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Shrinking Inertial Extragradient Methods for Solving Split Equilibrium and Fixed Point Problems

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Abstract This paper presents a shrinking inertial extragradient algorithm for finding a solution of the split equilibrium and fixed point problems involving nonexpansive mappings and pseudomonotone bifunctions that satisfy Lipschitz-type continuous in the setting of real Hilbert spaces. The strong convergence theorem of the introduced algorithm without the prior knowledge of the Lipschitz-type constants of bifunctions is presented under some constraint qualifications of the scalar sequences. Some numerical experiments are performed to demonstrate the computational effectiveness of the established algorithm.

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1. Introduction

The fixed point problem is a very useful tool for studying physics, chemistry, engineering, and economics in different mathematical models. Besides, the fixed point problem has many important applications, such as null point problem, variational inequality problem, equilibrium problem, optimization problem, see [1–5], and the references therein. The fixed point problem is a problem of finding a point $x \in H$ such that $Tx = x$, where H is a real Hilbert space and $T : H \to H$ is a mapping. The set of fixed points of the mapping T will be represented by $F(T)$.

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One of the most popular methods for finding fixed points of a nonexpansive mapping $T : C \to C$ was proposed by Mann [6] as followed:

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1-\alpha_k)x_k + \alpha_k T x_k, \end{cases} \tag{1.1}$$

where $\{\alpha_k\} \subset (0,1)$ and C is a nonempty closed convex subset of a real Hilbert space H. In [7], the author proved that if T has a fixed point and $\sum_{k=0}^{\infty} \alpha_k(1-\alpha_k) = \infty$, then the sequence $\{x_k\}$ generated by (1.1) converges weakly to a fixed point of T. In order to obtain a strong convergence result for Mann iterative method (1.1), Takahashi et al. [8] proposed the following so-called shrinking method for finding fixed points of a nonexpansive mapping T:

$$\begin{cases} u_0 \in H, C_1 = C, \\ x_1 = P_{C_1}(u_0), \\ y_k = \alpha_k x_k + (1-\alpha_k)T x_k, \\ C_{k+1} = \{x \in C_k : \|y_k - x\| \leq \|x_k - x\|\}, \\ x_{k+1} = P_{C_{k+1}}(x_0), \end{cases} \tag{1.2}$$

where $\{\alpha_k\} \subset [\underline{\alpha}, \overline{\alpha}]$ with $0 \leq \underline{\alpha} \leq \overline{\alpha} < 1$. They proved that the sequence $\{x_k\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

On the other hand, the equilibrium problem started to gain interest after the publication of a paper by Blum and Oettli [9] which has been used for studying a variety of mathematical problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, see [9–12], and the references therein. The equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1.3}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f : H \times H \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.3) will be denoted by $EP(f, C)$.

A famous method for solving the equilibrium problem (1.3), when f is a monotone bifunction, is the proximal point method, see [13]. However, if f satisfies a weaker assumption such as pseudomonotone, the proximal point method cannot be guaranteed in this situation. To overcome this drawback, the extragradient method was introduced for solving pseudomonotone equilibrium problem instead of the proximal point method. By using the idea of Korpelevich [14], Tran et al. [15] proposed the following extragradient method for solving the equilibrium problem, when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\left\{\lambda f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\right\}, \\ x_{k+1} = \arg\min\left\{\lambda f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\right\}, \end{cases} \tag{1.4}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by (1.4) converges weakly to a solution of the equilibrium problem. It points out that the extragradient method can compute effectively numerically by using the optimization tools.

Consequently, the extragradient method is the first of our interest for solving the equilibrium problem.

Second, we focus on the inertial type methods for finding a solution of the equilibrium problem. This method originates from the heavy ball method (an implicit discretization) of a second-order-in-time dissipative dynamical system [16, 17] and can be regarded as a method of speeding up the convergence properties. The inertial techniques have been proposed for solving the equilibrium problems, for instance, see [18–20], and the references therein. In 2020, by using both inertial and extragradient methods together with the shrinking method, Hieu et al. [21] proposed the following method for solving the equilibrium problem, when the bifunction f is pseudomonotone and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0, x_1 \in H, C_0 = H, \\ w_k = x_k + \theta_k(x_k - x_{k-1}), \\ y_k = \arg\min\{\lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\}, \\ z_k = \arg\min\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C\}, \\ \lambda_{k+1} = \min\left\{\lambda_k, \dfrac{\beta(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]_+}\right\}, \\ H_k = \{x \in H : \|z_k - x\|^2 \leq \|w_k - x\|^2\}, \\ C_{k+1} = C_k \cap H_k, \\ x_{k+1} = P_{C_{k+1}}(x_0), \end{cases} \tag{1.5}$$

where $\lambda_0 > 0$, $\beta \in (0, 1)$, $\theta_k \subset [-\theta, \theta]$ for some $\theta > 0$, and $[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)]_+$ $:= \max\{0, f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\}$. They proved that the sequence $\{x_k\}$ generated by (1.5) converges strongly to $P_{EP(f,C)}(x_0)$.

