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Solutions of Ginzburg-Landau-Type Equations Involving Variable Exponent

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Abstract In this article, we are interested in some class of Ginzburg-Landau-type equations involving variable exponent under the homogenous Dirichlet boundary conditions and settled in Musielak-Sobolev spaces. We look for nontrivial weak solutions, that is, critical points of the corresponding Ginzburg-Landau energy functional.

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1. Introduction

We are interested in the following Ginzburg-Landau-type (GL) equation

$$\begin{cases} -\mathrm{div}(a(x,|\nabla u|)\nabla u) + \alpha(x)\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]|u|^{q(x)-2}u & = & f(x,u) \text{ in } \Omega, \\ u & = & 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$; $u : \Omega \to \mathbb{R}$ denotes primal field; $q \in C(\overline{\Omega})$ with $\min_{x\in\overline{\Omega}} q(x) \geq 2$; $\alpha, \beta \in L^\infty(\Omega)$ with $\min_{x\in\overline{\Omega}} \alpha(x), \beta(x) > 0$; $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and the function $\varphi(x,t) := a(x,|t|)t$ is an increasing homeomorphism from $\Omega \times \mathbb{R}$ onto $\mathbb{R}$ such that $\Phi(x,t) = \int_0^t \varphi(x,s)ds$.
The corresponding variational formulation given by the functional $\mathcal{E} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ (called the Ginzburg-Landau energy) to the equation (1.1) is the following energy functional

$$\mathcal{E}(u) = \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{\alpha(x)}{2}\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2 dx - \int_\Omega F(x,u)dx. \tag{1.2}$$

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Then the problem will be to find some $u_0 \in W_0^{1,\Phi}(\Omega)$, which satisfies the equation (1.1), such that

$$\mathcal{E}(u_0) = \min_{u \in W_0^{1,\Phi}(\Omega)} \{\mathcal{E}(u)\}.$$

We want to remark that if we let $a(x,t) = |t|^{p(x)-2}$, where $p(x)$ is a continuous function on $\overline{\Omega}$, $\beta(x) = 1$, $\alpha(x) = \alpha = const > 0$, $f(x,t) = f(t)$, and $p(x) = q(x) = 2$, equation (1.1) turns into the well-known the GL equation

$$\begin{cases} -\nabla^2 \psi + \alpha(\frac{|\psi|^2}{2} - 1)\psi &= f(\psi) \text{ in } \Omega, \\ \psi &= 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Equation (1.3) is the Euler equation of the GL energy

$$\mathcal{E}_*(\psi) = \frac{1}{2}\int_\Omega |\nabla \psi|^2 dx + \frac{\alpha}{2}\int_\Omega \left[\frac{|\psi|^2}{2} - 1\right]^2 dx - \int_\Omega F(\psi)dx.$$

In the field of superconductivity, the GL equation has been playing an important role for the understanding of macroscopic superconducting phenomena. This equation was originally proposed in [22], where the magnetic effect caused by the current of superconducting electrons is taken account into the equation. In the Ginzburg-Landau theory, ψ denotes the macrowave function describing a superconducting state and $|\psi|^2$ is the density of superconducting electrons. Therefore, $|\psi| = 0$ corresponds to the normal state and a solution with zeros physically represents a mixed state of superconducting and normal ones. Then the zero of ψ is called a *vortex*. We refer the reader to [2, 6–8, 14, 26–28, 31, 32, 34, 36, 41–44] and the references therein for detailed background regarding the GL equations.

We also want to mention that equations like (1.1) particularly generalize the problems involving variable exponent. This kind of equations has been intensively studied by many authors over the past twenty years due to its significant role in many fields of mathematics, such as calculus of variations, non-linear potential theory, non-Newtonian fluids, image processing (see, e.g., [4, 5, 9, 10, 13, 19, 21, 25, 33, 35, 40, 45, 46]). Therefore, equations of type (1.1) may represent a variety of mathematical models corresponding to certain phenomenons.

For $\varphi(x,t) := \varphi(t) = |t|^{p-2}t$, we have:

- Nonlinear elasticity: $\varphi(t) = \left(1 + t^2\right)^\alpha - 1$, $\alpha > \frac{1}{2}$,
- Plasticity: $\varphi(t) = t^\alpha \left(\log(1+t)\right)^\beta$, $\alpha \geq 1, \beta > 0$,
- Generalized Newtonian fluids: $\varphi(t) = \int_0^t s^{1-\alpha}\left(\sinh^{-1} s\right)^\beta ds$, $0 \leq \alpha \leq 1, \beta > 0$.

For $\varphi(x,t) := |t|^{p(x)-2}t$, we have another example, which is a new model for image restoration given in [11]. In this model, main aim is to recover an image, u, from an observed, noisy image, I, where the two are related by $I = u + v$. The proposed model incorporates the strengths of the various types of diffusion arising from the minimization problem

$$\min_{I=u+v,\, u \in BV \cap L^2(\Omega)} \int_\Omega \varphi(x, \nabla u)dx + \lambda \|u\|_{L^2(\Omega)},$$

where $\Omega \subset \mathbb{R}^2$ is an open domain,

$$\varphi(x,t) = \begin{cases} \frac{1}{p(x)}|t|^{p(x)}, & \text{for } |t| \leq \beta, \\ |t| - \frac{\beta p(x) - \beta^{p(x)}}{p(x)}, & \text{for } |t| > \beta, \end{cases}$$

where $\beta > 0$ is fixed and $1 < \alpha \leq p(x) \leq 2$, where the function $p(x)$ depends on the location of x in the model. For instance, $p(x)$ can be chosen as

$$p(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * I|^2},$$

where $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/4\sigma^2)$ is the Gaussian filter and $k > 0$ and $\sigma > 0$ are fixed parameters.

