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On V-Regular Ordered LA-Semihypergroups

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Abstract In this paper, we introduce the concepts of fuzzy (bi-, generalized bi-) hyperideals, fuzzy interior hyperideals, fuzzy quasi-hyperideals in ordered LA-semihypergroups. Moreover we characterize V-regular class of ordered LA-semihypergroups in terms of fuzzy (left, right, two-sided, interior, bi-, generalized bi- and quasi-) hyperideals. Furthermore, we have shown that all these fuzzy hyperideals coincide in a V-regular ordered LA-semihypergroup with pure left identity.

MSC: 18B40; 20N20; 20M12

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1. Introduction

In 1934, Marty introduced the theory of hyperstructures [1]. He analyzed different properties of hypergroups and applied them to the theory of groups. Thus one can say that hypergroups are suitable generalization of classical groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus hyperstructures are natural extension of classical algebraic structures. After the pioneering work of Marty, algebraic hyperstructures have been intensively studied from the theoretical point of view and especially for their applications in other fields

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such as Euclidean and non-Euclidean geometries, graphs and hypergraphs, fuzzy sets, automata, cryptography, artificial intelligence, codes, probabilities, lattices and so on (see [2]). Several books and papers have been written on algebraic hyperstructures theory, for example, (see [3–10]).
Kazim and Naseerudin [11], introduced the concept of left almost semigroups (abbreviated as LA-semigroups). They generalized some useful results of semigroup theory. Later, Mushtaq and others further investigated the structure and added many useful results to the theory of LA-semigroups, also (see [12–15]). An LA-semigroup is the midway structure between a commutative semigroup and a groupoid. Despite the fact, the structure is non-associative and non-commutative. It nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures. Mushtaq and Yusuf produced useful results [16], on locally associative LA-semigroups in 1979. In this structure they defined powers of an element and congruences using these powers. They constructed quotient LA-semigroups using these congruences. It is a useful non-associative structure with wide applications in theory of flocks [17]. Several papers are written on LA-semigroups. There are lot of results which have been added to the theory of LA-semigroups by Mushtaq, Kamran, Shabir, Aslam, Davvaz, Madad, Faisal, Abdullah, Yaqoob, Hila, Rehman, Chinram, Holgate, Jezek, Protic and many other researchers. Hila and Dine[18], introduced the notion of LA-semihypergroups they investigated several properties of hyperideals of LA-semihypergroup and studied by many authors (see [19–21]). Zadeh introduced a mathematical framework called fuzzy sets [22], which plays a significant role in many fields of real life. Fuzzy set has several advantages over a Cantorian set because it has a clear demarcation about uncertainty and vagueness. A fuzzy set is actually characterized by a membership function with the range of $[0, 1]$. The membership of an element in a fuzzy set is a single value between 0 and 1. In [23], Khan et al. applied fuzzy set theory to LA-semihypergroups and introduced the notions of fuzzy left (resp. right) hyperideals in LA-semihypergroups.
In this paper, we introduce the notions of fuzzy left (resp. right, interior, bi-, generalized bi- and quasi-) hperideals of ordered LA-semihypergroups and investigate related properties. Moreover, we characterize a V-regular ordered LA-semihypergroups in terms of these fuzzy hyperideals

2. Preliminaries

A hypergroupoid is a nonempty set S equipped with a hyperoperation $\circ$, that is a map $\circ : S \times S \longrightarrow P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [1]). We shall denote by $x \circ y$, the hyperproduct of elements x, y of S. Let A, B be the nonempty subsets of S. Then the hyperproduct of A and B is defined as $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. We shall write $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$.

A hypergroupoid $(S, \circ)$ is called an LA-semihypergroup [18], if it satisfies the left invertive law:

$$(a \circ b) \circ c = (c \circ b) \circ a \text{ for all } a, b, c \in S.$$

Every LA-semihypergroup satisfies the following law:

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w),$$

for all $w, x, y, z \in S$. This law is known as medial law [18].

Definition 2.1. (see [19]). Let $(S, \circ)$ be an LA-semihypergroup then an element $e \in S$ is called

(i) left identity (resp. pure left identity) if $\forall\, a \in S$, $a \in e \circ a$ (resp. $a = e \circ a$).
(ii) right identity (resp. pure right identity) if $\forall\, a \in S$, $a \in a \circ e$ (resp. $a = a \circ e$).
(iii) identity (resp. pure identity) if $\forall\, a \in S$, $a \in e \circ a \cap a \circ e$ (resp. $a = e \circ a \cap a \circ e$).

An LA-semihypergroup $(S, \circ)$ with pure left identity e, paramedial law holds. That is $(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x)$ for all $w, x, y, z \in S$.

An LA-semihypergroup $(S, \circ)$ with pure left identity e, satisfies the following law

$$x \circ (y \circ z) = y \circ (x \circ z) \qquad (a).$$

An ordered LA-semihypergroup is a structure $(S, \circ, \leq)$ in which the following conditions hold:

(1) $(S, \circ)$ is an LA-semihypergroup.
(2) $(S, \leq)$ is a poset.
(3) $\forall\, a, b, x \in S$, $a \leq b \Longrightarrow a \circ x \leq b \circ x$ and $x \circ a \leq x \circ b$.

This means that there exist $u \in x \circ a$ and $v \in x \circ b$ such that $u \leq v$. Moreover

if $A, B \in P^*(S)$,then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. If $A = \{a\}$ then we write $a \preceq B$ instead of $\{a\} \preceq B$.

A non-empty subset A of an ordered LA-semihypergroup S is called an LA-*subsemihypergroup* of S if $A^2 \subseteq A$.

A non-empty subset A of S is called a *left* (resp. *right*) *hyperideal* of S if it satisfies the following conditions:

(1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$).
(2) If $a \in A, b \in S$ and $b \leq a$, implying $b \in A$.

By a two sided hyperideal or simply a *hyperideal* of S we mean a non-empty subset of S which is both a left hyperideal and a right hyperideal of S.

A non-empty subset I of an ordered LA-semihypergroup S is called an interior *hyperideal* of S if it satisfies the following conditions:

(1) $(S \circ I) \circ S \subseteq I$.
(2) If $a \in I, b \in S$ and $b \leq a$ imply $b \in I$.

A subsemihypergroup B of an ordered LA-semihypergroup S is called a *bi-hyperideal* of S if it satisfies the following conditions:

(1) $(B \circ S) \circ B \subseteq B$.
(2) If $a \in B, b \in S$ and $b \leq a$ imply $b \in B$.

A non-empty subset B of an ordered LA-semihypergroup S is called a generalized *bi-hyperideal* of S if it satisfies the following conditions:

(1) $(B \circ S) \circ B \subseteq B$.
(2) If $a \in B, b \in S$ and $b \leq a$ imply $b \in B$.

A nonempty subset Q of an ordered LA-semihypergroup $(S, \circ, \leq)$ is called a quasi-hyperideal of S if:

(1) $Q \circ S \cap S \circ Q \subseteq Q$.
(2) If $a \in Q$ and $S \ni b \leq a$ then $b \in Q$.

For $A \subseteq S$,we denote $(A] := \{t \in S : t \leq h \text{ for some } h \in A\}$.

For $A, B \subseteq S$, we have $A \circ B := \bigcup \{a \circ b : a \in A,\ b \in B\}$.

