Thai Journal of Mathematics
Volume 20 Number 1 (2022)
Pages 405–416

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Study of Generalized Clones of Rank k and Generalized k-Hypersubstitutions

Nareupanat Lekkoksung[1,*] **and Prakit Jampachon**[2]

[1] *Division of Mathematics, Faculty of Engineering, Rajamangala University of Technology Isan, Khon Kaen Campus, Khon Kaen 40000, Thailand*
e-mail : nareupanat.le@rmuti.ac.th

[2] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*
e-mail : prajam@kku.ac.th

Abstract The set of all n-ary terms of type τ_n together with an $(n+1)$-ary superposition and n nullary operation symbols forms an algebra, so-called a unitary Menger algebra of rank n. Generalizing this idea, let $k \geq n$, we study an algebraic structure consisting of the set of all k-ary terms of type τ_n, an $(n+1)$-ary generalized superposition and k nullary operation symbols. We call this algebra a generalized clone of rank k. We show that the generalized clone of rank k is a unitary Menger algebra of rank k. We use this concept to investigate the properties of a particular generalized hypersubstitution of type τ_n which maps each operation symbol of type τ_n to a k-ary term of the same type.

MSC: 08B15; 08B25
Keywords: hypersubstitution; generalized hypersubstitution; Menger algebra; clone; generalized superposition of terms

Submission date: 11.05.2018 / Acceptance date: 06.01.2022

1. Introduction

Let n be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an *alphabet* and its elements are called *variables*. Let $\{f_i : i \in I\}$ be the set of *operation symbols*, indexed by a set I. The sets X_n and $\{f_i : i \in I\}$ have to be disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$, called the *arity* of f_i. As in the definition of an algebra, the sequence $\tau = (n_i)_{i \in I}$ of all the arities is called the *type*. Classes of algebras can be described by logical expressions. This formal language is built up by variables from an n-element set. With this notation for operation symbols and variables, we can define terms of type τ, (see [1–3]).

An *n-ary term* of type τ, for simply an n-ary term, is defined in the following inductive way.

(i) Every variable $x_i \in X_n$ is an n-ary term.

*Corresponding author.

Published by The Mathematical Association of Thailand.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then the term $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

The set $W_\tau(X_n) = W_\tau(\{x_1, \ldots, x_n\})$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii). We denote the set of all terms of type τ by

$$W_\tau(X) := \bigcup_{m=1}^{\infty} W_\tau(X_m).$$

A type of algebras is called *n-ary* if all operation symbols of the type are n-ary, for some fixed natural number n. We let τ_n be such a fixed type with operation symbols $(f_i)_{i \in I}$, indexed by a nonempty set I.

For every $n \geq 1$. The $(n+1)$-ary superposition operation $\bar{S}^n$ on the set of all n-ary terms of type τ_n is defined in the following inductive way.

(i) If $t = x_j \in X_n$, then $S^n(x_j, t_1, \ldots, t_n) := t_j$.
(ii) If $t = f_i(s_1, \ldots, s_n)$, then

$$S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)).$$

On the set $W_{\tau_n}(X_n)$ of all n-ary terms of type τ_n together with the $(n+1)$-ary superposition operation $\bar{S}^n$ and the nullary operation symbols $x_1, \ldots, x_n$, one obtains an algebra

$$n\textbf{-clone}(\tau_n) := (W_{\tau_n}(X_n); \bar{S}^n, x_1, \ldots, x_n).$$

This algebra is an example of a *unitary Menger algebra of rank n*, (see [4, 5]). That is, the algebra n**-clone**(τ_n) satisfies the following identities.

(C1) $\tilde{S}^n(\tilde{Z}, \tilde{S}^n(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n(\tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n)) \approx \tilde{S}^n(\tilde{S}^n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n)\tilde{X}_1, \ldots, \tilde{X}_n)$.
(C2) $\tilde{S}^n(\lambda_i, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_i$ for all $1 \leq i \leq n$.
(C3) $\tilde{S}^n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}$.

Here $\tilde{Z}, \tilde{Y}_i, \tilde{X}_i$ are variables for terms for each $1 \leq i \leq n$, $\tilde{S}^n$ is an operation symbol, and λ_i is a variable for all $1 \leq i \leq n$.

Let V be a variety of some type $\tau := (n_i)_{i \in I}$. An identity $s \approx t$ satisfied in V is a hyperidentity of V if the identity $s \approx t$ holds also for all possible n_i-ary terms of the variety not only for the n_i-ary fundamental operations which occur in $s \approx t$. The study of hyperidentities was introduced by Taylor [6].

The hypersubstitution theory (of arbitrary type τ) was first initialed by Denecke et al. [7]. The concept of the superpositions is used to define the extension of a hypersubstitution. The authors used hypersubstitutions to make precise the concept of hyperidentities.

In 2000, the concept of hypersubstitutions was extended to generalized hypersubstitutions. This notion can be used to study strong hyperidentities [8]. The generalized hypersubstitutions are intensively investigated in the past decade, (see [9–12]).

In this paper, we generalize the study of the algebra n**-clone**(τ_n) by using a generalized superposition. The properties of this new algebra are investigated. Moreover, we introduce a particular kind of generalized hypersubstitutions of type τ_n and study their properties. Finally, the well-known connection between strong hyperidentities of a variety and identities satisfied by algebras in this variety is given in a restricted way.

2. Generalized Clone of Rank k

Leeratanavalee and Denecke [8] defined mappings from the set of all operation symbols to the set of all terms which may not preserve arities. These mappings are called generalized hypersubstitutions. To define their extensions, the authors defined the concept of a *generalized superposition* as follows. Let $m \geq 1$. Let $W_\tau(X)$ be the set of all terms of type τ. The *generalized superposition operation* S^m is an $(m+1)$-ary operation on $W_\tau(X)$ defined inductively by the following.