To the best of the author knowledge, problem (1.1) of the present paper has not been included in the related literature so far, and therefore, has a potential to contribute to it in some way. The main challenge regarding to problem (1.1) was to obtain the smoothness properties of the corresponding Ginzburg-Landau energy functional $\mathcal{E}$ because it contains the term $\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2$ which does not appear usually in such problems. Additionally, we use the theory of Musielak-Orlicz spaces since problem (1.1) contains a nonhomogeneous function φ in the differential operator, namely, $-\mathrm{div}(a(x, |\nabla \cdot|)\nabla \cdot)$, which makes equation (1.1) to be particularized to some well-known equations such as $p(x)$-Laplace equations in case we let $a(x,t) = |t|^{p(x)-2}$.

2. Preliminaries

We start with some basic concepts of Orlicz spaces. For more details we refer the readers to the monographs [1, 29, 30, 37, 39], and the papers [18, 19, 23, 24, 33].

The function $a(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that the mapping $\varphi(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x,t) = \begin{cases} a(x,|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases} \tag{2.1}$$

and for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism. For the function φ above, if we define

$$\Phi(x,t) = \int_0^t \varphi(x,s)ds, \ \forall x \in \Omega, \ t \geq 0, \tag{2.2}$$

then the function $\Phi : \Omega \times [0,+\infty) \to [0,+\infty)$ is called a *generalized N-function* if it satisfies the following conditions (see e.g., [1, 37, 39]):

(Φ_0) for almost all $x \in \Omega$, $\Phi(x,\cdot)$ is a *N-function*, i.e., convex, nondecreasing and continuous function of t such that, $\Phi(x,0) = 0$, $\Phi(x,t) > 0$ for all $t > 0$, and

$$\lim_{t\to 0} \frac{\Phi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\Phi(x,t)}{t} = +\infty.$$

(Φ_1) $\Phi(\cdot,t)$ is a measurable function on Ω for all $t \geq 0$.

The set of all generalized N-functions is denoted by $N(\Omega)$. The function $\bar{\Phi}$ defined by

$$\bar{\Phi}(x,t) = \int_0^t \varphi^{-1}(x,s)ds, \ \forall x \in \Omega, t \geq 0 \tag{2.3}$$

is called the *complementary (or conjugate) function* to Φ, where $\bar{\Phi}$ satisfies the following

$$\bar{\Phi}(x,t) = \sup_{s>0}\{st - \Phi(x,s) : \ s \in \mathbb{R}\}, \ \forall x \in \Omega, t \geq 0.$$

It is well known that $\bar{\Phi} \in N(\Omega)$, and then the following Young inequality holds

$$st \leq \Phi(x,t) + \bar{\Phi}(x,s) \text{ for } x \in \Omega \text{ and } t, s \in \mathbb{R}. \tag{2.4}$$

The function Φ allow us to define the *Musielak-Orlicz spaces*, also called *the generalized Orlicz spaces*, by

$$L^{\Phi}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable}; \exists \lambda > 0 \text{ such that } \int_{\Omega} \Phi(x, |u(x)|/\lambda)dx < +\infty\}.$$

Moreover, by Δ_2-*condition* (see below), $L^{\bar{\Phi}}(\Omega)$ is the dual space of $L^{\Phi}(\Omega)$, i.e., $(L^{\Phi}(\Omega))^* = L^{\bar{\Phi}}(\Omega)$.
In the sequel, we also use the following assumptions for Φ:

$$1 < \varphi_0 := \inf_{t>0} \frac{t\varphi(x,t)}{\Phi(x,t)} \leq \frac{t\varphi(x,t)}{\Phi(x,t)} \leq \varphi^0 := \sup_{t>0} \frac{t\varphi(x,t)}{\Phi(x,t)} < \infty, \ \forall x \in \Omega, t \geq 0; \tag{2.5}$$

$$\inf_{x\in\Omega} \Phi(x,t) > 0, \ \forall t > 0; \tag{2.6}$$

$$\text{the function } \ t \to \Phi(x, \sqrt{t}) \ \text{ is convex}, \ \ \forall x \in \Omega, t \geq 0. \tag{2.7}$$

By help of assumption (2.5), the Musielak-Orlicz spaces coincides with the equivalence classes of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} \Phi(x, |u(x)|)dx < \infty, \tag{2.8}$$

and is equipped with the Luxembourg norm

$$|u|_{\Phi} := \inf \left\{ \mu > 0 : \ \int_{\Omega} \Phi\left(x, \frac{|u(x)|}{\mu}\right) dx \leq 1 \right\}. \tag{2.9}$$

For the Musielak-Orlicz spaces, Hölder inequality reads as follows (see [1],[39])

$$\int_{\Omega} uv \, dx \leq 2\|u\|_{L^{\Phi}(\Omega)}\|v\|_{L^{\bar{\Phi}}(\Omega)} \quad \text{ for all } u \in L^{\Phi}(\Omega) \text{ and } v \in L^{\bar{\Phi}}(\Omega).$$

The *Musielak-Sobolev spaces* $W^{1,\Phi}(\Omega)$ is the space defined by

$$W^{1,\Phi}(\Omega) := \left\{ u \in L^{\Phi}(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), \ i = 1, 2, \ldots, N \right\}$$

under the norm

$$\|u\|_{1,\Phi} := |u|_{\Phi} + |\nabla u|_{\Phi}. \tag{2.10}$$

Now we introduce *Musielak-Sobolev spaces with zero boundary traces* $W_0^{1,\Phi}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ under the norm $\|u\|_{1,\Phi}$. Moreover, by help of the well-known *Poincaré inequality* (see [23]), we can define an equivalent norm $\|\cdot\|_{\Phi}$ on $W_0^{1,\Phi}(\Omega)$ by

$$\|u\|_{\Phi} := |\nabla u|_{\Phi}. \tag{2.11}$$

Remark 2.1. (1) For the case $\Phi(x,t) := \Phi(t)$, we obtain $L^{\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ called *Orlicz spaces* and *Orlicz-Sobolev spaces*, respectively (see [29, 30, 37, 39]).