If S is an ordered LA-semihypergroup then the following are true

(1) $A \subseteq (A] \ \forall \ A \subseteq S$.
(2) if $A \subseteq B$, then $(A] \subseteq (B]$.
(3) $(A] \circ (B] \subseteq (A \circ B] \ \forall \ A, B \subseteq S$.
(4) $(A] = ((A]] \ \forall \ A \subseteq S$.
(5) $((A] \circ (B]] = (A \circ B] \ \forall \ A, B \subseteq S$.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq (a \circ x) \circ a$. Equivalent Definitions:

(1) $a \in ((a \circ S) \circ a] \ \forall a \in S$. (2) $A \subseteq ((A \circ S) \circ A] \ \forall A \subseteq S$.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called left (resp. right) *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq x \circ a^2$ (resp. $a \leq a^2 \circ x$) Equivalent Definitions:

(1) $a \in (S \circ a^2] \ \forall a \in S$ (resp. $a \in (a^2 \circ S] \ \forall a \in S$). (2) $A \subseteq (S \circ A^2] \ \forall A \subseteq S$ (resp. $A \subseteq (A^2 \circ S] \ \forall A \subseteq S$).

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called completely regular if it is regular, left regular and right regular.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called intra-*regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq \left(x \circ a^2\right) \circ y$. Equivalent Definitions:

(1) $a \in (\left(S \circ a^2\right) \circ S] \ \forall a \in S$.
(2) $A \subseteq (\left(S \circ A^2\right) \circ S] \ \forall A \subseteq S$.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called a $(2,2)$-*regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq \left(a^2 \circ x\right) \circ a^2$. Equivalent Definitions:

(1) $a \in (\left(a^2 \circ S\right) \circ a^2] \ \forall a \in S$.
(2) $A \subseteq (\left(A^2 \circ S\right) \circ A^2] \ \forall A \subseteq S$.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called a weakly-*regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq (a \circ x) \circ (a \circ y)$.

An ordered LA-semihypergroup $(S, \circ, \leq)$ is called a left quasi-*regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq (x \circ a) \circ (y \circ a)$.

$\emptyset \neq A \subseteq S$ is called semiprime if $a \circ a \subseteq A \Longrightarrow a \in A$.

3. On Some Classes of an Ordered LA-semihypergroup

Lemma 3.1. *The left and completely regular classes of an ordered LA-semihypergroup S coincide in S with pure left identity e.*

Proof. Let S be a left regular ordered LA-semihypergroup with pure left identity then for every $a \in S$ there exists $x \in S$ such that $a \leq x \circ a^2$. Now

$$a \leq x \circ a^2 = (e \circ x) \circ (a \circ a) = (a \circ a) \circ (x \circ e) \leq a^2 \circ y,$$

for some $y \in x \circ e$. Also,

$$\begin{aligned} a &\leq x \circ a^2 = x \circ (a \circ a) = a \circ (x \circ a) \\ &\leq \left(x \circ a^2\right) \circ (x \circ a) = (a \circ x) \circ \left(a^2 \circ x\right) \\ &= (((a \circ a) \circ x) \circ x) \circ a \\ &= \left(x^2 \circ (a \circ a)\right) \circ a = \left(a \circ \left(x^2 \circ a\right)\right) \circ a \\ &\leq (a \circ z) \circ a, \end{aligned}$$

for some $z \in \left(x^2 \circ a\right)$.

The converse is simple. ■

Corollary 3.2. *The left and right regular classes of S coincide in S with pure left identity.*

Theorem 3.3. *An ordered LA-semihypergroup S with pure left identity is completely regular if and only if $a \in \left(\left(a^2 \circ S\right) \circ a^2\right]$.*

Proof. Let S be a completely regular with pure left identity then for each $a \in S$ there exist $u, v, w \in S$ such that

$$\begin{aligned} a &\leq (a \circ u) \circ a \leq \left(\left(a^2 \circ v\right) \circ u\right) \circ \left(w \circ a^2\right) \\ &= \left((u \circ v) \circ a^2\right) \circ \left(w \circ a^2\right) = \left(\left(w \circ a^2\right) \circ a^2\right) \circ (u \circ v) \\ &= \left(\left(a^2 \circ a^2\right) \circ w\right) \circ (u \circ v) = ((u \circ v) \circ w) \circ \left(a^2 \circ a^2\right) \\ &= a^2 \circ \left(((u \circ v) \circ w) \circ a^2\right) = \left(e \circ a^2\right) \circ \left(((u \circ v) \circ w) \circ a^2\right) \\ &= \left(a^2 \circ ((u \circ v) \circ w)\right) \circ \left(a^2 \circ e\right) = \left(a^2 \circ ((u \circ v) \circ w)\right) \circ ((a \circ a) \circ e) \\ &= \left(a^2 \circ ((u \circ v) \circ w)\right) \circ ((e \circ a) \circ a) \\ &= \left(a^2 \circ ((u \circ v) \circ w)\right) \circ a^2 \subseteq \left(a^2 \circ S\right) \circ a^2. \end{aligned}$$

Thus $a \leq \left(a^2 \circ ((u \circ v) \circ w)\right) \circ a^2$, which shows that $a \in \left(\left(a^2 \circ S\right) \circ a^2\right]$.

Conversely, assume that $a \in \left(\left(a^2 \circ S\right) \circ a^2\right]$, then $a \leq \left(a^2 \circ u\right) \circ a^2$ for some $u \in S$. Therefore,

$a \leq \left(a^2 \circ u\right) \circ (a \circ a) = (a \circ a) \circ \left(u \circ a^2\right) \leq a^2 \circ v$ for some $v \in u \circ a^2$. Now by Lemma 3.1, $a \leq w \circ a^2$ for some $w \in S$. Therefore,

$$\begin{aligned} a \in \left(\left(a^2 \circ S\right) \circ a^2\right] &= \left(\left(a^2 \circ S\right) \circ (a \circ a)\right] \\ &= \left((a \circ a) \circ \left(S \circ a^2\right)\right] = ((a \circ a) \circ (S \circ (a \circ a))] \\ &= ((a \circ a) \circ ((e \circ S) \circ (a \circ a))] \\ &= ((a \circ a) \circ ((a \circ a) \circ (S \circ e))] \subseteq ((a \circ a) \circ ((a \circ a) \circ S)] \\ &= \left((a \circ a) \circ \left(a^2 \circ S\right)\right] = \left(\left(\left(a^2 \circ S\right) \circ a\right) \circ a\right] \\ &= ((((a \circ a) \circ S) \circ a) \circ a] = (((a \circ S) \circ (a \circ a)) \circ a] \\ &= ((a \circ ((S \circ a) \circ a)) \circ a] \subseteq ((a \circ S) \circ a]. \end{aligned}$$

Thus S is completely regular. ■

Corollary 3.4. *An ordered LA-semihypergroup S with pure left identity is completely regular if and only if S is $(2,2)$-regular.*

Lemma 3.5. *The left regular and intra-regular classes of S coincide in S with pure left identity.*