In 2016, Dinh et al. [22] introduced the split equilibrium and fixed point problems as follows:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad Tx^* = x^*, \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* = Ax^* \in Q \quad \text{solves} \quad Su^* = u^*, \quad g(u^*, v) \geq 0, \forall v \in Q, \end{cases} \tag{1.6}$$

where C and Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ are bifunctions, $T : C \to C$ and $S : Q \to Q$ are mappings, and $A : H_1 \to H_2$ is a bounded linear operator. By using the ideas of proximal point and extragradient methods together with the shrinking method, Dinh et al. [22] proposed the following algorithm for solving the split equilibrium and fixed point problems (1.6), when S and T are nonexpansive mappings, g is monotone bifunction, f is pseudomonotone bifunction and satisfies Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_1 \in C_1 = C, \\ y_k = \arg\min\left\{\lambda_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\right\}, \\ z_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\right\}, \\ s_k = (1-\alpha)z_k + \alpha T z_k, \\ u_k = T^g_{r_k} A s_k, \\ t_k = P_C(s_k + \delta A^*(S u_k - A s_k)), \\ C_{k+1} = \{x \in C_k : \|x - t_k\| \leq \|x - s_k\| \leq \|x - x_k\|\}, \\ x_{k+1} = P_{C_{k+1}}(x_1), \end{cases} \tag{1.7}$$

where A^* is the adjoint operator of A, $\{\lambda_k\} \subset [\underline{\lambda}, \overline{\lambda}]$ with $0 < \underline{\lambda} \leq \overline{\lambda} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{r_k\} \subset (0,\infty)$ such that $\liminf\limits_{k\to\infty} r_k > 0$, $\alpha \in (0,1)$, $\delta \in \left(0, \frac{1}{\|A\|^2}\right)$, and $T^g_{r_k} A s_k := \left\{u \in Q : g(u,v) + \frac{1}{r_k}\langle v - u, u - A s_k\rangle \geq 0, \forall v \in Q\right\}$. They proved that the sequence $\{x_k\}$ generated by (1.7) converges strongly to a solution of the split equilibrium and fixed point problems (1.6). Here, the algorithm (1.7) will be called SEPM Algorithm.

In 2019, by using the extragradient method, Petrot et al. [23] proposed the following algorithm for finding a solution of the split equilibrium and fixed point problems (1.6), when the mappings S and T are nonexpansive, and the bifunctions f and g are pseudomonotone and satisfy Lipschitz-type continuous with some positive constants $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively:

$$\begin{cases} x_1 \in H_1, \\ u_k = \arg\min\left\{\mu_k g(P_Q(Ax_k), u) + \frac{1}{2}\|P_Q(Ax_k) - u\|^2 : u \in Q\right\}, \\ v_k = \arg\min\left\{\mu_k g(u_k, u) + \frac{1}{2}\|P_Q(Ax_k) - u\|^2 : u \in Q\right\}, \\ y_k = P_C\left(x_k + \delta_k A^*(S v_k - A x_k)\right), \\ t_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y_k - y\|^2 : y \in C\right\}, \\ z_k = \arg\min\left\{\lambda_k f(t_k, y) + \frac{1}{2}\|y_k - y\|^2 : y \in C\right\}, \\ x_{k+1} = \alpha_k h(x_k) + (1-\alpha_k)\left(\beta_k x_k + (1-\beta_k) T z_k\right), \end{cases} \tag{1.8}$$

where h is a ρ-contraction mapping, $\{\lambda_k\} \subset [\underline{\lambda}, \overline{\lambda}]$ with $0 < \underline{\lambda} \leq \overline{\lambda} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\{\mu_k\} \subset [\underline{\mu}, \overline{\mu}]$ with $0 < \underline{\mu} \leq \overline{\mu} < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\}$, $\{\delta_k\} \subset [\underline{\delta}, \overline{\delta}]$ with $0 < \underline{\delta} \leq \overline{\delta} < \frac{1}{\|A\|^2}$, $\{\beta_k\} \subset (0,1)$ with $0 < \liminf\limits_{k\to\infty} \beta_k \leq \limsup\limits_{k\to\infty} \beta_k < 1$, $\{\alpha_k\} \subset \left(0, \frac{1}{2-\rho}\right)$ such that $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$, and $\lim\limits_{k\to\infty} \alpha_k = 0$. They proved that the sequence $\{x_k\}$ generated by (1.8) converges strongly to a solution of the split equilibrium and fixed point problems (1.6). Here, the algorithm (1.8) will be called NEM Algorithm. We emphasize that the SEPM and NEM algorithms need to have prior knowledge of the Lipschitz-type constants of the bifunctions. This means that the SEPM and NEM algorithms used the stepsizes which depend on the Lipschitz-type constants of the bifunctions. This fact may give some restrictions in applications because the Lipschitz-type constants are often unknown or difficult to approximate.

In this paper, we will still focus on the methods for solving the split equilibrium and fixed point problems (1.6). That is, we will present a new iterative algorithm without the prior knowledge of the Lipschitz-type constants of the bifunctions for finding the solutions of the split equilibrium and fixed point problems, when the mappings are nonexpansive and the bifunctions are pseudomonotone and satisfy Lipschitz-type continuous. Some numerical examples and comparisons of the introduced algorithm with some appeared algorithms will be discussed.