(2) For the case $\Phi(x,t) := |t|^{p(x)}$, where $p(x)$ is a continuous function on $\overline{\Omega}$ with $p(x) > 1$, we replace $L^{\Phi}(\Omega)$ by $L^{p(x)}(\Omega)$ and $W^{1,\Phi}(\Omega)$ by $W^{1,p(x)}(\Omega)$ and call them *variable exponent Lebesgue spaces* and *variable exponent Sobolev spaces*, respectively (see [1, 15, 16, 40]).

Proposition 2.2 ([1, 19])**.** *If (2.5)-(2.7) hold then the spaces $L^{\Phi}(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.3 ([18, 33])**.** *Let define the modular $\rho(u) := \int_\Omega \Phi(x, |\nabla u|)dx : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$. Then for every $u_n, u \in W_0^{1,\varphi}(\Omega)$, we have*

(i) $\|u\|_\Phi^{\varphi^0} \le \rho(u) \le \|u\|_\Phi^{\varphi_0}$ *if* $\|u\|_\Phi < 1$.

(ii) $\|u\|_\Phi^{\varphi_0} \le \rho(u) \le \|u\|_\Phi^{\varphi^0}$ *if* $\|u\|_\Phi > 1$.

(iii) $\|u\|_\Phi \le \rho(u) + 1$.

(iv) $\|u_n - u\|_\Phi \to 0 \Leftrightarrow \rho(u_n - u) \to 0$.

(v) $\|u_n - u\|_\Phi \to \infty \Leftrightarrow \rho(u_n - u) \to \infty$.

In Proposition 2.3, the statements $(iv) - (v)$ mean that norm and modular topology coincide on $L^{\Phi}(\Omega)$ provided Φ satisfies (2.5), which enables that well-known Δ_2-*condition* holds (see below).

Remark 2.4. The functional ρ is from $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with the derivative

$$\langle \rho'(u), v\rangle = \int_\Omega a(x, |\nabla u|)\nabla u \cdot \nabla v dx,$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $W_0^{1,\Phi}(\Omega)$ and its dual $(W_0^{1,\Phi}(\Omega))^*$. Moreover, the operator ρ' is of type (S_+), that is, $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ and $\limsup \langle \rho'(u_n), u_n - u\rangle \le 0$ imply $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$ (see [33]).

It is said that Φ satisfies the Δ_2-*condition* if there is a positive constant M such that

$$\Phi(x, 2t) \le M\Phi(x,t), \quad \text{for all } x \in \Omega, t \ge 0. \tag{2.12}$$

If $\Psi, \Phi \in N(\Omega)$ and

$$\Psi(x,t) \le k_1 \Phi(x, k_2 t) + h(x), \quad \text{for all } x \in \Omega, t \ge 0, \tag{2.13}$$

holds, where $h \in L^1(\Omega)$ with $h(x) \ge 0$ a.e. $x \in \Omega$, k_1, k_2 are positive constants, then we have the following continuous embeddings (see [37]):

(i) $L^{\Phi}(\Omega) \hookrightarrow L^{\Psi}(\Omega)$.

(ii) $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Psi}(\Omega)$.

We also assume that the following condition hold for function Φ.
For every $t > 0$ there exists a constant $C_t > 0$ such that

(Φ_2) $C_t \le \Phi(x,t) \le C_t^{-1}$,

for a.e. $x \in \Omega$.

Proposition 2.5 ([17])**.** *Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is compact provided $r, p \in C(\overline{\Omega})$ such that $p^- > 1$, $1 \le r(x) < p^*(x)$, where $p^*(x) := \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) := +\infty$ if $p(x) \ge N$.*

Remark 2.6. First, we note that for $t > 1$ and $s > 0$ it holds $t^{\varphi_0}\Phi(x,s) \leq \Phi(x,ts) \leq t^{\varphi^0}\Phi(x,s)$. Indeed, from the assumption (2.5), we have

$$\varphi_0 \leq \frac{z\varphi(x,z)}{\Phi(x,z)} \leq \varphi^0, \ \ \forall x \in \Omega, z \geq 0.$$

Considering that for almost all $x \in \Omega$, $\Phi(x,z)$ is a convex, nondecreasing and continuous function of z, we can proceed as follows

$$\int_s^{ts} \frac{\varphi_0}{z}dz \leq \int_s^{ts} \frac{\varphi(x,z)}{\Phi(x,z)} \leq \int_s^{ts} \frac{\varphi^0}{z}dz,$$

$$\log t^{\varphi_0} \leq \log \Phi(x,ts) - \log \Phi(x,s) \leq \log t^{\varphi^0},$$

and hence

$$t^{\varphi_0}\Phi(x,s) \leq \Phi(x,ts) \leq t^{\varphi^0}\Phi(x,s). \tag{2.14}$$

Now, if we consider (Φ_2) and the inequality (2.14) together, we can obtain

$$C_t t^{\varphi_0} \leq \Phi(x,st) + C, \ \ C \geq 0. \tag{2.15}$$

Hence, if we consider (2.15) along with (2.13) where $\frac{1}{k_1} = C_t$, $k_2 = s$ and $h(x) = C \geq 0$, the Musielak-Sobolev space $W^{1,\Phi}(\Omega)$ is continuously embedded in the variable exponent Sobolev space $W^{1,\varphi_0}(\Omega)$. On the other hand, $W^{1,\varphi_0}(\Omega)$ is compactly embedded in the variable exponent Lebesgue space $L^{r(x)}(\Omega)$ for all $1 \leq r(x) < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$ with $r \in C(\overline{\Omega})$. As a result, $W^{1,\Phi}(\Omega)$ is continuously and compactly embedded in the variable exponent Lebesgue space $L^{r(x)}(\Omega)$ (see also Remark 2.1, [5]).