(i) If $t = x_j \in X_m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.
(ii) If $t = x_j \in X \setminus X_m$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.
(iii) If $t = f_i(s_1, \ldots, s_m)$, then

$$S^m(f_i(s_1, \ldots, s_m), t_1, \ldots, t_m) \\ := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_m, t_1, \ldots, t_m)).$$

By this definition, we generalize the algebra n-$\mathbf{clone}(\tau_n)$ in this way. Let k be a natural number such that $k \geq n$. We will use the set $W_{\tau_n}(X_k)$ of all k-ary terms of type τ_n as universe and a generalized superposition S^n as an operation on this carrier set. Together with nullary operation symbols $x_1, \ldots, x_k$, we obtain an algebra

$$k\text{-}\mathbf{clone_G}(\tau_n) := (W_{\tau_n}(X_k); S^n, x_1, \ldots, x_k)$$

of type $(n+1, 0, \ldots, 0)$. This algebra is called a *generalized clone of rank k*. Observe that if $n = k$, then

$$k\text{-}\mathbf{clone_G}(\tau_n) = n\text{-}\mathbf{clone}(\tau_n).$$

By this motivation, we call an algebra $(M; \widehat{S}^n, e_1, \ldots, e_k)$, where $\widehat{S}^n$ is an $(n+1)$-ary operation and nullary operation $e_1, \ldots, e_k$, a *unitary Menger algebra of rank k* if it satisfies the following identties.

(CG1) $\tilde{S}^n(T, \tilde{S}^n(F_1, T_1, \ldots, T_n), \ldots, \tilde{S}^n(F_n, T_1, \ldots, T_n))$
$\approx \tilde{S}^n(\tilde{S}^n(T, F_1, \ldots, F_n), T_1, \ldots, T_n)$.
(CG2) $\tilde{S}^n(T, \lambda_1, \ldots, \lambda_n) \approx T$.
(CG3) $\tilde{S}^n(\lambda_i, T_1, \ldots, T_n) \approx T_i$ for $1 \leq i \leq n$.
(CG4) $\tilde{S}^n(\lambda_i, T_1, \ldots, T_n) \approx \lambda_i$ for $i > n$.

Here T, T_i, F_i are variables for terms for each $1 \leq i \leq n$, $\tilde{S}^n$ is an operation symbol, and λ_i is a variable for all $1 \leq i \leq k$.

Theorem 2.1. *The algebra k-$\mathbf{clone_G}(\tau_n)$ is a unitary Menger algebra of rank k.*

Proof. (CG1): We replace the variables by arbitrary $t, t_1, \ldots, t_n, s_1, \ldots, s_n \in W_{\tau_n}(X_k)$ and the operation symbol by the generalized superposition S^n. Then we have

$$S^n(t, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \approx S^n(S^n(t, t_1, \ldots, t_n), s_1, \ldots, s_n).$$

We give a proof by induction on the complexity of the k-ary term t of type τ_n. If $t = x_i \in X_n$, then

$$\begin{aligned} &S^n(x_i, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad = S^n(t_i, s_1, \ldots, s_n) \\ &\quad = S^n(S^n(x_i, t_1, \ldots, t_n), s_1, \ldots, s_n). \end{aligned}$$

If $t = x \in X_k \setminus X_n$, then

$$\begin{aligned} &S^n(x, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad = x \\ &\quad = S^n(x, s_1, \ldots, s_n) \\ &\quad = S^n(S^n(x, t_1, \ldots, t_n), s_1, \ldots, s_n). \end{aligned}$$

If $t = f_i(u_1, \ldots, u_n)$ and assume that

$$\begin{aligned} &S^n(u_i, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad = S^n(S^n(u_i, t_1, \ldots, t_n), s_1, \ldots, s_n) \end{aligned}$$

for all $1 \leq i \leq n$, then

$$\begin{aligned} &S^n(f_i(u_1, \ldots, u_n), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad = f_i(S^n(u_1, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)), \\ &\qquad \ldots, S^n(u_n, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))) \\ &\quad = f_i(S^n(S^n(u_1, t_1, \ldots, t_n), s_1, \ldots, s_n), \\ &\qquad \ldots, S^n(S^n(u_n, t_1, \ldots, t_n), s_1, \ldots, s_n)) \\ &\quad = S^n(f_i(S^n(u_1, t_1, \ldots, t_n), \ldots, S^n(u_n, t_1, \ldots, t_n)), s_1, \ldots, s_n) \\ &\quad = S^n(S^n(f_i(u_1, \ldots, u_n), t_1, \ldots, t_n), s_1, \ldots, s_n). \end{aligned}$$

Thus, k-$\mathbf{clone_G}(\tau_n)$ satisfies (CG1).

(CG2): We replace the variable T by an arbitrary k-ary term t of type τ_n, $\tilde{S}^n$ by S^n and λ_i by $x_i \in X_n$ for all $1 \leq i \leq n$. Then we have

$$S^n(t, x_1, \ldots, x_n) \approx t.$$

We give a proof by induction on the complexity of the k-ary term t of type τ_n. If $t = x_i \in X_n$, then $S^n(x_i, x_1, \ldots, x_n) = x_i$. If $t = x \in X_k \setminus X_n$, then $S^n(x, x_1, \ldots, x_n) = x$. If $t = f_i(t_1, \ldots, t_n)$ and assume that $S^n(t_i, x_1, \ldots, x_n) = t_i$ for all $1 \leq i \leq n$, then

$$\begin{aligned} S^n(f_i(t_1, \ldots, t_n), x_1, \ldots, x_n) &= f_i(S^n(t_1, x_1, \ldots, x_n), \ldots, S^n(t_n, x_1, \ldots, x_n)) \\ &= f_i(t_1, \ldots, t_n). \end{aligned}$$

This implies that k-$\mathbf{clone_G}(\tau_n)$ satisfies (CG2).