This paper is organized as follows: In Section 2, some necessary definitions and properties will be reviewed for further use. In Section 3, we will present the shrinking inertial extragradient algorithm and prove the strong convergence theorem. In Section 4, we will discuss the performance of the introduced algorithm by comparing it to the aforesaid algorithms via numerical experiments.

2. Preliminaries

This section will present some definitions and properties that will be used in this paper. Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$, and its corresponding $\|\cdot\|$. The symbols $\to$ and $\rightharpoonup$ will be denoted for the strong convergence and the weak convergence in H, respectively.

First, we will recall some definitions and facts which are related to nonlinear mappings.

Definition 2.1. A mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H.$$

Remark 2.2. It is well-known that $F(T)$ is closed and convex, when T is a nonexpansive mapping, see [24].

Definition 2.3. A mapping $T : H \to H$ is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup x^* \in H$ and $Tx_k \to y$ imply $Tx^* = y$.

Lemma 2.4. (see [24]) *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, $I - T$ demiclosed at 0.*

Next, we will provide some definitions and results for concerning the equilibrium problems.

Definition 2.5. [9, 12, 25] Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is said to be:

(i) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \ \forall x, y \in C;$$

(iii) Lipshitz-type continuous on H with constants $c_1 > 0$ and $c_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y, z \in H.$$

Remark 2.6. We note that a monotone bifunction is a pseudomonotone bifunction. However, the converse may not be true in general, for instance, see [26].

Let C be a nonempty closed convex subset of H. For a bifunction $f : H \times H \to \mathbb{R}$, the following assumptions will be considered in this paper:

($A1$) f is weakly continuous on $C \times C$ in the sense that, if $x \in C$, $y \in C$, and $\{x_k\} \subset C$, $\{y_k\} \subset C$ are two sequences converge weakly to x and y respectively, then $f(x_k, y_k)$ converges to $f(x, y)$;

($A2$) $f(x, \cdot)$ is convex and subdifferentiable on C, for each fixed $x \in C$;

($A3$) f is psuedomonotone on C and $f(x, x) = 0$, for each $x \in C$;

($A4$) f is Lipshitz-type continuous on H with constants $c_1 > 0$ and $c_2 > 0$.

Remark 2.7. It is well-known that the solution set $EP(f, C)$ is closed and convex, when the bifunction f satisfies the assumptions $(A1) - (A3)$, see [15, 27, 28], for more detail.

We end this section by recalling the projection mapping and calculus concepts in Hilbert space.

Let C be a nonempty closed convex subset of H. For each $x \in H$, we denote the metric projection of x onto C by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|y - x\|, \forall y \in C.$$

Lemma 2.8. (see [29, 30]) *Let C be a nonempty closed convex subset of H. Then,*

(i) $P_C(x)$ *is singleton and well-defined for each* $x \in H$*;*

(ii) $z = P_C(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$*;*

(iii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2$, $\forall x, y \in C$*;*

(iv) P_C *is a nonexpansive mapping.*

For a function $f : H \to \mathbb{R}$, the subdifferential of f at $z \in H$ is defined by

$$\partial f(z) = \{w \in H : f(y) - f(z) \geq \langle w, y - z\rangle, \forall y \in H\}.$$

The function f is said to be subdifferentiable at z if $\partial f(z) \neq \emptyset$.

Lemma 2.9. (see [29]) *For any $z \in H$, the subdifferentiable $\partial f(z)$ of a continuous convex function f is a weakly closed and bounded convex set.*

Lemma 2.10. [11] *Let C be a convex subset of H and $f : C \to \mathbb{R}$ be subdifferentiable on C. Then, x^* is a solution to the following convex problem:*

$$\min \{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, *where* $N_C(x^*) := \{y \in H : \langle y, z - x^*\rangle \leq 0, \forall z \in C\}$ *is the normal cone of C at x^*.*

3. Main Results

Let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let us recall the split equilibrium and fixed point problems:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad Tx^* = x^*, \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* = Ax^* \in Q \quad \text{solves} \quad Su^* = u^*, \quad g(u^*, v) \geq 0, \forall v \in Q, \end{cases} \tag{3.1}$$

where $f \colon H_1 \times H_1 \to \mathbb{R}$ and $g \colon H_2 \times H_2 \to \mathbb{R}$ are bifunctions, $T \colon H_1 \to H_1$ and $S \colon H_2 \to H_2$ are mappings, and $A \colon H_1 \to H_2$ is a bounded linear operator with its adjoint operator

A^*. From now on, the solution set of problem (3.1) will be denoted by Ω. That is,

$$\Omega := \{p \in EP(f, C) \cap F(T) : \ Ap \in EP(g, Q) \cap F(S)\}.$$

Now, we introduce the algorithm for solving the split equilibrium and fixed point problems (3.1).

