatrix} = \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{\sqrt{k^2+4}F_{k,m}}{2} \\ \frac{\sqrt{k^2+4}F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\sqrt{k^2+4}F_{k,n}}{2} \\ \frac{\sqrt{k^2+4}F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}.$$

Therefore

$$2L_{k,m+n} = L_{k,m}L_{k,n} + \left(k^2+4\right) F_{k,m}F_{k,n},$$
$$2F_{k,m+n} = L_{k,n}F_{k,m} + L_{k,m}F_{k,n}.$$

This completes the proof of the lemma. ∎

Lemma 2.6. *For any integer n, the following equalities hold:*

(1) $\alpha^n = \alpha F_{k,n} + F_{k,n-1}$,
(2) $\beta^n = \beta F_{k,n} + F_{k,n-1}$.

Proof. Let $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, so that $A^2 = kA + I$. By Lemma 2.1, we then have that $A^n = F_{k,n}A + F_{k,n-1}I$. It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha F_{k,n} + F_{k,n-1} & 0 \\ 0 & \beta F_{k,n} + F_{k,n-1} \end{bmatrix},$$

which implies that $\alpha^n = \alpha F_{k,n} + F_{k,n-1}$ and $\beta^n = \beta F_{k,n} + F_{k,n-1}$, and thus completes the proof. ∎

By using Lemma 2.1 and Lemma 2.6, we obtain the following lemma.

Lemma 2.7. *Let $B = \begin{bmatrix} \alpha & 0 \\ 1 & \beta \end{bmatrix}$. Then $B^n = \begin{bmatrix} \alpha^n & 0 \\ F_{k,n} & \beta^n \end{bmatrix}$ for all $n \in \mathbb{Z}$.*

Proof. Since $B^2 = kB + I$, by Lemma 2.1 and Lemma 2.6 it follows that

$$\begin{aligned} B^n &= F_{k,n}B + F_{k,n-1}I \\ &= \begin{bmatrix} \alpha F_{k,n} + F_{k,n-1} & 0 \\ F_{k,n} & \beta F_{k,n} + F_{k,n-1} \end{bmatrix} = \begin{bmatrix} \alpha^n & 0 \\ F_{k,n} & \beta^n \end{bmatrix}. \end{aligned}$$

This completes the proof of the lemma. ∎

Remark 2.8. For any integer n,

$$\begin{aligned} F_{k,n+2} + 2F_{k,n} + F_{k,n-2} &= kF_{k,n+1} + 4F_{k,n} - kF_{k,n-1} \\ &= kF_{k,n+1} + 4F_{k,n} - k\left(F_{k,n+1} - kF_{k,n}\right) = \left(k^2+4\right)F_{k,n}. \end{aligned}$$

Next, using Lemma 2.7 and Remark 2.8 we obtain the following theorem:

Theorem 2.9. *Let m and n be arbitrary integers. Then the following equality holds:*

$$(-1)^{m+n}L_{k,m+n}^2 + (-1)^m L_{k,m}^2 + (-1)^n L_{k,n}^2 = (-1)^{m+n}L_{k,m}L_{k,n}L_{k,m+n} + 4.$$

Proof. Let B be the matrix of Lemma 2.7. Then

$$B^{n+1} + B^{n-1} = \begin{bmatrix} \sqrt{k^2+4}\alpha^n & 0 \\ L_{k,n} & -\sqrt{k^2+4}\beta^n \end{bmatrix}.$$

Since $\left(B^{m+1} + B^{m-1}\right)\left(B^{n+1} + B^{n-1}\right) = B^{m+n+2} + 2B^{m+n} + B^{m+n-2}$, we obtain by Remark 2.8 that

$$\sqrt{k^2+4}F_{k,m+n} = \alpha^n L_{k,m} - \beta^m L_{k,n}.$$

Hence,

$$\begin{aligned}\left(k^2+4\right)F_{k,m+n}^2 &= \left(\sqrt{k^2+4}F_{k,m+n}\right)\left(\sqrt{k^2+4}F_{k,n+m}\right)\\ &= \left(\alpha^n L_{k,m}-\beta^m L_{k,n}\right)\left(\alpha^m L_{k,n}-\beta^n L_{k,m}\right)\\ &= \left(\alpha^{m+n}+\beta^{m+n}\right)L_{k,m}L_{k,n}-(\alpha\beta)^n L_{k,m}^2-(\alpha\beta)^m L_{k,n}^2\\ &= L_{k,m}L_{k,n}L_{k,m+n}-(-1)^n L_{k,m}^2-(-1)^m L_{k,n}^2.\end{aligned}$$