3. Main Results

First, we will give the variational framework of the problem.

Definition 3.1. We say that $u \in W_0^{1,\Phi}(\Omega)$ is a weak solution of problem (1.1) iff

$$\langle \rho'(u), v\rangle + \int_\Omega \alpha(x)\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]|u|^{q(x)-2}uvdx = \int_\Omega f(x,u)vdx,$$

for all $v \in W_0^{1,\Phi}(\Omega)$.

The Ginzburg-Landau energy functional corresponding to problem (1.1) is defined as $\mathcal{E} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$,

$$\mathcal{E}(u) = \rho(u) + \int_\Omega \frac{\alpha(x)}{2}\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2 dx - \int_\Omega F(x,u)dx,$$

where $F(x,t) := \int_0^t f(x,s)ds$. We will study problem (1.1) under the following assumptions.We set

$$r^- = \min_{x\in\overline{\Omega}} r(x) \text{ and } r^+ = \max_{x\in\overline{\Omega}} r(x).$$

Throughout the paper we always assume that

$$2 \leq q^- \leq q(x) \leq q^+ \leq p^- \leq p(x) \leq p^+ < \infty.$$

$(f1)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $c_1 > 0$ such that

$$|f(x,t)| \leq c_1|t|^{s(x)-1}$$

where $s \in C(\overline{\Omega})$ such that $s(x) \leq s^+ < \varphi_0$.

$(f2)$ There exist constants $M, \theta > 0$ with $2q^+ < \varphi^0 < \theta < \varphi_0^*$ such that

$$0 < \theta F(x,t) \leq f(x,t)t, \ |t| \geq M, \ \forall x \in \Omega.$$

$(f3)$ $f(x,t) = o\left(|t|^{q^+-1}\right)$ as $t \to 0$ uniformly, for $x \in \Omega$.

$(f4)$ $f(x,-t) = -f(x,t)$.

Remark 3.2. The function $f(x,t) = |t|^{\sigma(x)-2}t$, where $\sigma^- > q^+$, satisfies assumptions $(f1)$-$(f4)$.

Remark 3.3. By assumption $(f2)$ there exists a constant $c > 0$ such that $F(x,t) \geq c|t|^\theta$ for all $x \in \Omega$ and $|t| \geq M$.

The main results of the present paper are the following.

Theorem 3.4. *Assume that* $(f1)$-$(f3)$ *hold. Then problem* (1.1) *has a nontrivial solution in* $W_0^{1,\Phi}(\Omega)$.

Theorem 3.5. *Suppose that in addition to the assumptions of Theorem* 3.4, $(f4)$ *holds. Then problem* (1.1) *has infinitely many solutions with arbitrary large action in* $W_0^{1,\Phi}(\Omega)$.

First, we need to show that functional $\mathcal{E}$ satisfies the main smoothness properties which are the essential part of the main proofs of the paper.

Lemma 3.6. *The functional* $\mathcal{E}$ *is well-defined on* $W_0^{1,\Phi}(\Omega)$ *and Fréchet differentiable, i.e.,* $\mathcal{E} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ *whose derivative is*

$$\langle \mathcal{E}'(u), v\rangle = \langle \rho'(u), v\rangle + \int_\Omega \alpha(x) \left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right] |u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx.$$

Proof. From the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{2q(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, for any $u \in W_0^{1,\Phi}(\Omega)$ it is easy to see that

$$\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2 \in L^1(\Omega). \tag{3.1}$$

By condition $(f1)$, we have $|F(x,u)| \leq \frac{c_1}{s^-}|u|^{s(x)}$. Therefore, considering the continuous embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, it follows

$$|\mathcal{E}(u)| \leq \rho(u) + \int_\Omega \frac{\alpha(x)}{2}\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2 dx + \int_\Omega |F(x,u)| dx < \infty,$$

which means that $\mathcal{E}$ is well-defined on $W_0^{1,\Phi}(\Omega)$.
Since $\rho \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$, it is enough to show that the operator Λ given by

$$\Lambda(u) = \int_\Omega \frac{\alpha(x)}{2}\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2 dx - \int_\Omega F(x,u) dx$$

is of class $C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$. To this end, first, it must be shown that for all $v \in W_0^{1,\Phi}(\Omega)$

$$\langle \Lambda'(u), v\rangle = \lim_{t\to 0} \frac{\Lambda(u+tv) - \Lambda(u)}{t} = \int_\Omega \alpha(x)\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right] |u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx,$$

and then it must be obtained that $\Lambda' : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ is continuous.
The continuity properties of $|\cdot|$ and f along with the definition of F allow us to apply the mean value theorem, that is,

$$\begin{aligned}\langle \Lambda'(u), v\rangle &= \lim_{t\to 0}\int_\Omega \frac{\alpha(x)}{2t}\left(\left[\frac{|u+tv|^{q(x)}}{q(x)} - \beta(x)\right]^2 - \left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]^2\right)dx \\ &\quad - \lim_{t\to 0}\int_\Omega \frac{F(x,u+tv) - F(x,u)}{t}dx \\ &= \lim_{t\to 0}\int_\Omega \alpha(x)\left[\frac{|u+t\tau v|^{q(x)}}{q(x)} - \beta(x)\right]|u+t\tau v|^{q(x)-2}(u+t\tau v)v dx \\ &\quad - \lim_{t\to 0}\int_\Omega f(u+t\tau v)v dx,\end{aligned}$$