Algorithm 1: Shrinking Inertial Extragradient Method (SIEM)

Initialization. Choose parameters $\lambda_1 > 0$, $\mu_1 > 0$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\eta \in \left(0, \frac{1}{\|A\|^2}\right)$, $\{\alpha_k\} \subset (0, 1)$, and $\{\theta_k\} \subset [-1, 1]$. Pick $x_0, x_1 \in C =: C_1$ and set $k = 1$.

Step 1. Compute

$$w_k = x_k + \theta_k(x_k - x_{k-1}).$$

Step 2. Solve the strongly convex program

$$y_k = \arg\min \left\{ \lambda_k f(w_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C \right\}.$$

Step 3. Solve the strongly convex program

$$z_k = \arg\min \left\{ \lambda_k f(y_k, y) + \frac{1}{2}\|y - w_k\|^2 : y \in C \right\}.$$

Step 4. Compute

$$s_k = \alpha_k z_k + (1 - \alpha_k) T z_k.$$

Step 5. Solve the strongly convex program

$$u_k = \arg\min \left\{ \mu_k g(As_k, u) + \frac{1}{2}\|u - As_k\|^2 : u \in Q \right\}.$$

Step 6. Solve the strongly convex program

$$v_k = \arg\min \left\{ \mu_k g(u_k, u) + \frac{1}{2}\|u - As_k\|^2 : u \in Q \right\}.$$

Step 7. Compute

$$t_k = P_C\left(s_k + \eta A^*(Sv_k - As_k)\right).$$

Step 8. Construct closed convex subsets of C:

$$C_{k+1} = \{x \in C_k : \|x - t_k\| \leq \|x - s_k\| \leq \|x - w_k\|\}.$$

Step 9. Compute

$$\lambda_{k+1} = \begin{cases} \min\left\{ \lambda_k, \dfrac{\beta(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\left[f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)\right]} \right\}, & \\ \qquad\qquad\qquad\qquad \text{if } f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) > 0, & \\ \lambda_k, \qquad\qquad\qquad\quad \text{otherwise}, & \end{cases}$$

and

$$\mu_{k+1} = \begin{cases} \min\left\{\mu_k, \dfrac{\gamma(\|As_k - u_k\|^2 + \|v_k - u_k\|^2)}{2\left[g(As_k, v_k) - g(As_k, u_k) - g(u_k, v_k)\right]}\right\}, \\ \qquad\qquad\qquad \text{if } g(As_k, v_k) - g(As_k, u_k) - g(u_k, v_k) > 0, \\ \mu_k, \qquad\qquad \text{otherwise.} \end{cases}$$

Step 10. The next approximation x_{k+1} is defined as the projection of x_1 onto C_{k+1}, i.e.,

$$x_{k+1} = P_{C_{k+1}}(x_1).$$

Step 11. Put $k := k + 1$ and go to **Step 1**.

Remark 3.1. We point out that the stepsizes of the SIEM Algorithm are independent of the Lipschitz-type constants of the bifunctions. This means that the SIEM Algorithm is constructed without prior knowledge of the Lipschitz-type constants of the bifunctions. We emphasize that the Lipschitz-type constants of the bifunctions are often unknown or difficult to estimate. Furthermore, the term $\theta_k(x_k - x_{k-1})$, which is included in the SIEM Algorithm, is called the inertial effect and intended to speed up the convergence properties. We observe that the parameter θ_k in the SIEM Algorithm can take negative values.

Theorem 3.2. *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $f\colon H_1 \times H_1 \to \mathbb{R}$ and $g\colon H_2 \times H_2 \to \mathbb{R}$ be bifunctions which satisfy $(A1) - (A4)$ with some positive constants $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively. Let $T\colon H_1 \to H_1$ and $S\colon H_2 \to H_2$ be nonexpansive mappings, and $A\colon H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ which is generated by the SIEM Algorithm converges strongly to $P_\Omega(x_1)$.*

Proof. The proof of Theorem 3.2 is divided into 3 steps.

Claim 1. The following hold:

$$\|z_k - p\|^2 \le \|w_k - p\|^2 - \epsilon_1\|w_k - y_k\|^2 - \epsilon_1\|y_k - z_k\|^2, \text{ for each fixed } \epsilon_1 \in (0, 1-\beta),$$

and

$$\|v_k - Ap\|^2 \le \|As_k - Ap\|^2 - \epsilon_2\|As_k - u_k\|^2 - \epsilon_2\|u_k - v_k\|^2, \text{ for each fixed } \epsilon_2 \in (0, 1-\gamma).$$

The proof of Claim 1. Let $p \in \Omega$. That is, $p \in F(T)$, $p \in EP(f, C)$, and $Ap \in F(S)$, $Ap \in EP(g, Q)$. By the definition of z_k and Lemma 2.10, we have

$$0 \in \partial_2\left\{\lambda_k f(y_k, z_k) + \frac{1}{2}\|z_k - w_k\|^2\right\} + N_C(z_k).$$

Thus, there exists $q \in \partial_2 f(y_k, z_k)$ and $\overline{q} \in N_C(z_k)$ such that

$$0 = \lambda_k q + z_k - w_k + \overline{q}. \tag{3.2}$$

So, it follows from the subdifferentiability of f that

$$f(y_k, y) - f(y_k, z_k) \ge \langle q, y - z_k\rangle, \forall y \in C. \tag{3.3}$$