Proof. Let S be an intra-regular with pure left identity, then for every $a \in S$ there exist $u, v, w \in S$ such that $a \leq \left(u \circ a^2\right) \circ v \leq w \circ a^2$. Indeed:

$$\begin{aligned} a &\leq \left(u \circ a^2\right) \circ v = (u \circ (a \circ a)) \circ v = (a \circ (u \circ a)) \circ v \\ &\leq (a \circ (u \circ a)) \circ (p \circ q) = (a \circ p) \circ ((u \circ a) \circ q) \\ &= (u \circ a) \circ ((a \circ p) \circ q) = (u \circ a) \circ ((q \circ p) \circ a) \\ &= (u \circ a) \circ (t \circ a) = (a \circ t) \circ (a \circ u) \\ &= (a \circ a) \circ (t \circ u) = (u \circ t) \circ (a \circ a) \\ &= (u \circ t) \circ a^2 \\ &\leq w \circ a^2, \end{aligned}$$

for some $w \in u \circ t$. Similarly, $a \leq w \circ a^2 \leq \left(u \circ a^2\right) \circ v$ also holds. ∎

Lemma 3.6. *The left and left quasi-regular classes of S coincide in S with pure left identity.*

Proof. Let S be a left regular with pure left identity, then for every $a \in S$ there exists $x \in S$ such that $a \leq x \circ a^2$. Now $a \leq x \circ a^2 = (e \circ x) \circ (a \circ a) = (e \circ a) \circ (x \circ a)$.

The converse is simple. ∎

Lemma 3.7. *The left and weakly regular classes of S coincide in S with pure left identity.*

Proof. It can be followed by Lemma 3.1. ∎

Definition 3.8. An element a of S is called a V-regular element of S if there exist some $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \in S$ ($x_1, x_2, ...x_9$ may be repeated) such that:

$$\begin{aligned} a &\leq x_1 \circ a^2 \leq a^2 \circ x_2 \leq \left(a^2 \circ x_3\right) \circ a^2 \leq \left(x_4 \circ a^2\right) \circ x_5 \\ &\leq (a \circ x_6) \circ (a \circ x_7) \leq (x_8 \circ a) \circ (x_9 \circ a). \end{aligned}$$

If every element of S is V-regular, then S is called V-regular.

Example 3.9. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$\{e_2\}$	$\{e_4\}$	$\{e_6\}$	$\{e_1\}$	$\{e_3\}$	$\{e_5\}$	$\{e_7\}$
e_2	$\{e_5\}$	$\{e_7\}$	$\{e_2\}$	$\{e_4\}$	$\{e_6\}$	$\{e_1\}$	$\{e_3\}$
e_3	$\{e_1\}$	$\{e_3\}$	$\{e_5\}$	$\{e_7\}$	$\{e_2\}$	$\{e_4\}$	$\{e_6\}$
e_4	$\{e_4\}$	$\{e_6\}$	$\{e_1\}$	$\{e_3\}$	$\{e_5\}$	$\{e_7\}$	$\{e_2\}$
e_5	$\{e_7\}$	$\{e_2\}$	$\{e_4\}$	$\{e_6\}$	$\{e_1\}$	$\{e_3\}$	$\{e_5\}$
e_6	$\{e_3\}$	$\{e_5\}$	$\{e_7\}$	$\{e_2\}$	$\{e_4\}$	$\{e_6\}$	$\{e_1\}$
e_7	$\{e_6\}$	$\{e_1\}$	$\{e_3\}$	$\{e_5\}$	$\{e_7\}$	$\{e_2\}$	$\{e_4\}$

$$\leq := \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_4, e_4), (e_5, e_5), (e_6, e_6), (e_7, e_7)\}.$$

By routine calculation it is easy to verify that S is V-regular.

Lemma 3.10. *Every V-regular S with pure left identity is regular.*

Proof. Let S be V-regular with pure left identity, then

$$\begin{aligned} a &\leq x \circ a^2 = (e \circ x) \circ (a \circ a) = (a \circ a) \circ (x \circ e) \\ &= (a \circ x) \circ (a \circ e) = ((a \circ e) \circ x) \circ a \\ &= ((x \circ e) \circ a) \circ a = (p \circ a) \circ a \\ &= (p \circ ((a \circ x) \circ (a \circ e))) \circ a \\ &= (p \circ ((a \circ a) \circ (x \circ e))) \circ a \\ &= (p \circ (x \circ (a \circ a))) \circ a \\ &= (p \circ (a \circ (x \circ a))) \circ a \\ &= (a \circ (p \circ (x \circ a))) \circ a \\ &\leq (a \circ q) \circ a, \end{aligned}$$

for some $q \in (p \circ (x \circ a))$. Thus S is regular. ∎

The converse is not true. We can illustrate by the following example.

Example 3.11. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	e_1	e_2	e_3	e_4
e_1	$\{e_2\}$	$\{e_2\}$	$\{e_2, e_4\}$	$\{e_2, e_4\}$
e_2	$\{e_2\}$	$\{e_2\}$	$\{e_2\}$	$\{e_2\}$
e_3	$\{e_1, e_2\}$	$\{e_2\}$	$\{e_3\}$	$\{e_2, e_4\}$
e_4	$\{e_1, e_2\}$	$\{e_2\}$	$\{e_1, e_2\}$	$\{e_2\}$

$$\leq := \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_4, e_4), (e_2, e_1), (e_2, e_4)\}.$$

Clearly S is regular. But S is not V-regular because $e_1 \not\leq (e_1 \circ x) \circ (e_1 \circ y)$ for any $x, y \in S$.

4. Fuzzy Hyperideals in Ordered LA-semihypergroups

Let S be an ordered LA-semihypergroup. A function f from a nonempty set X to the unit interval $[0, 1]$ is called a fuzzy subset of S.

Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then for every $t \in (0, 1]$ the set

$$U(f; t) = \{x \mid x \in S,\ f(x) \geq t\},$$

is called the level set of f.

For $x \in S$, define

$$A_x = \{(y, z) \in S \times S : x \leq y \circ z\}.$$

We denote by $F(S)$ the set of all fuzzy subsets of S.

Definition 4.1. Let S be an ordered LA-semihypergroup and f, g are any two fuzzy subsets of S. We define the product $f \diamond g$ of f and g as follows:

$$(f \diamond g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \{f(y) \wedge g(z)\}, & \text{if } A_x \neq \emptyset, \\ 0, & \text{if } A_x = \emptyset. \end{cases}$$

The fuzzy subsets defined by $\mathcal{S} : S \longrightarrow [0,1]$, $x \longrightarrow \mathcal{S}(x) = 1$ and $0 : S \longrightarrow [0,1], x \longrightarrow 0(x) = 0$ for all $x \in S$ are the greatest and least elements of $F(S)$.

Definition 4.2. Let S be an ordered LA-semihypergroup and $\emptyset \neq A \subseteq S$. Then the characteristic function χ_A of A is defined as:

$$\chi_A : S \longrightarrow [0,1], \longrightarrow \chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Definition 4.3. Let S be an ordrerd LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy LA-subsemihypergroup of S if:

$$(\forall x, y \in S) \bigwedge_{\alpha \in x \circ y} f(\alpha) \geq f(x) \wedge f(y).$$

Definition 4.4. Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy left (resp. right) hyperideal of S if:

(1) $(\forall x, y \in S) \bigwedge_{\alpha \in x \circ y} f(\alpha) \geq f(y)$ (resp. $\bigwedge_{\alpha \in x \circ y} f(\alpha) \geq f(x)$).