Equations (CG3) and (CG4) correspond to the definition of S^n. Therefore, the algebra k-$\mathbf{clone_G}(\tau_n)$ is a unitary Menger algebra of rank k. ∎

Since (CG1), (CG2), (CG3) and (CG4) are identities, the class of all unitary Menger algebras of rank k forms a variety which is denoted by $V_{\mathcal{M}_k}$. Let $\mathcal{F}_{V_{\mathcal{M}_k}}(Y)$ be the free algebra with respect to $V_{\mathcal{M}_k}$, freely generated by $Y = \{y_i : i \in I\}$ where y_i is a new alphabet of individual variables indexed by a nonempty set I of the operation symbol f_i. We denote the $(n+1)$-ary operation and nullary operations defined on $\mathcal{F}_{V_{\mathcal{M}_k}}(Y)$ by $\tilde{S}^n$ and $\lambda_1, \ldots, \lambda_k$, respectively.

Let $\widehat{F}_{\tau_n} := \{f_i(x_1, \ldots, x_n) : i \in I\}$. Observe that $\widehat{F}_{\tau_n} \subseteq W_{\tau_n}(X_k)$. Then we have:

Lemma 2.2. *The set $\widehat{F}_{\tau_n}$ is a generating system of k-$\mathbf{clone_G}(\tau_n)$.*

Proof. We show by induction on the complexity of the k-ary term t of type τ_n that $W_{\tau_n}(X_k)$ is generated by $\widehat{F}_{\tau_n}$. Let $x_i \in X_k$. It is clear that x_i is generated since it belongs to the type of k-$\mathbf{clone_G}(\tau_n)$. Assume that $t = f_i(t_1, \ldots, t_n) \in W_{\tau_n}(X_k)$ and t_i is generated for all $1 \leq i \leq n$. Then

$$S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n) = f_i(t_1, \ldots, t_n).$$

This shows that $f_i(t_1, \ldots, t_n)$ is generated. Therefore, we obtain our result. ∎

Theorem 2.3. *The algebra k-$\mathbf{clone_G}(\tau_n)$ is free with respect to the variety $V_{\mathcal{M}_k}$ of unitary Menger algebras of rank k.*

Proof. We prove that k-$\mathbf{clone_G}(\tau_n)$ is isomorphic with $\mathcal{F}_{V_{\mathcal{M}_k}}(Y)$. Define $\varphi : W_{\tau_n}(X_k) \to \mathcal{F}_{V_{\mathcal{M}_k}}(\{y_i : i \in I\})$ by

(i) $\varphi(x_i) = \lambda_i$ for all $1 \leq i \leq k$,

(ii) $\varphi(f_i(t_1, \ldots, t_n)) = \tilde{S}^n(y_i, \varphi(t_1), \ldots, \varphi(t_n))$.

We prove the homomorphism property by induction on the complexity of the k-ary term t of type τ_n. If $x_i \in X_n$, then we have

$$\begin{aligned}\varphi(S^n(x_i, t_1, \ldots, t_n)) &= \varphi(t_i)\\ &= \tilde{S}^n(\lambda_i, \varphi(t_1), \ldots, \varphi(t_n))\\ &= \tilde{S}^n(\varphi(x_i), \varphi(t_1), \ldots, \varphi(t_n)).\end{aligned}$$

If $x_j \in X_k \setminus X_n$, then

$$\begin{aligned}\varphi(S^n(x_j, t_1, \ldots, t_n)) &= \lambda_j\\ &= \tilde{S}^n(\lambda_j, \varphi(t_1), \ldots, \varphi(t_n))\\ &= \tilde{S}^n(\varphi(x_j), \varphi(t_1), \ldots, \varphi(t_n)).\end{aligned}$$

Assume that $t = f_i(s_1, \ldots, s_n)$ and

$$\varphi(S^n(s_i, t_1, \ldots, t_n)) = \tilde{S}^n(\varphi(s_i), \varphi(t_1), \ldots, \varphi(t_n))$$

for all $1 \leq i \leq n$. Then

$$\begin{aligned}&\varphi(S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n))\\ &\quad= \varphi(f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)))\\ &\quad= \tilde{S}^n(y_i, \varphi(S^n(s_1, t_1, \ldots, t_n)), \ldots, \varphi(S^n(s_n, t_1, \ldots, t_n)))\\ &\quad= \tilde{S}^n(y_i, \tilde{S}^n(\varphi(s_1), \varphi(t_1), \ldots, \varphi(t_n)), \ldots, \tilde{S}^n(\varphi(s_n), \varphi(t_1), \ldots, \varphi(t_n)))\\ &\quad= \tilde{S}^n(\tilde{S}^n(y_i, \varphi(s_1), \ldots, \varphi(s_n)), \varphi(t_1), \ldots, \varphi(t_n))\\ &\quad= \tilde{S}^n(\varphi(f_i(s_1, \ldots, s_n), \varphi(t_1), \ldots, \varphi(t_n))).\end{aligned}$$

This shows that φ is a homomorphism. Let $y_i \in Y$. Then there exists $f_i(x_1, \ldots, x_n) \in \widehat{F}_{\tau_n}$ such that $\varphi(f_i(x_1, \ldots, x_n)) = y_i$. That is, φ is surjective. Moreover, φ is injective since $\{y_i : i \in I\}$ is free independent, we have

$$y_i = y_j \Rightarrow i = j \Rightarrow f_i(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n).$$

Therefore, φ is an isomorphism. ∎

On the set $W_{\tau_n}(X_k)$ of all k-ary terms of type τ_n with the set of all n-ary operation symbols $F^n := \{f_i : i \in I\}$, we can define an n-ary operation $\overline{f}_i : (W_{\tau_n}(X_k))^n \to W_{\tau_n}(X_k)$ by

$$\overline{f}_i(t_1, \ldots, t_n) := f_i(t_1, \ldots, t_n).$$

Then we obtain the *absolutely free algebra* $\mathcal{F}_{\tau_n}(X_k) := (W_{\tau_n}(X_k); (\overline{f}_i)_{i\in I})$ of type τ_n.