Then by Lemma 2.4,

$$L_{k,m+n}^2-4(-1)^{m+n}=L_{k,m}L_{k,n}L_{k,m+n}-(-1)^n L_{k,m}^2-(-1)^m L_{k,n}^2. \tag{2.1}$$

Hence, we can rewrite the above equation as follows:

$$(-1)^{m+n}L_{k,m+n}^2+(-1)^m L_{k,m}^2+(-1)^n L_{k,n}^2=(-1)^{m+n}L_{k,m}L_{k,n}L_{k,m+n}+4.$$

This completes the proof of the theorem. ∎

Theorem 2.10. *Let m and n be arbitrary integers. Then the following equality holds:*

$$\begin{aligned}&(-1)^{m+n}L_{k,m+n}^2+(-1)^{m+1}F_{k,m}^2+(-1)^{n+1}\left(k^2+4\right)F_{k,n}^2\\ &\qquad=(-1)^{m+n}\left(k^2+4\right)F_{k,m}F_{k,n}L_{k,m+n}+4.\end{aligned}$$

Proof. By (2.1), Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned}L_{k,m+n}^2-4(-1)^{m+n} &= \left(2L_{k,m+n}-\left(k^2+4\right)F_{k,m}F_{k,n}\right)L_{k,m+n}\\ &\quad-(-1)^n\left(\left(k^2+4\right)F_{k,m}^2+4(-1)^m\right)-(-1)^m\left(\left(k^2+4\right)F_{k,n}^2+4(-1)^n\right)\end{aligned}$$

which may be rewritten as

$$\begin{aligned}&L_{k,m+n}^2-(-1)^m\left(k^2+4\right)L_{k,m}-(-1)^n\left(k^2+4\right)L_{k,n}\\ &\qquad=(-1)^{m+n}\left(k^2+4\right)F_{k,m}F_{k,n}L_{k,m+n}+4(-1)^{m+n}.\end{aligned}$$

Thus we have

$$\begin{aligned}&(-1)^{m+n}L_{k,m+n}^2+(-1)^{m+1}F_{k,m}^2+(-1)^{n+1}\left(k^2+4\right)F_{k,n}^2\\ &\qquad=(-1)^{m+n}\left(k^2+4\right)F_{k,m}F_{k,n}L_{k,m+n}+4.\end{aligned}$$

This completes the proof of the theorem. ∎

Theorem 2.11. *Let m and n be arbitrary integers. Then the following equality holds:*

$$\begin{aligned}&(-1)^{m+n}\left(k^2+4\right)F_{k,m+n}^2+(-1)^{n+1}L_{k,n}^2+(-1)^m\left(k^2+4\right)F_{k,m}^2\\ &\qquad=(-1)^{m+n}\left(k^2+4\right)L_{k,n}F_{k,m}F_{k,m+n}-4.\end{aligned}$$

Proof. By a similar argument as in Theorem 2.9, and since

$$\left(B^{n+1}+B^{n-1}\right)B^m=B^{m+n+1}+B^{m+n-1}=B^m\left(B^{n+1}+B^{n-1}\right)$$

we obtain

$$\begin{aligned}L_{k,m+n} &= \alpha^m L_{k,n}-\sqrt{k^2+4}\beta^n F_{k,m} \quad \text{and}\\ L_{k,m+n} &= \sqrt{k^2+4}\alpha^n F_{k,m}+\beta^m L_{k,n}.\end{aligned}$$