(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f(x) \geq f(y)$.

Example 4.5. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a,c\}$
c	$\{a\}$	$\{a\}$	$\{a\}$

$$\leq := \{(a,a), (b,b), (c,c), (a,c)\}$$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.9 \text{ if } x = a, \\ 0.3 \text{ if } x = b, \\ 0.6 \text{ if } x = c. \end{cases}$$

Then it is easy to see that f is a fuzzy hyperideal of S.

Definition 4.6. Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy interior hyperideal of S if:

(1) $(\forall x, y, z \in S) \bigwedge_{\alpha \in ((x \circ y) \circ z)} f(\alpha) \geq f(y)$.

(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f(x) \geq f(y)$.

Example 4.7. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	$\{a,c\}$	$\{b\}$	$\{b,c\}$
b	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
c	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$

$$\leq := \{(a,a), (b,b), (c,c), (b,c)\}.$$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.4 \text{ if } x = a. \\ 0.9 \text{ if } x = b, c. \end{cases}$$

Then it is easy to see that f is a fuzzy interior hyperideal of S.

Definition 4.8. Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy bi-hyperideal of S if:

(1) $(\forall x, y \in S) \bigwedge_{\alpha \in x \circ y} f(\alpha) \geq f(x) \wedge f(y)$.

(2) $(\forall x, y, z \in S) \bigwedge_{\alpha \in ((x \circ y) \circ z)} f(\alpha) \geq f(x) \wedge f(z)$.

(3) $(\forall x, y \in S)\ x \leq y \Longrightarrow f(x) \geq f(y)$.

Definition 4.9. Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy generalized bi-hyperideal of S if:

(1) $(\forall x, y, z \in S) \bigwedge_{\alpha \in ((x \circ y) \circ z)} f(\alpha) \geq f(x) \wedge f(z)$.

(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f(x) \geq f(y)$.

Example 4.10. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	$\{b\}$	$\{b\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{b\}$	$\{a, b\}$	$\{b\}$

$$\leq := \{(a,a), (b,b), (c,c), (a,b)\}.$$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.7 \text{ if } x = a, b, \\ 0.5 \quad \text{if } x = c. \end{cases}$$

Then f is a fuzzy bi-hyperideal of S.

Definition 4.11. Let S be an ordered LA-semihypergroup and f be a fuzzy subset of S. Then f is called a fuzzy quasi-hyperideal of S if:

(1) $f \supseteq (f \diamond \mathcal{S}) \cap (\mathcal{S} \diamond f)$.

(2) $(\forall x, y \in S)\ x \leq y \Longrightarrow f(x) \geq f(y)$.

Example 4.12. Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	$\{a, c\}$	$\{c\}$	$\{b, c\}$
b	$\{b, c\}$	$\{c\}$	$\{c\}$
c	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$

$$\leq := \{(a,a), (b,b), (c,c), (b,c)\}.$$

Let us define a fuzzy subset $f : S \longrightarrow [0,1]$ as follows

$$f(x) = \begin{cases} 0.2 \text{ if } x = a, \\ 0.8 \text{ if } x = b, c. \end{cases}$$

Then it is easy to see that f is a fuzzy quasi-hyperideal of S.

Definition 4.13. A fuzzy subset f of S is called idempotent if:

$$f \diamond f = f.$$

Definition 4.14. A fuzzy subset f of S is called fuzzy semiprime if:

$$f(a) \geq \bigwedge_{\alpha \in a \circ a} f(\alpha).$$

Proposition 4.15. *Let χ_A and χ_B be fuzzy subsets of an ordered LA-semihypergroup S, where A and B are nonempty subsets of S. Then the following properties hold:*

(1) *If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.*
(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$.
(3) $\chi_A \diamond \chi_B = \chi_{(A \circ B]}$.

Proof. Straightforward proof. ■

Proposition 4.16. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup. Let A be a nonempty subset of S. Then A is a left (resp. right, bi-,generalized bi-, quasi-) hyperideal of S if and only if χ_A is a fuzzy left (resp. right, bi-, generalized bi-, quasi-)hyperideal of S.*

Proof. The proofs are straightforward therefore we omit it. ■

5. Characterizations of V-regular Ordered LA-semihypergroups in Terms of Fuzzy Hyperideals

Theorem 5.1. *For V-regular S with pure left identity the following statements are true:*

(1) *A fuzzy subset f is a fuzzy left hyperideal of S if and only if f is a fuzzy right hyperideal of S.*
(2) *A fuzzy subset f is a fuzzy interior hyperideal of S if and only if f is a fuzzy hyperideal of S.*
(3) *A fuzzy subset f is a fuzzy two-sided hyperideal of S if and only if f is a fuzzy interior hyperideal of S.*
(4) *A fuzzy subset f is a fuzzy generalized bi-hyperideal of S if and only if f is a fuzzy bi-hyperideal of S.*
(5) *A fuzzy subset f is a fuzzy bi-hyperideal of S if and only if f is a fuzzy two-sided hyperideal of S.*
(6) *A fuzzy subset f is a fuzzy quasi-hyperideal of S if and only if f is a fuzzy two-sided hyperideal of S.*

Proof. (1) and (2) are straightforward.

(3). Let f be a fuzzy interior hyperideal of V-regular S, then for each $a \in S$ there exists $x \in S$ such that $a \leq \left(a^2 \circ x\right) \circ a^2$. Then $a \circ b \leq \left(\left(a^2 \circ x\right) \circ a^2\right) \circ b$. So there exist

$\alpha \in a \circ b$ and $\gamma \in \left(\left(\left(a^2 \circ x\right) \circ a^2\right) \circ b\right)$ such that $\alpha \leq \gamma$. Then

$$f(\alpha) \geq f(\gamma) \geq \bigwedge_{\gamma \in (((a^2 \circ x) \circ a^2) \circ b)} f(\gamma) = \bigwedge_{\substack{\gamma \in \left(\left(\lambda \circ a^2\right) \circ b\right) \\ \lambda \in a^2 \circ x}} f(\gamma) \geq f\left(a^2\right)$$
$$\geq \bigwedge_{\beta \in a^2} f(\beta) \geq f(a) \wedge f(a) = f(a).$$

Thus $\bigwedge_{\alpha \in a \circ b} f(\alpha) \geq f(a)$. Also since S is V-regular, therefor $b \leq \left(b^2 \circ y\right) \circ b^2$. Then $a \circ b \leq a \circ \left(\left(b^2 \circ y\right) \circ b^2\right)$. Then there exist $\alpha \in a \circ b$ and $v \in \left(a \circ \left(\left(b^2 \circ y\right) \circ b^2\right)\right)$ such that $\alpha \leq v$. Thus

$$f(\alpha) \geq f(v) \geq \bigwedge_{v \in (a \circ ((b^2 \circ y) \circ b^2))} f(v) = \bigwedge_{v \in ((b^2 \circ y) \circ (a \circ b^2))} f(v) = \bigwedge_{v \in (((b \circ b) \circ y) \circ (a \circ b^2))} f(v)$$
$$= \bigwedge_{v \in (((y \circ b) \circ b) \circ (a \circ b^2))} f(v) = \bigwedge_{\substack{v \in ((p \circ b) \circ q) \\ p \in y \circ b, q \in a \circ b^2}} f(v) \geq f(b).$$

Thus $\bigwedge_{\alpha \in a \circ b} f(\alpha) \geq f(b)$.