where $u, v \in W_0^{1,\Phi}(\Omega)$ and $0 \le \tau \le 1$. Now, if we apply the Young inequality along with the inequality $|a+b|^m \le 2^{m-1}(|a|^m + |b|^m)$, for all $a, b \in \mathbb{R}^N$ and $m \ge 1$, consecutively to all integrands on the right-hand side of the above expression, and use condition $(f1)$, it reads

$$\begin{aligned}&\left|\alpha(x)\left[\frac{|u+t\tau v|^{q(x)}}{q(x)} - \beta(x)\right]|u+t\tau v|^{q(x)-2}(u+t\tau v)v\right| \\ &\le \alpha(x)\left[\frac{|u+t\tau v|^{q(x)}}{q^-} + \beta(x)\right]|u+t\tau v|^{q(x)-1}||v| \\ &\le \alpha(x)\left[\frac{|u+t\tau v|^{2q(x)-1}}{q^-}|v| + \beta(x)|u+t\tau v|^{q(x)-1}|v|\right].\end{aligned}$$

However, by the Young inequality, it reads

$$\frac{|u+t\tau v|^{2q(x)-1}}{q^-}|v| \le \frac{(2q(x)-1)2^{2q(x)-1}}{2q(x)q^-}\left[|u|^{2q(x)} + |v|^{2q(x)}\right] + \frac{1}{2q(x)}|v|^{2q(x)} \tag{3.2}$$

and

$$|u+t\tau v|^{q(x)-1}|v| \le \frac{2^{q(x)-1}(q(x)-1)}{q(x)q^-}\left[|u|^{q(x)} + |v|^{q(x)}\right] + \frac{1}{q(x)}|v|^{q(x)}. \tag{3.3}$$

Similarly,

$$|f(x,u+t\tau v)v| \le c\left(\frac{2^{s(x)-1}(s(x)-1)}{s(x)}|u|^{s(x)} + (\frac{2^{s(x)-1}(s(x)-1)+1}{s(x)})|v|^{s(x)}\right). \tag{3.4}$$

The right-hand sides of the inequalities of (3.2)-(3.4) belong to $L^1(\Omega)$. Therefore, by the Lebesgue dominated convergence theorem, which make it possible to change the order of lim and integral signs, along with the continuity properties of f and $|\cdot|$, it reads that

$$\begin{aligned}\langle \Lambda'(u), v\rangle &= \int_\Omega \alpha(x)\lim_{t\to 0}\left[\frac{|u+t\tau v|^{q(x)}}{q(x)} - \beta(x)\right]|u+t\tau v|^{q(x)-2}(u+t\tau v)v dx \\ &\quad - \int_\Omega \lim_{t\to 0} f(x,u+t\tau v)v dx \\ &= \int_\Omega \alpha(x)\left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]|u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx.\end{aligned}$$

Since the right-hand side of the above expression, as a function of v, is a continuous linear functional on $W_0^{1,\Phi}(\Omega)$, it is the Gâteaux differential of Λ.
Next, we proceed to the continuity of Λ'. To this end, we assume, for a sequence $(u_n) \subset W_0^{1,\Phi}(\Omega)$, that $u_n \to u \in W_0^{1,\Phi}(\Omega)$. Then,

$$|\langle \Lambda'(u_n) - \Lambda'(u), v\rangle| \leq \left|\int_\Omega \alpha(x) I_n v dx\right| + \left|\int_\Omega (f(x,u) - f(x,u_n)) v dx\right|,$$

where

$$I_n := \Theta(u_n) - \Theta(u) = \left[\left(\frac{|u_n|^{q(x)}}{q(x)} - \beta(x)\right)|u_n|^{q(x)-2}u_n - \left(\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right)|u|^{q(x)-2}u\right]$$

and

$$\Theta(u) := \left[\frac{|u|^{q(x)}}{q(x)} - \beta(x)\right]|u|^{q(x)-2}u.$$

By the Hölder inequality, it reads

$$\left|\int_\Omega \alpha(x) I_n v dx\right| \leq c|I_n|_{\frac{q^-}{q^- -1}} |v|_{q^-}. \tag{3.5}$$

Note that because of the embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{q^-}(\Omega) \hookrightarrow L^{\frac{q^-}{q^- -1}}(\Omega)$, we can apply $u_n \to u \in W_0^{1,\Phi}(\Omega)$ to (3.5).
On the other hand, we can write

$$\begin{aligned}
|I_n| &= |\Theta(u_n) - \Theta(u)| \\
&\leq \beta(x)\left(|u_n|^{q(x)-1} + |u|^{q(x)-1}\right) + \frac{1}{q(x)}\left(|u_n|^{2q(x)-1} + |u|^{2q(x)-1}\right) \\
&\leq C\left(|u_n|^{q(x)-1} + |u|^{q(x)-1} + |u_n|^{2q(x)-1} + |u|^{2q(x)-1}\right),
\end{aligned} \tag{3.6}$$

where $C := \max(\frac{1}{p^-}, \max_{x\in\overline{\Omega}} \beta(x))$. Since $u_n \to u \in W_0^{1,\Phi}(\Omega)$, by the compact embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\frac{(q(x)-1)q^-}{q^- -1}}(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\frac{(2q(x)-1)q^-}{q^- -1}}(\Omega)$ and $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, up to a subsequence still denoted by (u_n), we have