Let V be a variety of type τ_n. The *equation* $s \approx t \in W_{\tau_n}(X) \times W_{\tau_n}(X)$ is satisfied in the variety V if the term operations induced by s and t on every algebra from V are equal. We call the equation $s \approx t$ an *identity* in the variety V if $s \approx t$ is satisfied in V, denoted by $s \approx t \in \operatorname{Id} V$. We define

$$\operatorname{Id}_k V := \{s \approx t : s, t \in W_{\tau_n}(X_k) \text{ and } s \approx t \in \operatorname{Id} V\}.$$

That is, $\operatorname{Id}_k V = (W_{\tau_n}(X_k))^2 \cap \operatorname{Id} V$. It is clear that $\operatorname{Id}_k V$ is an equivalence relation on $W_{\tau_n}(X_k)$. If V is a variety of type τ_n, then $\operatorname{Id}_k V$ forms a fully invariant congruence relation on $\mathcal{F}_{\tau_n}(X_k)$. Then we obtain the following lemma.

Proposition 2.4. *Let V be a variety of type τ_n. Then $\operatorname{Id}_k V$ is a congruence relation on k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$.*

Proof. Assume that $s_1 \approx t_1, \ldots, s_n \approx t_n \in \operatorname{Id}_k V$. We show by induction on the complexity of the k-ary term t of type τ_n that

$$S^n(t, s_1, \ldots, s_n) \approx S^n(t, t_1, \ldots, t_n) \in \operatorname{Id}_k V.$$

If $x_i \in X_n$, then

$$S^n(t, s_1, \ldots, s_n) = s_i \approx t_i = S^n(t, t_1, \ldots, t_n) \in \operatorname{Id}_k V.$$

If $t = x_j \in X_k \setminus X_n$, then

$$S^n(t, s_1, \ldots, s_n) = x_j \approx x_j = S^n(t, t_1, \ldots, t_n) \in \operatorname{Id}_k V,$$

since $\operatorname{Id}_k V$ is an equivalence relation on $W_{\tau_n}(X_k)$. If $t = f_i(u_1, \ldots, u_n)$ and assume that

$$S^n(u_i, s_1, \ldots, s_n) \approx S^n(u_i, t_1, \ldots, t_n) \in \operatorname{Id}_k V$$

for all $1 \le i \le n$, then

$$\begin{aligned} S^n(f_i(u_1, \ldots, u_n), s_1, \ldots, s_n) &= f_i(S^n(u_1, s_1, \ldots, s_n), \ldots, S^n(u_n, s_1, \ldots, s_n)) \\ &\approx f_i(S^n(u_1, t_1, \ldots, t_n), \ldots, S^n(u_n, t_1, \ldots, t_n)) \\ &= S^n(f_i(u_1, \ldots, u_n), t_1, \ldots, t_n) \\ &\in \operatorname{Id}_k V \end{aligned}$$

by the fact that $\operatorname{Id}_k V$ is a fully invariant congruence relation on the algebra $\mathcal{F}_{\tau_n}(X_k)$. Next, we prove that if $t \approx s \in \operatorname{Id}_k V$, then

$$S^n(t, u_1, \ldots, u_n) \approx S^n(s, u_1, \ldots, u_n) \in \operatorname{Id}_k V.$$

This assertion holds since $\operatorname{Id}_k V$ is a fully invariant congruence relation of the absolutely free algebra $\mathcal{F}_{\tau_n}(X_k)$. Finally, assume that $t \approx s, t_1 \approx s_1, \ldots, t_n \approx s_n \in \operatorname{Id}_k V$. Then

$$S^n(t, t_1, \ldots, t_n) \approx S^n(s, t_1, \ldots, t_n) \approx S^n(s, s_1, \ldots, s_n) \in \operatorname{Id}_k V.$$

Therefore, $\operatorname{Id}_k V$ is a congruence relation on k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$. ∎

Now, we can form the quotient algebra

$$k\textbf{-clone}_{\tau_n} V := k\textbf{-clone}_{\mathbf{G}}(\tau_n)/\mathrm{Id}_k V,$$

with the $(n+1)$-ary operation

$$\widehat{S}^n : (W_{\tau_n}(X_k)/\mathrm{Id}_k V)^{n+1} \to W_{\tau_n}(X_k)/\mathrm{Id}_k V$$

on k-**clone**$_{\tau_n} V$ defined by

$$\widehat{S}^n([t]_{\mathrm{Id}_k V}, [t_1]_{\mathrm{Id}_k V}, \ldots, [t_n]_{\mathrm{Id}_k V}) := [S^n(t, t_1, \ldots, t_n)]_{\mathrm{Id}_k V}.$$

The quotient algebra

$$k\textbf{-clone}_{\tau_n} V := (W_{\tau_n}(X_k)/\mathrm{Id}_k V; \widehat{S}^n, [x_1]_{\mathrm{Id}_k V}, \ldots, [x_k]_{\mathrm{Id}_k V})$$

satisfies also (CG1) – (CG4). Thus, k-**clone**$_{\tau_n} V$ is a unitary Menger algebra of rank k as homomorphic image of k-**clone**$_{\mathbf{G}}(\tau_n)$.

Let $\widehat{F}_{\tau_n}^{\mathrm{Id}_k V} := \{[f_i(x_1, \ldots, x_n)]_{\mathrm{Id}_k V} : i \in I\}$. Then we prove the following.