The converse is obvious.

(4). Let f be a fuzzy generalized bi-hyperideal of a V-regular S, then for each $a \in S$ there exists $x \in S$ such that $a \leq \left(a^2 \circ x\right) \circ a^2$. Then $a \circ b \leq \left(\left(a^2 \circ x\right) \circ a^2\right) \circ b$. So there exist $\alpha \in a \circ b$ and $w \in \left(\left(\left(a^2 \circ x\right) \circ a^2\right) \circ b\right)$ such that $\alpha \leq w$. Then by using eq (a), we have

$$f(\alpha) \geq f(w) \geq \bigwedge_{w \in (((a^2 \circ x) \circ a^2) \circ b)} f(w) = \bigwedge_{w \in (((a^2 \circ x) \circ (a \circ a)) \circ b)} f(w)$$
$$= \bigwedge_{w \in ((a \circ ((a^2 \circ x) \circ a)) \circ b)} f(w) = \bigwedge_{\substack{w \in ((a \circ m) \circ b) \\ m \in \left(\left(a^2 \circ x\right) \circ a\right)}} f(w) \geq f(a) \wedge f(b).$$

The converse is obvious.

(5). Let f be a fuzzy bi-hyperideal of a V-regular S, then for each $a \in S$ there exists $x \in S$ such that $a \leq \left(a^2 \circ x\right) \circ a^2$. Then $a \circ b \leq \left(\left(a^2 \circ x\right) \circ a^2\right) \circ b$. So there exist $\alpha \in a \circ b$ and $p \in \left(\left(\left(a^2 \circ x\right) \circ a^2\right) \circ b\right)$ such that $\alpha \leq p$. Then we have,

$$
\begin{aligned}
f(\alpha) \geq f(p) &\geq \bigwedge_{p\in(((a^2\circ x)\circ a^2)\circ b)} f(p) \\
&= \bigwedge_{p\in((((a\circ a)\circ x)\circ a^2)\circ b)} f(p) = \bigwedge_{p\in((((x\circ a)\circ a)\circ a^2)\circ b)} f(p) \\
&= \bigwedge_{p\in((b\circ a^2)\circ((x\circ a)\circ a))} f(p) = \bigwedge_{p\in((b\circ(a\circ a))\circ((a\circ a)\circ x))} f(p) \\
&= \bigwedge_{p\in((a\circ(b\circ a))\circ((a\circ a)\circ x))} f(p) = \bigwedge_{p\in((a\circ a)\circ((a\circ(b\circ a))\circ x))} f(p) \\
&= \bigwedge_{p\in((((a\circ(b\circ a))\circ x)\circ a)\circ a)} f(p) = \bigwedge_{p\in((((b\circ(a\circ a))\circ x)\circ a)\circ a)} f(p) \\
&= \bigwedge_{p\in((((x\circ(a\circ a))\circ b)\circ a)\circ a)} f(p) = \bigwedge_{p\in((((a\circ(x\circ a))\circ b)\circ a)\circ a)} f(p) \\
&= \bigwedge_{p\in(((a\circ b)\circ(a\circ(x\circ a)))\circ a)} f(p) = \bigwedge_{p\in((a\circ((a\circ b)\circ(x\circ a)))\circ a)} f(p) \\
&= \bigwedge_{\substack{p\in((a\circ\lambda)\circ a) \\ \lambda\in((a\circ b)\circ(x\circ a))}} f(p) \geq f(a) \wedge f(a) = f(a).
\end{aligned}
$$

Since V-regular S, then for each $b \in S$ there exists $y \in S$ such that $b \leq \left(b^2 \circ y\right) \circ b^2$. Then $a \circ b \leq a \circ \left(\left(b^2 \circ y\right) \circ b^2\right)$. Then there exist $u \in a \circ b$ and $q \in \left(a \circ \left(\left(b^2 \circ y\right) \circ b^2\right)\right)$ such that $u \leq q$. Hence

$$\begin{aligned} f(u) &\geq f(q) \geq \bigwedge_{q\in(a\circ((b^2\circ y)\circ b^2))} f(q) \\ &= \bigwedge_{q\in((b^2\circ y)\circ(a\circ b^2))} f(q) = \bigwedge_{q\in(((b\circ b)\circ y)\circ(a\circ(b\circ b)))} f(q) \\ &= \bigwedge_{q\in((a\circ(b\circ b)\circ y)\circ(b\circ b))} f(q) = \bigwedge_{q\in(((a\circ(b\circ b))\circ(e\circ y))\circ(b\circ b))} f(q) \\ &= \bigwedge_{q\in(((y\circ e)\circ((b\circ b)\circ a))\circ(b\circ b))} f(q) = \bigwedge_{q\in(((b\circ b)\circ((y\circ e)\circ a))\circ(b\circ b))} f(q) \\ &= \bigwedge_{\substack{q\in((\lambda\circ\gamma)\circ\lambda) \\ \lambda\in b\circ b,\gamma\in((y\circ e)\circ a)}} f(q) \geq f(\gamma) \geq \bigwedge_{\gamma\in b\circ b} f(\gamma) \geq f(b)\wedge f(b) = f(b). \end{aligned}$$

The converse is obvious.

(6). Let f be a fuzzy quasi-hyperideal of V-regular S. Now let $a \in S$, there exist $x, y \in S$ such that $a\circ b \leq a\circ\big(\big(x\circ b^2\big)\circ y\big) = \big(x\circ b^2\big)\circ(a\circ y) = (y\circ a)\circ\big(b^2\circ x\big) = (y\circ a)\circ ((x\circ b)\circ b) = (y\circ a)\circ((b\circ e)\circ(b\circ x)) = (y\circ a)\circ(b\circ((b\circ e)\circ x)) = b\circ((y\circ a)\circ((b\circ e)\circ x))$ $\leq b\circ c$ where $c \in ((y\circ a)\circ((b\circ e)\circ x))$. So there exists $\alpha \in a\circ b$ such that $\alpha \leq b\circ c$, Then $(b,c) \in A_\alpha$. Thus

$$\begin{aligned} f(\alpha) &\geq (\mathcal{S}\diamond f)(\alpha)\wedge(f\diamond\mathcal{S})(\alpha) \\ &= \left\{\bigvee_{(b,c)\in A_\alpha}(\mathcal{S}(b)\wedge f(c))\right\}\wedge\left\{\bigvee_{(b,c)\in A_\alpha}(f(b)\wedge\mathcal{S}(c))\right\} \\ &\geq \{\mathcal{S}(b)\wedge f(c)\}\wedge\{f(b)\wedge\mathcal{S}(c)\} \\ &\geq \{1\wedge f(b)\}\wedge\{f(b)\wedge 1\} \\ &= f(b)\wedge f(b) = f(b), \end{aligned}$$

which shows that f is a fuzzy left hyperideal of S. Similarly we can prove that f is a fuzzy right hyperideal of S. Therefore f is a fuzzy hyperideal of S.

The converse is obvious. ∎

Theorem 5.2. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup with pure left identity. Then the following statements are equivalent:*

(1) S is V-regular.
(2) $Q\cap L \subseteq (L\circ Q]$ for every quasi-hyperideal Q and left hyperideal L of S.
(3) $f\cap g \subseteq g\diamond f$ for every fuzzy quasi-hyperideal f and every fuzzy left hyperideal g of S.
(4) $f\cap g \subseteq g\diamond f$ for every fuzzy quasi-hyperideals f and g of S.