Hence,

$$\begin{aligned}
L_{k,m+n}^2 &= \left(\alpha^m L_{k,n} - \sqrt{k^2+4}\beta^n F_{k,m}\right)\left(\sqrt{k^2+4}\alpha^n F_{k,m} + \beta^m L_{k,n}\right) \\
&= \sqrt{k^2+4}\left(\alpha^{m+n} - \beta^{m+n}\right) L_{k,n}F_{k,m} + (\alpha\beta)^m L_{k,n}^2 - \left(k^2+4\right)(\alpha\beta)^n F_{k,m}^2 \\
&= \left(k^2+4\right)\left(\frac{\alpha^{m+n} - \beta^{m+n}}{\alpha-\beta}\right) L_{k,n}F_{k,m} + (\alpha\beta)^m L_{k,n}^2 - \left(k^2+4\right)(\alpha\beta)^n F_{k,m}^2 \\
&= \left(k^2+4\right) L_{k,n}F_{k,m}F_{k,m+n} + (-1)^m L_{k,n}^2 - (-1)^n\left(k^2+4\right) F_{k,m}^2.
\end{aligned}$$

Then by Lemma 2.4,

$$\begin{aligned}
&\left(k^2+4\right) F_{k,m+n}^2 + 4(-1)^{m+n} \\
&\qquad = \left(k^2+4\right) L_{k,n}F_{k,m}F_{k,m+n} + (-1)^m L_{k,n}^2 - (-1)^n\left(k^2+4\right) F_{k,m}^2,
\end{aligned}$$

and we can write

$$\begin{aligned}
&(-1)^{m+n}\left(k^2+4\right) F_{k,m+n}^2 + (-1)^{n+1} L_{k,n}^2 + (-1)^m\left(k^2+4\right) F_{k,m}^2 \\
&\qquad = (-1)^{m+n}\left(k^2+4\right) L_{k,n}F_{k,m}F_{k,m+n} - 4.
\end{aligned}$$

This completes the proof of the theorem. ∎

3. Applications

In this section we give the solutions of some Diophantine equations by applying the identities of Theorems 2.9–2.11.

Theorem 3.1. *Let m and n be integers.*

(1) *If m and n are both even, then $(x, y, z) = (L_{k,m}, L_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 + x^2 + y^2 = xyz + 4$.*
(2) *If m and n are both odd, then $(x, y, z) = (L_{k,m}, L_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 - x^2 - y^2 = xyz + 4$.*
(3) *If m is even and n is odd, then $(x, y, z) = (L_{k,m}, L_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 - x^2 + y^2 = xyz - 4$.*
(4) *If m is odd and n is even, then $(x, y, z) = (L_{k,m}, L_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 + x^2 - y^2 = xyz - 4$.*

Proof. The assertion follows from Theorem 2.9. ∎

Theorem 3.2. *Let m and n be integers.*

(1) *If m and n are both even, then $(x, y, z) = (F_{k,m}, F_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 - x^2 - \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz + 4$.*
(2) *If m and n are both odd, then $(x, y, z) = (F_{k,m}, F_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 + x^2 + \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz + 4$.*
(3) *If m is even and n is odd, then $(x, y, z) = (F_{k,m}, F_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 + x^2 - \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz - 4$.*
(4) *If m is odd and n is even, then $(x, y, z) = (F_{k,m}, F_{k,n}, L_{k,m+n})$ is a solution of the equation $z^2 - x^2 + \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz - 4$.*

Proof. The assertion follows from Theorem 2.10. ∎

Theorem 3.3. *Let m and n be integers.*

(1) *If m and n are both even, then* $(x, y, z) = (L_{k,n}, F_{k,m}, F_{k,m+n})$ *is a solution of the equation* $\left(k^2+4\right) z^2 - x^2 + \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz - 4$.
(2) *If m and n are both odd, then* $(x, y, z) = (L_{k,n}, F_{k,m}, F_{k,m+n})$ *is a solution of the equation* $\left(k^2+4\right) z^2 + x^2 - \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz - 4$.
(3) *If m is even and n is odd, then* $(x, y, z) = (L_{k,n}, F_{k,m}, F_{k,m+n})$ *is a solution of the equation* $\left(k^2+4\right) z^2 - x^2 - \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz + 4$.
(4) *If m is odd and n is even, then* $(x, y, z) = (L_{k,n}, F_{k,m}, F_{k,m+n})$ *is a solution of the equation* $\left(k^2+4\right) z^2 + x^2 + \left(k^2+4\right) y^2 = \left(k^2+4\right) xyz + 4$.

Proof. The assertion follows from Theorem 2.11. ■

4. Conclusions

In this research, some identities for k-Fibonacci and k-Lucas numbers were studied and discovered by using a matrix approach. Furthermore, these identities were applied to present the solutions of some Diophantine equations.

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