Lemma 2.5. *The set $\widehat{F}_{\tau_n}^{\mathrm{Id}_k V}$ is a generating system of k-**clone**$_{\tau_n} V$.*

Proof. It is clear that $[x_i]_{\mathrm{Id}_k V}$ is generated for all $1 \leq i \leq k$ since it belongs to the type of k-**clone**$_{\tau_n} V$. Assume that $t = f_i(t_1, \ldots, t_n)$ and $[t_i]_{\mathrm{Id}_k V}$ is generated for all $1 \leq i \leq n$. Then

$$\begin{aligned}
&\widehat{S}^n([t]_{\mathrm{Id}_k V}, [t_1]_{\mathrm{Id}_k V}, \ldots, [t_n]_{\mathrm{Id}_k V}) \\
&\quad = \widehat{S}^n([f_i(x_1, \ldots, x_n)]_{\mathrm{Id}_k V}, [t_1]_{\mathrm{Id}_k V}, \ldots, [t_n]_{\mathrm{Id}_k V}) \\
&\quad = [S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n)]_{\mathrm{Id}_k V} \\
&\quad = [f_i(t_1, \ldots, t_n)]_{\mathrm{Id}_k V}.
\end{aligned}$$

Hence, the algebra k-**clone**$_{\tau_n} V$ is generated by $\widehat{F}_{\tau_n}^{\mathrm{Id}_k V}$. ∎

3. Generalized k-hypersubstitutions

A *generalized hypersubstitution* of type τ is a mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ which maps each operation symbol of type τ to a term of the same type which may not preserve arity. We denote the set of all generalized hypersubstitutions of type τ by $\mathrm{Hyp}_{\mathrm{G}}(\tau)$.

The generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ on the set of all terms of type τ inductively defined as follows:

(i) $\widehat{\sigma}[x] := x$ for any variable $x \in X$;

(ii) $\widehat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ for every n_i-ary operation symbol f_i, assumed that $\widehat{\sigma}[t_j]$ is already defined for all $1 \leq j \leq n_i$.

More detail about generalized hypersubstitutions of arbitrary type τ can be found in [8].

Denecke [13] studied a particular type τ_n of generalized hypersubstitutions. The author defined a binary operation $\circ_{\mathrm{G}}$ on $\mathrm{Hyp}_{\mathrm{G}}(\tau_n)$ by

$$\sigma_1 \circ_{\mathrm{G}} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$$

where $\circ$ denotes the usual composition of functions, and the identity generalized hypersubstitution of type τ_n which maps the operation symbol f_i to the term $f_i(x_1, \ldots, x_n)$ is denoted by σ_{id}, (see also [8]). As a consequence, we obtain the following interesting results.

Theorem 3.1. [13] *The algebra* $\mathbf{Hyp_G}(\tau_n)$ *is a monoid.*

The theory of strong hyperidentities is based on the monoid $(\mathrm{Hyp_G}(\tau);\circ_{\mathrm{G}},\sigma_{\mathrm{id}})$, denoted by $\mathbf{Hyp_G}(\tau)$, of a fixed type τ. These reasons demonstrate the importance of studying the monoid properties of $\mathbf{Hyp_G}(\tau)$ and its submonoids of a fixed type τ, (see [8]).

A *generalized k-hypersubstitution σ of type τ_n* is a generalized hypersubstitution of type τ_n which maps every operation symbol of type n to $W_{\tau_n}(X_k)$. We denote the set of all generalized k-hypersubstitutions of type τ_n by $\mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. We observe that if $k = n$, then σ is a usual hypersubstitution of type τ_n.

It is clear that $\mathrm{Hyp}^k_{\mathrm{G}}(\tau_n) \subseteq \mathrm{Hyp_G}(\tau_n)$. The product $\sigma_1 \circ_{\mathrm{G}} \sigma_2$ of two generalized k-hypersubstitutions of type τ_n is again a generalized k-hypersubstitution of type τ_n and σ_{id} is a generalized k-hypersubstitution of type τ_n. Thus, we obtain the following proposition immediately.

Proposition 3.2. *The algebra* $\mathbf{Hyp}^k_{\mathbf{G}}(\tau_n) := (\mathrm{Hyp}^k_{\mathrm{G}}(\tau_n);\circ_{\mathrm{G}},\sigma_{\mathrm{id}})$ *is a submonoid of* $\mathbf{Hyp_G}(\tau_n)$.

To study the properties of strong hyperidentities of a fixed type τ_n by using variables only from X_k, $k \geq n$ we will develop the theory of generalized k-hypersubstitutions of type τ_n.

Lemma 3.3. *Let* $\sigma \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. *Then*

$$\widehat{\sigma}[S^n(t,t_1,\ldots,t_n)] = S^n(\widehat{\sigma}[t],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]).$$

That is, $\widehat{\sigma}$ is an endomorphism on k-$\mathbf{clone_G}(\tau_n)$.