Proof. $(1) \Longrightarrow (4)$. Let f and g be any fuzzy quasi-hyperideals of V-regular S with pure left identity. Now for $a \in S$, there exist some $x, y \in S$ such that

$a \leq (x \circ a^2) \circ y = (a \circ (x \circ a)) \circ y = (y \circ (x \circ a)) \circ a$. Then there exists $\alpha \in (y \circ (x \circ a))$ such that $a \leq \alpha \circ a$. Thus $(\alpha, a) \in A_a$. Therefore by using Theorem 5.1, we have

$$\begin{aligned}(g \diamond f)(a) &= \bigvee_{(u,v)\in A_a} \{g(u) \wedge f(v)\} \\ &\geq g(\alpha) \wedge f(a) \\ &\geq g(a) \wedge f(a) \\ &= (f \cap g)(a),\end{aligned}$$

which shows that $g \diamond f \supseteq f \cap g$.

(4) $\Longrightarrow$ (3) is simple.

(3) $\Longrightarrow$ (2). Let Q and L be any quasi and left hyperideals of S respectively. Then by Proposition 4.16, χ_Q is a fuzzy quasi-hyperideal and χ_L is a fuzzy left hyperideal of S. Then Proposition 4.15, $\chi_{Q\cap L} = \chi_Q \cap \chi_L \subseteq \chi_L \diamond \chi_Q = \chi_{(L\circ Q]}$. Thus $Q \cap L \subseteq (L \circ Q]$.

(2) $\Longrightarrow$ (1). Since $(S \circ a]$ are both left hyperideal and quasi-hyperideal of S with pure left identity and clearly $a \in (S \circ a]$, then

$$a \in (S \circ a] \cap (S \circ a] = ((S \circ a) \circ (S \circ a)] = \left((S \circ a)^2\right] = \left(S \circ a^2\right].$$

Hence S is V-regular. ∎

Theorem 5.3. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup with pure left identity. Then the following statements are equivalent:*

(1) S is V-regular.
(2) $L \cap B \subseteq ((L \circ B] \circ B]$ for every left hyperideal L and bi-hyperideal B of S.
(3) $f \cap g \subseteq (f \diamond g) \diamond g$ for every fuzzy left hyperideals f and fuzzy bi-hyperideal g of S.

Proof. (1) $\Longrightarrow$ (3). Let f be a fuzzy left hyperideal and g be a fuzzy bi-hyperideal of a V-regular S with pure left identity, then for any $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned}a &\leq (x \circ a^2) \circ y = (a \circ (x \circ a)) \circ y = (y \circ (x \circ a)) \circ a \\ &\leq \left(y \circ \left(x \circ \left((x \circ a^2) \circ y\right)\right)\right) \circ a = (x \circ (y \circ (a \circ (x \circ a) \circ y))) \circ a \\ &= \left(x \circ (a \circ (x \circ a)) \circ y^2\right) \circ a = \left((a \circ (x \circ a)) \circ \left(x \circ y^2\right)\right) \circ a \\ &= \left(\left(\left(x \circ y^2\right) \circ (x \circ a)\right) \circ a\right) \circ a \\ &= \left(\left(\left(a \circ y^2\right) \circ x^2\right) \circ a\right) \circ a \\ &= \left(\left(\left(x^2 \circ y^2\right) \circ a\right) \circ a\right) \circ a \\ &\leq (b \circ a) \circ a,\end{aligned}$$

where $b \in \left(\left(x^2 \circ y^2\right) \circ a\right)$ and $\alpha \in b \circ a$ such that $a \leq \alpha \circ a$. Then $(\alpha, a) \in A_a$. Therefore

$$\begin{aligned}((f \diamond g) \diamond g)(a) &= \bigvee_{(p,q)\in A_a} \{(f \diamond g)(p) \wedge g(q)\} \\ &\geq \{(f \diamond g)(\alpha) \wedge g(a)\}.\end{aligned}$$

Since $\alpha \in b \circ a$ then $\alpha \leq b \circ a$. Thus

$$(f \diamond g)(\alpha) = \bigvee_{(u,v) \in A_\alpha} \{f(u) \wedge g(v)\} \geq \{f(b) \wedge g(a)\}.$$

Since f is a fuzzy left hyperideal of S, so $\bigwedge_{b \in ((x^2 \circ y^2) \circ a)} f(b) = \bigwedge_{\substack{b \in \beta \circ a \\ \beta \in x^2 \circ y^2}} f(b) \geq f(a)$. Thus

$$(f \diamond g)(\alpha) \geq \{f(b) \wedge g(a)\} \geq \{f(a) \wedge g(a)\}.$$

Hence

$$((f \diamond g) \diamond g)(a) \geq f(a) \wedge g(a) \wedge g(a) = f(a) \wedge g(a).$$

Thus $f \cap g \subseteq (f \diamond g) \diamond g$.

(3) $\Longrightarrow$ (2). Let L be a left hyperideal and B be a bi-hyperideal of S with pure left identity, then by Proposition 4.16, χ_L is a fuzzy left hyperideal andχ_B is a fuzzy bi-hyperideal of S. Therefore by Proposition 4.15, we get

$$\chi_{L \cap B} = \chi_L \cap \chi_B \subseteq (\chi_L \diamond \chi_B) \diamond \chi_B \subseteq \left(\chi_{(L \circ B]}\right) \diamond \chi_B = \chi_{((L \circ B] \circ B]}.$$

Hence $L \cap B \subseteq ((L \circ B] \circ B]$.

(2) $\Longrightarrow$ (1). As $(S \circ a]$ is both left and bi-hyperideal of S with pure left identity. Then

$$a \in (S \circ a] \cap (S \circ a] = (((S \circ a] \circ (S \circ a]] \circ (S \circ a]] = (((S \circ a] \circ (S \circ a]) \circ (S \circ a)] = (((S \circ a) \circ (S \circ a)) \circ (S \circ a)] = (((S \circ S) \circ (a \circ a)) \circ (S \circ a)] \subseteq \left(\left(S \circ a^2\right) \circ S\right].$$

Therefore S is V-regular. ∎

Lemma 5.4. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup with pure left identity. Then the following statements are equivalent:*

(1) *S is V-regular.*
(2) *$f \cap g = f \diamond g$ for every fuzzy right hyperideal f, every fuzzy left hyperideal g of S such that f is semiprime.*

Proof. We omit the proof. ∎

Theorem 5.5. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup with pure left identity. Then the following statements are equivalent:*

(1) *S is V-regular.*
(2) *$h \cap f \cap g \subseteq (h \diamond f) \diamond g$ for any fuzzy right hyperideal h, fuzzy bi-hyperideal f and fuzzy left hyperideal g of S such that h is fuzzy semiprime.*
(3) *$h \cap f \cap g \subseteq (h \diamond f) \diamond g$ for any fuzzy right hyperideal h, fuzzy generalized bi-hyperideal f and fuzzy left hyperideal g of S such that h is fuzzy semiprime.*