Moreover, from $\overline{q} \in N_C(z_k)$, we have

$$\langle \overline{q}, z_k - y\rangle \geq 0, \forall y \in C.$$

Using this one together with (3.2), we get

$$\langle z_k - w_k, y - z_k\rangle \geq \lambda_k \langle q, z_k - y\rangle, \forall y \in C. \tag{3.4}$$

Thus, the relations (3.3) and (3.4) imply that

$$\langle z_k - w_k, y - z_k\rangle \geq \lambda_k [f(y_k, z_k) - f(y_k, y)], \forall y \in C. \tag{3.5}$$

Now, from $p \in C$, we see that

$$\langle z_k - w_k, p - z_k\rangle \geq \lambda_k [f(y_k, z_k) - f(y_k, p)].$$

So, it follows from the pseudomonotonic of f that

$$\langle z_k - w_k, p - z_k\rangle \geq \lambda_k f(y_k, z_k). \tag{3.6}$$

Similarly, by the definition of y_k and Lemma 2.10, we can show that

$$\lambda_k [f(w_k, y) - f(w_k, y_k)] \geq \langle y_k - w_k, y_k - y\rangle, \forall y \in C. \tag{3.7}$$

Note that, since $z_k \in C$, we have

$$\lambda_k [f(w_k, z_k) - f(w_k, y_k)] \geq \langle y_k - w_k, y_k - z_k\rangle. \tag{3.8}$$

Thus, in view of (3.6) and (3.8), we obtain

$$\lambda_k [f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k)] \geq \langle w_k - z_k, p - z_k\rangle + \langle y_k - w_k, y_k - z_k\rangle. \tag{3.9}$$

On the other hand, by the definition of λ_{k+1}, we see that

$$f(w_k, z_k) - f(w_k, y_k) - f(y_k, z_k) \leq \frac{\beta(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\lambda_{k+1}}.$$

Combining with (3.9) implies that

$$\langle z_k - w_k, p - z_k\rangle \geq \langle y_k - w_k, y_k - z_k\rangle - \frac{\beta\lambda_k(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{2\lambda_{k+1}}.$$

Due to the above inequality, we observe that

$$\begin{aligned}
\|w_k - p\|^2 - \|z_k - w_k\|^2 - \|p - z_k\|^2 &= 2\langle z_k - w_k, p - z_k\rangle \\
&\geq 2\langle y_k - w_k, y_k - z_k\rangle \\
&\quad - \frac{\beta\lambda_k(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{\lambda_{k+1}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|z_k - p\|^2 &\leq \|w_k - p\|^2 - \|z_k - w_k\|^2 - 2\langle y_k - w_k, y_k - z_k\rangle \\
&\quad + \frac{\beta\lambda_k(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{\lambda_{k+1}} \\
&= \|w_k - p\|^2 - \|z_k - y_k\|^2 - \|y_k - w_k\|^2 - 2\langle z_k - y_k, y_k - w_k\rangle \\
&\quad - 2\langle y_k - w_k, y_k - z_k\rangle + \frac{\beta\lambda_k(\|w_k - y_k\|^2 + \|z_k - y_k\|^2)}{\lambda_{k+1}} \\
&= \|w_k - p\|^2 - \left(1 - \frac{\beta\lambda_k}{\lambda_{k+1}}\right)\|w_k - y_k\|^2
\end{aligned}$$

$$-\left(1-\frac{\beta\lambda_k}{\lambda_{k+1}}\right)\|y_k-z_k\|^2. \quad (3.10)$$

Now, let us consider the definition of λ_{k+1}. We observe that $\lambda_{k+1} \leq \lambda_k$, for each $k \in \mathbb{N}$. This means that $\{\lambda_k\}$ is a nonincreasing sequence. In addition, by the Lipschitz-type continuity of f, we have

$$\begin{aligned} f(w_k,z_k)-f(w_k,y_k)-f(y_k,z_k) &\leq c_1\|w_k-y_k\|^2+c_2\|z_k-y_k\|^2 \\ &\leq \max\{c_1,c_2\}(\|w_k-y_k\|^2+\|z_k-y_k\|^2). \end{aligned}$$

Using this one together with the definition of λ_{k+1}, we see that

$$\lambda_{k+1} \geq \min\left\{\lambda_k, \frac{\beta}{2\max\{c_1,c_2\}}\right\} \geq \ldots \geq \min\left\{\lambda_1, \frac{\beta}{2\max\{c_1,c_2\}}\right\}.$$

This means that $\{\lambda_k\}$ is bounded from below. Consequently, we have the limit of $\{\lambda_k\}$ exists. Next, let $\epsilon_1 \in (0, 1-\beta)$ be fixed. These imply that

$$\lim_{k\to\infty}\left(1-\frac{\beta\lambda_k}{\lambda_{k+1}}\right)=1-\beta>\epsilon_1>0.$$

Thus, there exists $k_1 \in \mathbb{N}$ such that

$$1-\frac{\beta\lambda_k}{\lambda_{k+1}} \geq \epsilon_1 > 0, \forall k \geq k_1. \quad (3.11)$$