Proof. We will give a proof by induction on the complexity of the k-ary term t of type τ_n. If $t = x_i \in X_n$, then

$$\widehat{\sigma}[S^n(x_i,t_1,\ldots,t_n)] = \widehat{\sigma}[t_i] = S^n(x_i,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]) = S^n(\widehat{\sigma}[x_i],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]).$$

If $t = x \in X_k \setminus X_n$, then

$$\widehat{\sigma}[S^n(x,t_1,\ldots,t_n)] = x = S^n(x,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]) = S^n(\widehat{\sigma}[x],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]).$$

If $t = f_i(s_1,\ldots,s_n)$ and assume that

$$\widehat{\sigma}[S^n(s_i,t_1,\ldots,t_n)] = S^n(\widehat{\sigma}[s_i],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])$$

for all $1 \leq i \leq n$, then

$$\begin{aligned}
&\widehat{\sigma}[S^n(f_i(s_1,\ldots,s_n),t_1,\ldots,t_n)]\\
&\quad= \widehat{\sigma}[f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))]\\
&\quad= S^n(\sigma(f_i),\widehat{\sigma}[S^n(s_1,t_1,\ldots,t_n)],\ldots,\widehat{\sigma}[S^n(s_n,t_1,\ldots,t_n)])\\
&\quad= S^n(\sigma(f_i),S^n(\widehat{\sigma}[s_1],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]),\ldots,S^n(\widehat{\sigma}[s_n],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]))\\
&\quad= S^n(S^n(\sigma(f_i),\widehat{\sigma}[s_1],\ldots,\widehat{\sigma}[s_n]),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])\\
&\quad= S^n(\widehat{\sigma}[f_i(s_1,\ldots,s_n)],\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]).
\end{aligned}$$

Therefore, we have as desire. ∎

Proposition 3.4. *Let* $\sigma_1,\sigma_2 \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. *Then* $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ } = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Proof. It is not difficult to see that $\widehat{\sigma}_1 \circ \sigma_2 \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. We give a proof by induction on the complexity of the k-ary term t of type τ_n that $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ }[t] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)(t)$. It is clear that this equality holds if t is a variable. If $t = f_i(t_1, \dots, t_n)$ and assume that $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ }[t_i] = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)(t_i)$ for all $1 \leq i \leq n$, then

$$
\begin{aligned}
&(\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ }[f_i(t_1, \dots, t_n)] \\
&\qquad = S^n((\widehat{\sigma}_1 \circ \sigma_2)(f_i), (\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ }[t_1], \dots, (\widehat{\sigma}_1 \circ \sigma_2)\widehat{\ }[t_n]) \\
&\qquad = S^n(\widehat{\sigma}_1[\sigma_2(f_i)], \widehat{\sigma}_1[\widehat{\sigma}_2[t_1]], \dots, \widehat{\sigma}_1[\widehat{\sigma}_2[t_n]]) \\
&\qquad = \widehat{\sigma}_1[S^n(\sigma_2(f_i), \widehat{\sigma}_2[t_1], \dots, \widehat{\sigma}_2[t_n])] \\
&\qquad = (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)(f_i(t_1, \dots, t_n)).
\end{aligned}
$$

Therefore, the extension of the composition of any two elements in $\mathbf{Hyp}^k_{\mathbf{G}}(\tau_n)$ is a composition of their extensions. ∎

Proposition 3.5. *Every endomorphism on k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$ is the extension of a generalized k-hypersubstitution of type τ_n.*

Proof. Let $\varphi : W_{\tau_n}(X_k) \to W_{\tau_n}(X_k)$ be an endomorphism. We observe that $\varphi \circ \sigma_{\mathrm{id}} \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. Thus, we claim that $\varphi = (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }$. Let t be a k-ary term of type τ_n. We give a proof by induction on the complexity of the k-ary term t of type τ_n that $\varphi(t) = (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[t]$. Since every endomorphism fixed constants, we have that $\varphi(t) = (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[t]$ whenever t is a variable. If $t = f_i(t_1, \dots, t_n)$ and assume that $\varphi(t_i) = (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[t_i]$ for all $1 \leq i \leq n$, then

$$
\begin{aligned}
&(\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[f_i(t_1, \dots, t_n)] \\
&\qquad = S^n((\varphi \circ \sigma_{\mathrm{id}})(f_i), (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[t_1], \dots, (\varphi \circ \sigma_{\mathrm{id}})\widehat{\ }[t_n]) \\
&\qquad = S^n(\varphi(f_i(x_1, \dots, x_n)), \varphi(t_1), \dots, \varphi(t_n)) \\
&\qquad = \varphi(S^n(f_i(x_1, \dots, x_n), t_1, \dots, x_n)) \\
&\qquad = \varphi(f_i(t_1, \dots, t_n)).
\end{aligned}
$$

Thus, the proof is completed. ∎

By Lemma 2.2 and Theorem 2.3, $\widehat{F}_{\tau_n}$ is a generating set of k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$ and the algebra k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$ is free with respect to the variety $V_{\mathcal{M}_k}$. Since k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$ belongs to the variety $V_{\mathcal{M}_k}$, we have that any mapping η from $\widehat{F}_{\tau_n}$ into $W_{\tau_n}(X_k)$ can be uniquely extended to an endomorphism $\widehat{\eta}$ of k-$\mathbf{clone}_{\mathbf{G}}(\tau_n)$. We call such mapping *generalized clone k-substitutions* of type τ_n. We denote the set of all generalized clone k-substitutions of type τ_n by $\mathrm{Subst}^k_{\mathrm{G}}(\tau_n)$. Define a binary operation $\odot_{\mathrm{G}}$ on $\mathrm{Subst}^k_{\mathrm{G}}(\tau_n)$ by $\eta_1 \odot_{\mathrm{G}} \eta_2 := \widehat{\eta}_1 \circ \eta_2$, where $\circ$ is the usual composition of functions. An identity generalized k-substitution of type τ_n is defined by $\mathrm{id}(f_i(x_1, \dots, x_n)) = f_i(x_1, \dots, x_n)$. By this setting, we see that the algebra $\mathbf{Subst}^k_{\mathbf{G}}(\tau_n) := (\mathrm{Subst}^k_{\mathrm{G}}(\tau_n); \odot_{\mathrm{G}}, \mathrm{id})$ is a monoid.