$u_n \to u$ in $L^{\frac{(q(x)-1)q^-}{q^- -1}}(\Omega)$,
$u_n \to u$ in $L^{\frac{(2q(x)-1)q^-}{q^- -1}}(\Omega)$,
$u_n \to u$ in $L^{s(x)}(\Omega)$,
$u_n(x) \to u(x)$ a.e. $x \in \Omega$,

and there exist $w_1 \in L^{\frac{(q(x)-1)q^-}{q^- -1}}(\Omega)$, $w_2 \in L^{\frac{(2q(x)-1)q^-}{q^- -1}}(\Omega)$ and $w_3 \in L^{s(x)}(\Omega)$ such that $|u_n(x)| \leq w_1(x)$, $|u_n(x)| \leq w_2(x)$, and $|u_n(x)| \leq w_3(x)$, a.e. $x \in \Omega$, respectively, for all $n \in \mathbb{N}$. Therefore, using this information in (3.6), we obtain

$$|I_n|_{\frac{q^-}{q^- -1}} = |\Theta(u_n) - \Theta(u)|_{\frac{q^-}{q^- -1}} = \left(\int_\Omega |\Theta(u_n) - \Theta(u)|^{\frac{q^-}{q^- -1}} dx\right)^{\frac{q^- -1}{q^-}}$$

and

$$
\begin{aligned}
&|\Theta(u_n)-\Theta(u)|^{\frac{q^-}{q^--1}}\\
&\leq c\left(1+|u_n|^{q(x)-1}+|u|^{q(x)-1}+|u_n|^{2q(x)-1}+|u|^{2q(x)-1}\right)^{\frac{q^-}{q^--1}}\\
&\leq c\left(1+|w_1|^{\frac{(q(x)-1)q^-}{q^--1}}+|u|^{\frac{(q(x)-1)q^-}{q^--1}}+|w_2|^{\frac{(2q(x)-1)q^-}{q^--1}}+|u|^{\frac{(2q(x)-1)q^-}{q^--1}}\right)\in L^1(\Omega).
\end{aligned}
$$

Now, we show that $|\Theta(u_n(x))-\Theta(u(x))|\to 0$ as $n\to\infty$. Indeed,

$$
\begin{aligned}
&|\Theta(u_n)-\Theta(u)|\\
&=\left|\left(\frac{|u_n|^{q(x)}}{q(x)}-\beta(x)\right)|u_n|^{q(x)-2}u_n-\left(\frac{|u|^{q(x)}}{q(x)}-\beta(x)\right)|u|^{q(x)-2}u\right|\\
&\leq\frac{1}{q^-}\left||u_n|^{2q(x)-2}u_n-|u|^{2q(x)-2}u\right|+\beta(x)\left||u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\right|.
\end{aligned}
$$

Next, we apply the following inequality given in [12]: for $1<k<\infty$ there exist constants $C_k>0$ such that

$$||\xi|^{k-2}\xi-|\zeta|^{k-2}\zeta||\leq C_k|\xi-\zeta|(|\xi|+|\zeta|)^{k-2},\ \ \forall\xi,\zeta\in\mathbb{R}^N.$$

Therefore, since $u_n\to u\in W_0^{1,\Phi}(\Omega)$, we obtain that

$$\lim_{n\to\infty}|\Theta(u_n(x))-\Theta(u(x))|=0.$$

As for the term $\left|\int_\Omega(f(x,u)-f(x,u_n))vdx\right|$, using (f1), the Hölder inequality and the continuous embeddings $W_0^{1,\Phi}(\Omega)\hookrightarrow L^{s(x)}(\Omega)\hookrightarrow L^{s(x)-1}(\Omega)$, we have

$$
\begin{aligned}
&\left|\int_\Omega(f(x,u)-f(x,u_n))vdx\right|\\
&\leq c_2\int_\Omega(|w_3|^{s(x)-1}+|u|^{s(x)-1})|v|dx\\
&\leq c_3\left(||w_3|^{s(x)-1}|_{\frac{s(x)}{s(x)-1}}+||u|^{s(x)-1}|_{\frac{s(x)}{s(x)-1}}\right)|v|_{s(x)}\in L^1(\Omega).
\end{aligned}
$$

Moreover, considering that $u_n(x)\to u(x)$ a.e. $x\in\Omega$ and f is continuous, we obtain that

$$\lim_{n\to\infty}|f(x,u_n(x))-f(x,u(x))|=0.$$

If we take into account all information obtained above and apply the Lebesgue dominated convergence theorem, it reads

$$\lim_{n\to\infty}\int_\Omega|f(x,u_n)-f(x,u)|dx=0$$

and

$$\lim_{n\to\infty}\int_\Omega|\Theta(u_n)-\Theta(u)|^{\frac{q^-}{q^--1}}dx=0,$$

where these two results together mean, as a conclusion, that

$$\lim_{n\to\infty}\sup\|\Lambda'(u_n)-\Lambda'(u)\|_{(W_0^{1,\Phi}(\Omega))^*}=\lim_{n\to\infty}\sup_{\|v\|_\Phi\leq 1}|\langle\Lambda'(u_n)-\Lambda'(u),v\rangle|=0.$$

Therefore, $\Lambda':W_0^{1,\Phi}(\Omega)\to(W_0^{1,\Phi}(\Omega))^*$ is continuous. ∎

Lemma 3.7. *Assume that (f1) and (f3) hold. Then,*
(i) there exist two positive real numbers η and μ such that $\mathcal{E}(u) \geq \mu > 0$, for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_\Phi = \eta < 1$;
(ii) there exists $e \in W_0^{1,\Phi}(\Omega)$ such that $\|e\|_\Phi > \eta$, $\mathcal{E}(e) < 0$.