Similarly, we can show that

$$\mu_k[g(As_k,u)-g(As_k,u_k)] \geq \langle u_k-As_k, u_k-u\rangle, \forall u \in Q, \quad (3.12)$$

and

$$\begin{aligned} \|v_k-Ap\|^2 &\leq \|As_k-Ap\|^2-\left(1-\frac{\gamma\mu_k}{\mu_{k+1}}\right)\|As_k-u_k\|^2 \\ &\quad -\left(1-\frac{\gamma\mu_k}{\mu_{k+1}}\right)\|u_k-v_k\|^2. \end{aligned} \quad (3.13)$$

Moreover, we also have the limit of $\{\mu_k\}$ exists. Let $\epsilon_2 \in (0, 1-\gamma)$ be fixed. These imply that

$$\lim_{k\to\infty}\left(1-\frac{\gamma\mu_k}{\mu_{k+1}}\right)=1-\gamma>\epsilon_2>0.$$

Thus, there exists $k_2 \in \mathbb{N}$ such that

$$1-\frac{\gamma\mu_k}{\mu_{k+1}} \geq \epsilon_2 > 0, \forall k \geq k_2. \quad (3.14)$$

Choose $k_0 = \max\{k_1, k_2\}$. Then, by using (3.10), (3.11), (3.13), and (3.14), we have

$$\|z_k-p\|^2 \leq \|w_k-p\|^2-\epsilon_1\|w_k-y_k\|^2-\epsilon_1\|y_k-z_k\|^2, \quad (3.15)$$

and

$$\|v_k-Ap\|^2 \leq \|As_k-Ap\|^2-\epsilon_2\|As_k-u_k\|^2-\epsilon_2\|u_k-v_k\|^2, \quad (3.16)$$

for each $k \geq k_0$.

Claim 2. The sequence $\{x_k\}$ is well-defined.
The proof of Claim 2. It suffices to show that C_k is a nonempty closed convex subset of

H, for each $k \in \mathbb{N}$. Firstly, we will claim the non-emptiness by showing that $\Omega \subset C_k$, for each $k \in \mathbb{N}$. Obviously, $\Omega \subset C_1$.

Now, by using (3.15) and (3.16), we have

$$\|z_k - p\| \leq \|w_k - p\|, \tag{3.17}$$

and

$$\|v_k - Ap\| \leq \|As_k - Ap\|. \tag{3.18}$$

By the definition of t_k and the nonexpansivity of P_C, we have

$$\begin{aligned}\|t_k - p\|^2 &\leq \|(s_k - p) + \eta A^*(Sv_k - As_k)\|^2 \\ &= \|s_k - p\|^2 + \eta^2\|A\|^2\|Sv_k - As_k\|^2 + 2\eta\langle As_k - Ap, Sv_k - As_k\rangle. \end{aligned} \tag{3.19}$$

Consider,

$$\begin{aligned} 2\langle As_k - Ap, Sv_k - As_k\rangle &= 2\langle Sv_k - Ap, Sv_k - As_k\rangle - 2\|Sv_k - As_k\|^2 \\ &= \|Sv_k - Ap\|^2 - \|Sv_k - As_k\|^2 - \|As_k - Ap\|^2. \end{aligned}$$

Using this one together with (3.19) and the nonexpansivity of S, we get

$$\|t_k - p\|^2 \leq \|s_k - p\|^2 - \eta(1 - \eta\|A\|^2)\|Sv_k - As_k\|^2 + \eta(\|v_k - Ap\|^2 - \|As_k - Ap\|^2).$$

Combining with (3.18) implies that

$$\|t_k - p\|^2 \leq \|s_k - p\|^2 - \eta(1 - \eta\|A\|^2)\|Sv_k - As_k\|^2. \tag{3.20}$$

Thus, by the choice of η, we have

$$\|t_k - p\| \leq \|s_k - p\|. \tag{3.21}$$

On the other hand, by the definition of s_k and the nonexpansivity of T, we obtain

$$\begin{aligned}\|s_k - p\| &\leq \alpha_k\|z_k - p\| + (1 - \alpha_k)\|Tz_k - p\| \\ &\leq \|z_k - p\|. \end{aligned} \tag{3.22}$$

So, the relations (3.17), (3.21) and (3.22) imply that

$$\|t_k - p\| \leq \|s_k - p\| \leq \|w_k - p\|. \tag{3.23}$$

Now, let $k \geq k_0$ and suppose that $\Omega \subset C_k$. Thus, by using (3.23), we see that $\Omega \subset C_{k+1}$. Using this one together with the fact that $C_{k+1} \subset C_k$, for each $k \in \mathbb{N}$, we get $\Omega \subset C_k$, for each $k \in \mathbb{N}$. Consequently, from Ω is a nonempty set, we can conclude that C_k is a nonempty set, for each $k \in \mathbb{N}$.