Lemma 3.6. *Let t be a k-ary term of type τ_n and η a generalized k-substitution of type τ_n. Then $(\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t] = \widehat{\eta}(t)$.*

Proof. We prove this equation by induction on the complexity of the k-ary term t of type τ_n. It is clear that $(\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t] = \widehat{\eta}(t)$ if t is a variable since the endomorphism $\widehat{\eta}$ fixes constants. Assume that $t = f_i(t_1, \ldots, t_n)$ and $(\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t_i] = \widehat{\eta}(t_i)$ for all $1 \le i \le n$. Then

$$\begin{aligned}(\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[f_i(t_1, \ldots, t_n)] &= S^n((\eta \circ \sigma_{\mathrm{id}})(f_i), (\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t_1], \ldots, (\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t_n]) \\ &= S^n(\eta(f_i(x_1, \ldots, x_n)), \widehat{\eta}(t_1), \ldots, \widehat{\eta}(t_n)) \\ &= S^n(\widehat{\eta}(f_i(x_1, \ldots, x_n)), \widehat{\eta}(t_1), \ldots, \widehat{\eta}(t_n)) \\ &= \widehat{\eta}(S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n)) \\ &= \widehat{\eta}(f_i(t_1, \ldots, t_n)).\end{aligned}$$

Therefore, $(\eta \circ \sigma_{\mathrm{id}})\widehat{\ }[t] = \widehat{\varphi}(t)$. ∎

Theorem 3.7. *The monoids* $\mathbf{Hyp}^k_{\mathbf{G}}(\tau_n)$ *and* $\mathbf{Subst}^k_{\mathbf{G}}(\tau_n)$ *are isomorphic.*

Proof. Define $\varphi : \mathrm{Subst}^k_{\mathrm{G}}(\tau_n) \to \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$ by $\eta \mapsto \eta \circ \sigma_{\mathrm{id}}$. It is clear that $\eta \circ \sigma_{\mathrm{id}} \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$ and φ is well-defined. Let $\sigma \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. Then $\sigma \circ \sigma_{\mathrm{id}}^{-1} \in \mathrm{Subst}^k_{\mathrm{G}}(\tau_n)$ and $\varphi(\sigma \circ \sigma_{\mathrm{id}}^{-1}) = \sigma$. This shows that φ is surjective. Injectivity is clear since

$$\varphi(\eta_1) = \varphi(\eta_2) \Rightarrow \eta_1 \circ \sigma_{\mathrm{id}} = \eta_2 \circ \sigma_{\mathrm{id}} \Rightarrow \eta_1 = \eta_2.$$

Finally, we show that φ is a homomorphism. Let $\eta_1, \eta_2 \in \mathrm{Subst}^k_{\mathrm{G}}(\tau_n)$. Then

$$\begin{aligned}\varphi(\eta_1) \circ_{\mathrm{G}} \varphi(\eta_2) &= (\eta_1 \circ \sigma_{\mathrm{id}}) \circ_{\mathrm{G}} (\eta_2 \circ \sigma_{\mathrm{id}}) \\ &= (\eta_1 \circ \sigma_{\mathrm{id}})\widehat{\ } \circ (\eta_2 \circ \sigma_{\mathrm{id}}) \\ &= \widehat{\eta}_1 \circ (\eta_2 \circ \sigma_{\mathrm{id}}) \\ &= (\widehat{\eta}_1 \circ \eta_2) \circ \sigma_{\mathrm{id}} \\ &= (\eta_1 \odot_{\mathrm{G}} \eta_2) \circ \sigma_{\mathrm{id}} \\ &= \varphi(\eta_1 \odot_{\mathrm{G}} \eta_2).\end{aligned}$$

Therefore, we obtain our result. ∎

Let V be a variety of type τ_n. An identity $s \approx t \in \mathrm{Id}\, V$ is said to be a *strong hyperidentity* [8] in V if $\widehat{\sigma}[t] \approx \widehat{\sigma}[s] \in \mathrm{Id}\, V$ for $s, t \in W_{\tau_n}(X)$ and $\sigma \in \mathrm{Hyp}_{\mathrm{G}}(\tau_n)$. The set of all strong hyperidentities in V is denoted by $\text{H-Id}\, V$. A variety in which each of its identities holds as a strong hyperidentity is called a *strongly solid variety.* We define

$$\text{H-Id}_k\, V := \{s \approx t : s, t \in W_{\tau_n}(X_k), \widehat{\sigma}[t] \approx \widehat{\sigma}[s] \in \mathrm{Id}\, V \text{ for all } \sigma \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)\}.$$

Then $\text{H-Id}_k\, V$ is an equivalence relation on $W_{\tau_n}(X_k)$.

Proposition 3.8. *Let V be a variety of type τ_n. Then $\text{H-Id}_k\, V$ is a congruence relation on k-$\mathbf{clone_G}(\tau_n)$.*

Proof. Assume that $s \approx t, s_1 \approx t_1, \ldots, s_n \approx t_n \in \text{H-Id}_k\, V$. Let $\sigma \in \mathrm{Hyp}^k_{\mathrm{G}}(\tau_n)$. Then $\widehat{\sigma}[s] \approx \widehat{\sigma}[t], \widehat{\sigma}[s_1] \approx \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[s_n] \approx \widehat{\sigma}[t_n] \in \mathrm{Id}_k\, V$. By Proposition 2.4, we have that $S^n(\widehat{\sigma}[s], \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_n]) \approx S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) \in \mathrm{Id}_k\, V$. This implies that $\widehat{\sigma}[S^n(s, s_1, \ldots, s_n)] \approx \widehat{\sigma}[S^n(t, t_1, \ldots, t_n)] \in \mathrm{Id}_k\, V$ by Lemma 3.3. Therefore, we obtain that $S^n(s, s_1, \ldots, s_n) \approx S^n(t, t_1, \ldots, t_n) \in \text{H-Id}_k\, V$. This shows our claim. ∎