Proof. (i) By assumption (f3), given $\epsilon \in \left(0, \frac{\delta^{\varphi^0}}{2c_0^{q^+}}\right)$, with $\delta \in (0,1)$, we can write

$$|F(x,t)| \leq \frac{\epsilon\,|t|^{q^+}}{q^+}, \quad \forall x \in \Omega,\ |t| \leq \delta.$$

Let $u \in W_0^{1,\Phi}(\Omega)$ be such that

$$\|u\|_\Phi = \eta := \left(\frac{1}{q^+}\right)^{1/\varphi^0 - q^+} \delta^{\varphi^0/(\varphi^0 - q^+)} < 1.$$

Then, by Proposition 2.3 and the continuous embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{q^+}(\Omega)$, i.e., $\exists\, c_0 = c(|\Omega|) > 0$ such that $|u|_{q^+} \leq c_0\|u\|_\Phi\ \forall\, u \in W_0^{1,\Phi}(\Omega)$, it follows

$$\begin{aligned}
\mathcal{E}(u) &\geq \int_\Omega \Phi(x, |\nabla u|)dx - \frac{\epsilon}{q^+}\int_\Omega |u|^{q^+}dx \\
&\geq \|u\|_\Phi^{\varphi^0} - \frac{\epsilon}{q^+}c_0^{q^+}\|u\|_\Phi^{q^+} \\
&\geq (\|u\|_\Phi^{\varphi^0 - q^+} - \frac{\epsilon}{q^+}c_0^{q^+})\|u\|_\Phi^{q^+} = \left(\frac{1}{q^+}\delta^{\varphi^0} - \frac{\epsilon}{q^+}c_0^{q^+}\right)\eta^{q^+} = \mu
\end{aligned}$$

i.e., we obtain that $\mathcal{E}(u) \geq \mu > 0$.
(ii) Let $0 \neq \phi \in W^{1,\Phi}(\Omega)$ and $1 < t \in \mathbb{R}$. By Remark 3.3 and Remark 2.6, we have

$$\begin{aligned}
\mathcal{E}(t\phi) \leq t^{\varphi^0}\int_\Omega \Phi(x, |\nabla\phi|)dx &+ \frac{t^{2q^+}}{2(q^-)^2}\int_\Omega \alpha(x)|\phi|^{2q(x)}dx + t^{q^+}\int_\Omega \alpha(x)\beta(x)|\phi|^{q(x)}dx \\
&+ \int_\Omega \alpha(x)\beta^2(x)dx - ct^\theta\int_\Omega |\phi|^\theta dx.
\end{aligned}$$

Since $\theta > \varphi^0 > 2q^+$, we obtain that $\mathcal{E}(t\phi) \to -\infty$ as $t \to +\infty$. Then, for $t > 1$ large enough, if we set $t\phi = e$ with $\|e\|_\Phi > \eta$ we obtain that $\mathcal{E}(e) < 0$. ∎

Definition 3.8. Let X be a Banach space and $I : X \to \mathbb{R}$ be a C^1-functional. We say that a functional I satisfies the Palais-Smale condition (shortly, (PS)-condition), if any Palais-Smale sequence, i.e., a sequence $\{u_n\} \subset X$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$, contains a convergent subsequence.

Lemma 3.9. *Assume that (f1) and (f2) hold. Then, $\mathcal{E}$ satisfies the (PS)-condition.*

Proof. From the proof of Lemma 3.7, $\mathcal{E}$ satisfies the Mountain-Pass geometry which assures the existence of a Palais-Smale sequence $(u_n) \subset W_0^{1,\Phi}(\Omega)$ such that

$$\mathcal{E}(u_n) \to \widehat{c} \quad \text{and} \quad \|\mathcal{E}'(u_n)\|_{(W_0^{1,\Phi}(\Omega))^*} \to 0, \tag{3.7}$$

where $\widehat{c} > 0$ is a critical value of $\mathcal{E}$ and characterized in Theorem 3.10 (ii).

First, let us show that (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Assume the contrary. Then, along a subsequence, $\|u_n\|_\Phi \to \infty$ and, in addition, we may assume that $\|u_n\|_\Phi > 1$. Then, from (3.7) and Proposition 2.3, there exists a real number $C > 0$ such that

$$\begin{aligned}
C + \|u_n\| \geq \mathcal{E}(u_n) &= \rho(u_n) + \int_\Omega \frac{\alpha(x)}{2}\left[\frac{|u_n|^{q(x)}}{q(x)} - \beta(x)\right]^2 dx - \int_\Omega F(x,u_n)dx \\
&\geq \|u_n\|_\Phi^{\varphi_0} - \frac{c_1}{s^-}\int_\Omega |u_n|^{s(x)}dx \\
&\geq \|u_n\|_\Phi^{\varphi_0} - c\|u_n\|_\Phi^{s^+}.
\end{aligned}$$

Since $\varphi_0 > s^+ > 1$, if we divide above inequality by $\|u_n\|_\Phi^{s^+}$ and take the limit as $n \to +\infty$, we obtain a contradiction. Therefore, (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Since $W_0^{1,\Phi}(\Omega)$ is reflexive, there exists a subsequence, still denoted by (u_n), which converges weakly to a $u \in W_0^{1,\Phi}(\Omega)$. Then, by (3.7) it reads

$$\begin{aligned}
&\langle \mathcal{E}'(u_n), u_n - u\rangle \\
&= \langle \rho'(u_n), u_n - u\rangle + \int_\Omega \alpha(x)\left[\frac{|u_n|^{q(x)}}{q(x)} - \beta(x)\right]|u_n|^{q(x)-2}u_n(u_n-u)dx \\
&\quad - \int_\Omega f(x,u_n)(u_n-u)dx \to 0.
\end{aligned}$$

By $(f1)$, the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ and Hölder inequality we have

$$\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| \leq \int_\Omega |u_n|^{s(x)-1}|u_n-u|dx \leq ||u_n|^{s(x)-1}|_{\frac{s(x)}{s(x)-1}}|u_n-u|_{s(x)} \to 0. \tag{3.8}$$