Next, we will claim that C_k is a closed and convex subset, for each $k \in \mathbb{N}$, by induction. Note that we already have that C_1 is a closed and convex subset. Now, suppose that C_k is a closed and convex subset. Let us consider the sets $B_k^1 = \{x \in H_1 : \|t_k - x\| \leq \|s_k - x\|\}$, $B_k^2 = \{x \in H_1 : \|s_k - x\| \leq \|w_k - x\|\}$, and $D_k = \{x \in H_1 : \|t_k - x\| \leq \|s_k - x\| \leq \|w_k - x\|\}$. Thus, we see that

$$B_k^1 = \left\{x \in H_1 : \langle s_k - t_k, x\rangle \leq \frac{1}{2}(\|s_k\|^2 - \|t_k\|^2)\right\},$$

and

$$B_k^2 = \left\{x \in H_1 : \langle w_k - s_k, x\rangle \leq \frac{1}{2}(\|w_k\|^2 - \|s_k\|^2)\right\}.$$

This means that B_k^1 and B_k^2 are halfspaces. We observe that $D_k = B_k^1 \cap B_k^2$ and $C_{k+1} = C_k \cap D_k$. These imply that C_{k+1} is a closed convex subset. Then, by induction, we can conclude that C_k is a closed convex subset, for each $k \in \mathbb{N}$. Consequently, we can guarantee that $\{x_k\}$ is well-defined.

Claim 3. The sequence $\{x_k\}$ converges strongly to $P_\Omega(x_1)$.
The proof of Claim 3. By the definition of x_{k+1}, we see that $x_{k+1} \in C_{k+1} \subset C_k$, for each $k \in \mathbb{N}$. Since $x_k = P_{C_k}(x_1)$ and $x_{k+1} \in C_k$, we have

$$\|x_k - x_1\| \leq \|x_{k+1} - x_1\|.$$

This means that $\{\|x_k - x_1\|\}$ is a nondecreasing sequence. Similarly, for each $p \in \Omega \subset C_{k+1}$, we get

$$\|x_{k+1} - x_1\| \leq \|p - x_1\|.$$

Thus, by the above inequalities, we have

$$\|x_k - x_1\| \leq \|p - x_1\|. \tag{3.24}$$

Then, $\{\|x_k - x_1\|\}$ is a bounded sequence. Consequently, we can conclude that $\{\|x_k - x_1\|\}$ is a convergent sequence. Moreover, we see that $\{x_k\}$ is bounded. Suppose $k, j \in \mathbb{N}$ such that $k > j$. It follows that $x_k \in C_k \subset C_j$. Then, by Lemma 2.8 (iii), we have

$$\|P_{C_j}(x_k) - P_{C_j}(x_1)\|^2 \leq \|x_1 - x_k\|^2 - \|P_{C_j}(x_k) - x_k + x_1 - P_{C_j}(x_1)\|^2.$$

This implies that

$$\|x_k - x_j\|^2 \leq \|x_1 - x_k\|^2 - \|x_j - x_1\|^2.$$

Thus, by using the existence of $\lim_{k\to\infty} \|x_k - x_1\|$, we get

$$\lim_{k,j\to\infty} \|x_k - x_j\| = 0.$$

That is $\{x_k\}$ is a Cauchy sequence in C. Since C is closed, there exists $x^* \in C$ such that

$$\lim_{k\to\infty} x_k = x^*. \tag{3.25}$$

So, it follows from the definition of w_k that

$$\lim_{k\to\infty} \|w_k - x_k\| = 0. \tag{3.26}$$

Additionally, by the definition of C_{k+1} and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - t_k\| \leq \|x_{k+1} - s_k\| \leq \|x_{k+1} - w_k\|. \tag{3.27}$$

This implies that

$$\begin{aligned} \|t_k - w_k\| &\leq \|t_k - x_{k+1}\| + \|x_{k+1} - w_k\| \\ &\leq 2\|x_{k+1} - w_k\| \\ &\leq 2(\|x_{k+1} - x_k\| + \|x_k - w_k\|). \end{aligned}$$

Thus, in view of (3.25) and (3.26), we get

$$\lim_{k\to\infty} \|t_k - w_k\| = 0. \tag{3.28}$$

Similarly, from (3.27), we have

$$\begin{aligned} \|s_k - w_k\| &\leq \|s_k - x_{k+1}\| + \|x_{k+1} - w_k\| \\ &\leq 2\|x_{k+1} - w_k\| \\ &\leq 2(\|x_{k+1} - x_k\| + \|x_k - w_k\|). \end{aligned}$$

Thus, by using (3.25) and (3.26), we obtain

$$\lim_{k\to\infty} \|s_k - w_k\| = 0. \tag{3.29}$$

Combining with (3.28) implies that

$$\lim_{k\to\infty} \|s_k - t_k\| = 0. \tag{3.30}$$

On the other hand, in view of (3.15) and (3.22), we get that

$$\begin{aligned} \epsilon_1\|w_k - y_k\|^2 + \epsilon_1\|y_k - z_k\|^2 &\leq \|w_k - p\|^2 - \|z_k - p\|^2 \\ &\leq \|w_k - p\|^2 - \|s_k - p\|^2 \\ &= \|w_k - s_k\|(\|w_k - p\| + \|s_k - p\|). \end{aligned}$$