Lemma 3.9. *Let V be a variety of type τ_n and $s, t \in W_{\tau_n}(X_k)$. If $s \approx t \in \text{H-Id}_k\, V$, then $\varphi(s) \approx \varphi(t) \in \text{H-Id}_k\, V$ for all endomorphism φ on $W_{\tau_n}(X_k)$.*

Proof. Assume that $s \approx t \in$ H-Id$_k V$. Let φ be an endomorphism φ on $W_{\tau_n}(X_k)$. By Proposition 3.5, we have that $\varphi = (\varphi \circ \sigma_{\text{id}})\widehat{\ }$. Now, we let $\sigma \in \text{Hyp}_G^k(\tau_n)$. Then

$$\begin{aligned}
\widehat{\sigma}[\varphi(s)] &= \widehat{\sigma}[(\varphi \circ \sigma_{\text{id}})\widehat{\ }[s]] \\
&= (\widehat{\sigma} \circ (\varphi \circ \sigma_{\text{id}})\widehat{\ })[s] \\
&\approx (\widehat{\sigma} \circ (\varphi \circ \sigma_{\text{id}})\widehat{\ })[t] \\
&= \widehat{\sigma}[(\varphi \circ \sigma_{\text{id}})\widehat{\ }[t]] \\
&= \widehat{\sigma}[\varphi(t)] \\
&\in \text{Id}_k V.
\end{aligned}$$

Therefore, $\varphi(s) \approx \varphi(t) \in$ H-Id$_k V$. ■

By above two results, we obtain the following theorem.

Theorem 3.10. *Let V be a variety of type τ_n. Then* H-Id$_k V$ *is a fully invariant congruence relation on k-**clone**$_{\mathbf{G}}(\tau_n)$.*

Next, we give a connection between strong hyperidentities of a variety of type τ_n and identities satisfied in V.

Theorem 3.11. *Let V be a variety of type τ_n. Then V is strongly solid if and only if* Id$_k V$ *is a fully invariant congruence relation on k-**clone**$_{\mathbf{G}}(\tau_n)$.*

Proof. We assume that Id$_k V$ is a fully invariant congruence relation on k-**clone**$_{\mathbf{G}}(\tau_n)$. Let $s \approx t \in \text{Id}_k V$. We show that $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in \text{Id}_k V$ for all $\sigma \in \text{Hyp}_G^k(\tau_n)$. Let $\sigma \in \text{Hyp}_G^k(\tau_n)$. By Lemma 3.3, the extension of σ is an endomorphism on k-**clone**$_{\mathbf{G}}(\tau_n)$. This implies that $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in \text{Id}_k V$ since Id$_k V$ is preserved by every endomorphism on k-**clone**$_{\mathbf{G}}(\tau_n)$. Conversely, assume that V is strongly solid. It is clear that Id$_k V$ is a congruence relation on k-**clone**$_{\mathbf{G}}(\tau_n)$ by Proposition 2.4. Assume that $s \approx t \in \text{Id}_k V$. Let φ is an endomorphism on k-**clone**$_{\mathbf{G}}(\tau_n)$. By Proposition 3.5, $\varphi = (\varphi \circ \sigma_{\text{id}})\widehat{\ }$. It is clear that $\varphi \circ \sigma_{\text{id}} \in \text{Hyp}_G^k(\tau_n)$. Then

$$\varphi(s) = (\varphi \circ \sigma_{\text{id}})\widehat{\ }[s] \approx (\varphi \circ \sigma_{\text{id}})\widehat{\ }[t] = \varphi(t) \in \text{Id}_k V.$$

Therefore, Id$_k V$ is a fully invariant congruence relation on k-**clone**$_{\mathbf{G}}(\tau_n)$. ■

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] K. Denecke, S.L. Wismath, Hyperidentities and Clones, Gordon and Breach Science Publishers, 2000.

[2] K. Denecke, S.L. Wismath, Universal Algebra and Applications in Theoretical Computer Science, Boca Raton: Chapman & Hall/CRC, 2002.

[3] J. Koppitz, K. Denecke, *M*-Solid Varieties of Algebras, Springer Science+Business Media, Inc., 2006.

[4] K. Denecke, P. Jampachon, *N*-full varieties and clone of *N*-full terms, Southeast Asian Bull. Math. 29 (2005) 455–468.

[5] K. Denecke, P. Jampachon, S.L. Wismath, Clones of n-ary algebras, J. Appl. Algebra Discrete Struct. 2 (2003) 141–158.

[6] W. Taylor, Hyperidentities and hypervarieties, AequationesMath. 23 (1981) 111–127.

[7] K. Denecke, D. Lau, R. Pöschel, D. Schweigert, *Hyperidentities, hyperequational clases and clone congruences*, Contribution to General Algebra 7, Verlag Hölder-Pichler-Tempsky, Wien (1991) 97–118.

[8] S. Leeratanavalee, K. Denecke, *Generalized hypersubstitutions and strongly solid varieties*, General Algebra and Applications, Proceeding of the "59th workshop on general algebra 15th conference for young algebraists Potsdam 2000" (Shaker Verlag, 2000) 135–145.

[9] S. Leeratanavalee, Submonoids of generalized hypersubstitutions, Demonstratio Math. XL (2007) 13–22.

[10] N. Lekkoksung, P. Jampachon, Some properties of generalized full hypersubstitutions, Far East J. Math. Sci. 100 (2016) 343 – 359.

[11] N. Lekkoksung, P. Jampachon, Idempotent elements of pre-generalized hypersubstitutions of type (m, n), Ital. J. Pure Appl. Math. 36 (2016) 827–842.

[12] W. Puninagool, S. Leeratanavalee, Complexity of terms, superpositions, and generalized hypersubstitutions, Comput Math Appl 59 (2010) 1038–1045.

[13] K. Denecke, Strongly Solid Varieties and Free Generalized Clones, Kyungpook Math. J. 45 (2005) 33–43.