Similarly, by the compact embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{2q(x)}(\Omega)$, $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and Hölder inequality we have

$$\begin{aligned}
&\left|\int_\Omega \alpha(x)\left[\frac{|u_n|^{q(x)}}{q(x)} - \beta(x)\right]|u_n|^{q(x)-2}u_n(u_n-u)dx\right| \\
&\leq \overline{c}\int_\Omega (|u_n|^{q(x)}+1)|u_n|^{q(x)-1}|u_n-u|dx \\
&\leq \overline{c}\left(\int_\Omega |u_n|^{2q(x)-1}|u_n-u|dx + \int_\Omega |u_n|^{q(x)-1}|u_n-u|dx\right) \\
&\leq \overline{c}\left(||u_n|^{2q(x)-1}|_{\frac{2q(x)}{2q(x)-1}}|u_n-u|_{2q(x)} + ||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{q(x)}\right) \\
&\to 0,
\end{aligned}$$

where $\overline{c} := 2\max\left(\max_{x\in\overline{\Omega}}\alpha(x) \times \max_{x\in\overline{\Omega}}\beta(x), \frac{1}{q^-}\right)$. ■

From (3.8) and (3.9), we must have

$$\langle \rho'(u_n), u_n - u\rangle \to 0.$$

However, from Remark 2.4, we know that the operator ρ' is of type (S_+), which means that (u_n) converges strongly to $u \in W_0^{1,\Phi}(\Omega)$. As a conclusion, $\mathcal{E}$ satisfies the (PS)-condition.

Proof. (**Proof of Theorem 3.4**) By Lemmas 3.7 and 3.9, the functional $\mathcal{E}$ satisfies the assumptions of Mountain-Pass theorem ([3]). Hence, there exists a nontrivial critical point which is a solution of problem (1.1), if we take into account Definition 3.1 and Lemma 3.6. ∎

In the rest of the paper, we prove Theorem 3.5. To do this, we apply the following symmetric version of the classical Mountain-Pass Theorem (see, [3, 38]).

Theorem 3.10. *Let $\mathcal{E}$ be a C^1 functional on a Banach space $W_0^{1,\Phi}(\Omega)$ that satisfies the (PS) condition and $\mathcal{E}(0) \leq 0$.*

(a) Suppose that $\mathcal{E}(0) \leq 0$ and

(i) There exist $\eta > 0$ and $\tau > 0$ such that $\mathcal{E}(u) \geq \tau$ for all $u \in W_0^{\Phi}(\Omega)$ with $\|u\|_\Phi = \eta$;

(ii) There exists $e \in W_0^{1,\Phi}(\Omega)$, with $\|e\|_\Phi > \eta$, such that $\mathcal{E}(e) \leq 0$.

Then

$$\widehat{c} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}(\gamma(t)) \geq \tau,$$

where

$$\Gamma = \left\{ \gamma \in C([0,1], W_0^{1,\Phi}(\Omega)) : \gamma(0) = 0, \gamma(1) = e \right\},$$

is a critical value of $\mathcal{E}$.

(b) Suppose that the functional $\mathcal{E}$ is even, $\mathcal{E}(-u) = \mathcal{E}(u)$, assumption (i) is satisfied, and

(ii′) For every finite dimensional subspace $E \subset W_0^{1,\Phi}(\Omega)$ there exists $R = R(E) > 0$ such that $\mathcal{E}(u) \leq 0$ for all $u \in E$ with $\|u\|_\Phi \geq R(E)$.

Then the functional $\mathcal{E}$ possesses an infinite sequence of critical values accumulating to $+\infty$.

Proof. (**Proof of Theorem 3.5**) Considering the result of Lemmas 3.7 and 3.9 along with the facts that $\mathcal{E}$ is even and $\mathcal{E}(0) = 0$, it is enough to verify only assumption (ii') of Theorem 3.10.
Let E be a finite dimensional subspace of $W_0^{1,\Phi}(\Omega)$. The functional $|\cdot|_\theta : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$|u|_\theta = \left(\int_\Omega |u|^\theta dx \right)^{1/\theta}$$

is a norm in $W_0^{1,\Phi}(\Omega)$ because of the inclusion $W_0^{1,\Phi}(\Omega) \subset L^\theta(\Omega)$. Since in the finite dimensional subspace E the norms $|u|_\theta$ and $\|u\|_\Phi$ are equivalent, there exists a constant $C = C(E) > 0$ such that

$$\|u\|_\Phi \leq C|u|_\theta, \quad \forall u \in E.$$

Next, we follow the same steps as we did in the proof of Lemma 3.7 (ii). To this end, for $0 \neq \phi \in E$ and $1 < t \in \mathbb{R}$, we obtain that

$$\begin{aligned}
\mathcal{E}(t\phi) &\leq t^{\varphi^0} \int_\Omega \Phi(x, |\nabla \phi|) dx + \frac{t^{2q^+}}{2(q^-)^2} \int_\Omega \alpha(x)|\phi|^{2q(x)} dx + t^{q^+} \int_\Omega \alpha(x)\beta(x)|\phi|^{q(x)} dx \\
&\quad + \int_\Omega \alpha(x)\beta^2(x) dx - ct^\theta \int_\Omega |\phi|^\theta dx \\
&\leq t^{\varphi^0} \|\phi\|_\Phi^{\varphi^0} + c_1 t^{2q^+} \|\phi\|_\Phi^{2q^+} + c_2 t^{q^+} \|\phi\|_\Phi^{q^+} + c_3|\Omega| - c_4 t^\theta \|\phi\|_\Phi^\theta,
\end{aligned}$$

where $c_i > 0$, $i = \overline{1,4}$, are generic constants and independent of ϕ. Since $\theta > \varphi^0 > 2q^+$, we obtain that $\mathcal{E}(t\phi) \to -\infty$ as $t \to \infty$. On the other hand, considering that any non-zero vector $u \in E$ has a unique representation $u = t\phi$, where $t = \|u\|_\Phi$ and ϕ is a vector on the unit sphere S of E, we conclude that $\mathcal{E}(u) \leq 0$ for all $u \in E$ with $\|u\|_\Phi \geq R$. ∎

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