Proof. $(1) \Longrightarrow (3)$. Assume that S is V-regular with pure left identity. Let h, f and g be any fuzzy right hyperideal, fuzzy bi-hyperideal and fuzzy left hyperideal of S respectively. Now for any $a \in S$, there exist $x, y, u \in S$ such that $a \leq (a \circ u) \circ a$ and $a \leq \left(x \circ a^2\right) \circ y$. Now

$$\begin{aligned}
a \circ u &\leq \left(\left(x \circ a^2\right) \circ y\right) \circ u = (u \circ y) \circ \left(x \circ a^2\right) \\
&= (u \circ x) \circ \left(y \circ a^2\right) = \left(a^2 \circ x\right) \circ (y \circ u) \\
&= ((y \circ u) \circ x) \circ (a \circ a) = ((y \circ u) \circ a) \circ (x \circ a) \\
&= (a \circ x) \circ (a \circ (y \circ u)) = (a \circ a) \circ (x \circ (y \circ u)) \\
&= ((x \circ (y \circ u)) \circ a) \circ a = ((x \circ (y \circ u)) \circ (e \circ a)) \circ a \\
&= ((x \circ e) \circ ((y \circ u) \circ a)) \circ a \\
&= ((a \circ e) \circ ((y \circ u) \circ x)) \circ a.
\end{aligned}$$

Then there exist $\alpha \in a \circ u$, $\beta \in a \circ e$ and $\gamma \in (\beta \circ ((y \circ u) \circ x))$ such that $\alpha \leq \gamma \circ a$. Then $(\gamma, a) \in A_\alpha$. As $\alpha \in a \circ u$ then $a \leq \alpha \circ a$. Hence $(\alpha, a) \in A_a$. So

$$\begin{aligned}
((h \diamond f) \diamond g)(a) &= \bigvee_{(a,u) \in A_a} \{(h \diamond f)(\alpha) \wedge g(a)\} \\
&= \left[\bigvee_{(a,u) \in A_a} \left\{\bigvee_{(\gamma,a) \in A_\alpha} (h(\gamma) \wedge f(a))\right\} \wedge g(a)\right] \\
&\geq h(\gamma) \wedge f(a) \wedge g(a).
\end{aligned}$$

Since h is fuzzy right hyperideal of S, so $\bigwedge_{\gamma \in (\beta \circ ((y \circ u) \circ x))} h(\gamma) \geq h(\beta) \geq \bigwedge_{\beta \in a \circ e} \geq h(a)$. Hence $h(\gamma) \geq h(a)$. Thus

$$\begin{aligned}
h(a) \geq h(\delta) &\geq \bigwedge_{\delta \in ((x \circ a^2) \circ y)} h(\delta) \\
&= \bigwedge_{\delta \in ((a \circ (x \circ a)) \circ y)} h(\delta) = \bigwedge_{\delta \in (((e \circ a) \circ (x \circ a)) \circ y)} h(\delta) \\
&= \bigwedge_{\delta \in (((e \circ x) \circ (a \circ a)) \circ y)} h(\delta) = \bigwedge_{\delta \in (((a \circ x) \circ (a \circ e)) \circ y)} h(\delta) \\
&= \bigwedge_{\delta \in ((a^2 \circ (x \circ e)) \circ y)} h(\delta) = \bigwedge_{\substack{\delta \in \lambda \circ y \\ \lambda \in a^2 \circ v, v \in x \circ e}} h(\delta) \\
&\geq \bigwedge_{\lambda \in a^2 \circ v} h(\lambda) \geq h\left(a^2\right) \geq \bigwedge_{w \in a^2} h(w).
\end{aligned}$$

Thus $h(a) \geq \bigwedge_{w \in a^2} h(w)$. Therefore h is semiprime.

$(3) \Longrightarrow (2)$ is straightforward.

$(2) \Longrightarrow (1)$. Let h be any fuzzy right hyperideal and g be any fuzzy left hyperideal of S respectively. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}
(h \wedge g)(a) &= ((h \wedge \mathcal{S}) \wedge g)(a) \leq ((h \diamond \mathcal{S}) \diamond g)(a) \\
&= \bigvee_{a \leq u \circ v} \{(h \diamond \mathcal{S})(u) \wedge g(v)\} \\
&= \left[\bigvee_{a \leq u \circ v} \left\{\bigvee_{u \leq r \circ s} (h(r) \wedge \mathcal{S}(s))\right\} \wedge g(v)\right] \\
&\leq \bigvee_{a \leq u \circ v} \left(\bigvee_{u \leq r \circ s} \{h(r \circ s)\} \wedge g(v)\right) \\
&= \bigvee_{a \leq u \circ v} \{h(u) \wedge g(v)\} \\
&= (h \diamond g)(a).
\end{aligned}$$

Thus $h \diamond g \supseteq h \cap g$. Now

$$\begin{aligned}
(h \diamond g)(a) &= \bigvee_{a \leq p \circ q} \{h(p) \wedge g(q)\} \\
&\leq \bigvee_{a \leq p \circ q} \{h(\alpha) \wedge g(\alpha)\}, \text{ (where } \alpha \in p \circ q) \\
&\leq h(a) \wedge g(a) \\
&= (h \wedge g)(a).
\end{aligned}$$

Thus $h \diamond g \subseteq h \cap g$. So $h \diamond g = h \cap g$ by Lemma 5.4, S is is V-regular. ■

Theorem 5.6. *Let $(S, \circ, \leq)$ be an ordered LA-semihypergroup with pure left identity. Then the following statements are equivalent:*

(1) *S is V-regular.*
(2) *$(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$ for any fuzzy right hyperideal f, any fuzzy left hyperideal g of S such that f is fuzzy semiprime.*
(3) *$(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$ for any fuzzy right hyperideal f any fuzzy bi-hyperideal g of S such that f is fuzzy semiprime.*
(4) *$(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$ for any fuzzy right hyperideal f any fuzzy generalized bi-hyperideal g of S such that f is fuzzy semiprime.*
(5) *$(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$ for any fuzzy bi-hyperideals f and g of S such that f and g are fuzzy semiprime.*
(6) *$(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$ for any fuzzy generalized bi-hyperideals f and g of S such that f and g are fuzzy semiprime.*

Proof. $(1) \Longrightarrow (4)$. Let f and g be fuzzy generalized bi-hyperideals of a V-regular S with pure left identity. Now for $a \in S$ there exist $x, y, u \in S$ such that $a \leq (x \circ a^2) \circ y$ and $a \leq (a \circ u) \circ a$.

$a \leq \left(x \circ a^2\right) \circ y = (a \circ (x \circ a)) \circ y = (y \circ (x \circ a)) \circ a$, and

$$
\begin{aligned}
y \circ (x \circ a) &= y \circ \left(x \circ \left(\left(x \circ a^2\right) \circ y\right)\right) = y \circ \left(\left(x \circ a^2\right) \circ (x \circ y)\right) \\
&= \left(x \circ a^2\right) \circ (y \circ (x \circ y)) = \left(x \circ a^2\right) \circ \left(x \circ y^2\right) \\
&= (x \circ x) \circ \left(a^2 \circ y^2\right) = a^2 \circ \left(x^2 \circ y^2\right) \\
&= (a \circ a) \circ \left(x^2 \circ y^2\right) = \left(\left(x^2 \circ y^2\right) \circ a\right) \circ a \\
&= \left(\left(x^2 \circ y^2\right) \circ \left(\left(x \circ a^2\right) \circ y\right)\right) \circ a \\
&= \left(\left(x \circ a^2\right) \circ \left(\left(x^2 \circ y^2\right) \circ y\right)\right) \circ a \\
&= \left(\left(x \circ \left(x^2 \circ y^2\right)\right) \circ \left(a^2 \circ y\right)\right) \circ a \\
&= \left(\left(y \circ \left(x^2 \circ y^2\right)\right) \circ \left(a^2 \circ x\right)\right) \circ a \\
&= \left(a^2 \circ \left(\left(y \circ \left(x^2 \circ y^2\right)\right) \circ x\right)\right) \circ a \\
&= \left(\left(x \circ \left(y \circ \left(x^2 \circ y^2\right)\right)\right) \circ a^2\right) \circ a \\
&= \left(a \circ \left(x \circ \left(y \circ \left(x^2 \circ y^2\right)\right)\right) \circ a\right) \circ a \\
&\leq (a \circ \alpha) \circ a,
\end{aligned}
$$

where $\alpha \in \left(x \circ \left(y \circ \left(x^2 \circ y^2\right)\right)\right) \circ a$ and $\beta \in a \circ \alpha$ such that $a \leq \beta \circ a$. Then $(\beta, a) \in A_a$. Thus

$$
\begin{aligned}
(f \diamond g)(a) &= \bigvee_{(p,q) \in A_a} \{f(p) \wedge g(q)\} \\
&\geq \{f(\beta) \wedge g(a)\}.
\end{aligned}
$$

Since f is a fuzzy generalized bi-hyperideal of S, so

$$\bigwedge_{\beta \in a \circ \alpha} f(\beta) = \bigwedge_{\beta \in (a \circ (x \circ (y \circ (x^2 \circ y^2))) \circ a)} f(\beta) \geq f(a) \wedge f(a) = f(a).$$

So $f(\beta) \geq f(a)$. Hence

$$
\begin{aligned}
(f \diamond g)(a) &\geq \{f(\beta) \wedge g(a)\} \\
&\geq f(a) \wedge g(a).
\end{aligned}
$$

Hence $f \diamond g \supseteq f \cap g$. Similarly we can show that $g \diamond f \supseteq f \cap g$. Thus $(f \diamond g) \cap (g \diamond f) \supseteq f \cap g$.

$(6) \Longrightarrow (5) \Longrightarrow (4) \Longrightarrow (3) \Longrightarrow (2)$ are obvious cases.

$(2) \Longrightarrow (1)$. Let f and g be fuzzy generalized bi-hyperideals of S. Let $(p, q) \in A_a$. Then $a \leq p \circ q$. So there exists $\alpha \in p \circ q$ such that $a \leq \alpha$. Thus

$$
\begin{aligned}
(f \diamond g)(a) &= \bigvee_{(p,q) \in A_a} \{f(p) \wedge g(q)\} \\
&\leq \bigvee_{(p,q) \in A_a} \{f(\alpha) \wedge g(\alpha)\} \\
&= f(a) \wedge g(a).
\end{aligned}
$$

Therefore by using Lemma 5.4, S is V-regular. ∎

6. Conclusion

The main theme of this paper is to introduce the basic concepts of algebraic ordered LA-semihypergroups, together with the additional ideas that are to be developed in this monograph. In this paper we introduced the notions of fuzzy left (resp. right) hyperideals, (bi-, generalized bi-) hyperideals, interior hyperideals and quasi-hyperideals in ordered LA-semihypergroups. Furthermore, we characterized a V-regular class of an ordered LA-semihypergroup in terms of these fuzzy hyperideals. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered LA-semihypergroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

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References

[1] F. Marty, Sur Une generalization de la notion de group, 8^{iem}congress, Math. Scandinaves Stockholm (1934), 45–49.

[2] P. Corsini, V. Leoreanu-Fotea, Applications of hyperstructure theory, Advances in Mathematics; Kluwer Academic Publisher, 2003.

[3] P. Corsini, Prolegomena of Hypergroup Theory, Aviani Editore; Italy, 1993.

[4] B. Davvaz, Semihypergroup Theory, Elsevier Academic Press; London, 2016.

[5] B. Davvaz and I. Cristea, Fuzzy Algebraic Hyperstructures-An Introduction, Springer; 2015.

[6] B. Davvaz and V. Leoreanu-Fotea, Hyperring Theory and Applications, International Academic Press; USA, 2007.

[7] A. Khan, M. Farooq and B. Davvaz, Int-soft interior-hyperideals of ordered semihypergroups, International Journal of Analysis and Applications 14 (2) (2017) 193–202.

[8] A. Khan, M. Farooq and B. Davvaz, Characterizations of ordered semihypergroups by the properties of their intersectional-soft generalized bi-hyperideals, Soft Computing 22 (2018) 3001–3010.

[9] T. Vougiouklis, Hyperstructures and their Representations, Hadronic Press, Florida USA, 1994.

[10] J. Zhan, B. Davvaz and K.P. Shum, A new view of fuzzy hyperquasigroups, Journal Intelligent and Fuzzy Systems 20(4–5) (2009) 147–157.

[11] M. A. Kazim and M. Naseeruddin, On almost semigroups, The Aligarh Bulletin of Mathematics 2 (1972) 1–7.

[12] Q. Mushtaq and S. M. Yusuf, On AG-groupoids, The Aligarh Bulletin of Mathematics 8 (1978) 65–70.

[13] M. Khan and T. Asif, Characterizations of Intra-Regular Left Almost Semigroups by Their Fuzzy Ideals, J. Math. Research 2 (3) (2010) 87–96.

[14] F. Yousafzai, A. Khan, V. Amjad and A. Zeb, On fuzzy fully regular AG-groupoids, J. Intell. & Fuzzy Syst. 26 (2014) 2973–2982.

[15] M. Khan, Faisal and A. Manan, Intra-regular AG-groupoids characterized by their intuitionistic fuzzy ideals, J. Adv. Res. Dyn. Control Syst 3 (2) (2011) 17–33.

[16] Q. Mushtaq and S. M. Yusuf, On locally associative LAsemigroups, The Journal of Natural Sciences and Mathematics 19 (1) (1979) 57–62.

[17] M. Naseeruddin, Some studies on almost semigroups and flocks, Ph.D. Thesis, The Aligarh Muslim University India, 1970.

[18] K. Hila and J. Dine, On hyperideals in left almost semihypergroups. ISRN Algebra, Article ID953124 (2011).

[19] N. Yaqoob, P. Corsini and F. Yousafzai, On intra-regular left almost semihypergroups with pure left identity, J. Math. doi:10.1155/2013/510790.

[20] F. Yousafzai and P. Corsini, Some characterization problems in LA-semihypergroups, J. Algebra Number Theory Adv. Appl. 10 (2013) 41–55.

[21] N. Yaqoob and M. Gulistan, Partially ordered left almost semihypergroups, Journal of the Egyptian Mathematical Society 23 (2015) 231–235.

[22] L.A. Zadeh,Fuzzy sets, Inform. Control 8 (1965) 338–353.

[23] A. Khan, M. Farooq, M. Izhar and B. Davvaz, fuzzy hyperideals of left almost semihypergroups, International Journal of Analysis and Applications 2 (2017) 155–171.