Thus, by applying (3.29) to the above inequality, we have

$$\lim_{k\to\infty} \|w_k - y_k\| = 0, \tag{3.31}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{3.32}$$

These imply that

$$\lim_{k\to\infty} \|w_k - z_k\| = 0. \tag{3.33}$$

So, it follows from (3.29) that

$$\lim_{k\to\infty} \|s_k - z_k\| = 0. \tag{3.34}$$

Moreover, in view of (3.26) and (3.31), we obtain

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.35}$$

Combining with (3.32) implies that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.36}$$

Using this one together with (3.34), we get

$$\lim_{k\to\infty} \|x_k - s_k\| = 0. \tag{3.37}$$

Due to the definition of s_k, we observe that

$$(1-\alpha_k)\|Tz_k - z_k\| = \|s_k - z_k\|.$$

Thus, by using (3.34), we have

$$\lim_{k\to\infty} \|Tz_k - z_k\| = 0. \tag{3.38}$$

On the other hand, in view of (3.20), we see that

$$\begin{aligned} \eta(1-\eta\|A\|^2)\|Sv_k - As_k\|^2 &\leq \|s_k - p\|^2 - \|t_k - p\|^2 \\ &\leq \|s_k - t_k\|(\|s_k - p\| + \|t_k - p\|). \end{aligned}$$

Thus, by applying (3.30) to the above inequality, we get

$$\lim_{k\to\infty} \|Sv_k - As_k\| = 0. \tag{3.39}$$

Furthermore, the relation (3.16) and the nonexpansivity of S, we have

$$\epsilon_2\|As_k - u_k\|^2 + \epsilon_2\|u_k - v_k\|^2 \leq \|As_k - Ap\|^2 - \|v_k - Ap\|^2$$

$$\leq (\|As_k - Sv_k\| + \|Sv_k - Ap\| - \|v_k - Ap\|)(\|As_k - Ap\| + \|v_k - Ap\|)$$
$$\leq \|As_k - Sv_k\|(\|As_k - Ap\| + \|v_k - Ap\|).$$

Using this one together with (3.39), we have

$$\lim_{k\to\infty} \|As_k - u_k\| = 0, \tag{3.40}$$

and

$$\lim_{k\to\infty} \|u_k - v_k\| = 0. \tag{3.41}$$

These imply that

$$\lim_{k\to\infty} \|As_k - v_k\| = 0. \tag{3.42}$$

Thus, it follows from (3.39) that

$$\lim_{k\to\infty} \|Sv_k - v_k\| = 0. \tag{3.43}$$

Next, we will claim that $x^* \in \Omega$. Since $x_k \to x^*$, as $k \to \infty$, by using (3.26), (3.35), (3.36) and (3.37), we also have $w_k \to x^*$, $y_k \to x^*$, $z_k \to x^*$, and $s_k \to x^*$, as $k \to \infty$. The latter fact implies that $As_k \to Ax^*$, as $k \to \infty$. Using this one together with (3.40) and (3.42), we also have $u_k \to Ax^*$, and $v_k \to Ax^*$, as $k \to \infty$. Since Q is closed and $\{v_k\}$ is a sequence in Q, we have $Ax^* \in Q$. Moreover, from (3.7) and (3.12), we get that

$$f(w_k, y) - f(w_k, y_k) \geq -\frac{1}{\lambda_k}\|y_k - w_k\|\|y_k - y\|, \forall y \in C,$$

and

$$g(As_k, u) - g(As_k, u_k) \geq -\frac{1}{\mu_k}\|u_k - As_k\|\|u_k - u\|, \forall u \in Q.$$

Thus, by using (3.31), (3.40), and the weak continuity of f and g, we have

$$f(x^*, y) \geq 0, \forall y \in C,$$

and

$$g(Ax^*, u) \geq 0, \forall u \in Q.$$

On the other hand, since $z_k \to x^*$, as $k \to \infty$, and (3.38), then by the demiclosedness at 0 of $I - T$, we have $x^* \in F(T)$. Similarly, since $v_k \to Ax^*$, as $k \to \infty$, and (3.43), it follows from the demiclosedness at 0 of $I - S$ that $Ax^* \in F(S)$. Then, we had shown that $x^* \in \Omega$.

Finally, we will show that $x^* = P_\Omega(x_1)$. In fact, since $P_\Omega(x_1) \in \Omega$, it follows from (3.24) that

$$\|x_k - x_1\| \leq \|P_\Omega(x_1) - x_1\|.$$

Thus, by using the continuity of norm and $x_k \to x^*$, as $k \to \infty$, we get

$$\|x^* - x_1\| = \lim_{k\to\infty} \|x_k - x_1\| \leq \|P_\Omega(x_1) - x_1\|.$$

Then, by the definition of $P_\Omega(x_1)$ and $x^* \in \Omega$, we have $x^* = P_\Omega(x_1)$. This completes the proof. ∎

4. Numerical Experiments

This section will consider some examples and numerical results to illustrate the convergence of the introduced algorithm. We will compare the SIEM Algorithm with the SEPM Algorithm (1.7) in Example 4.1, the algorithm that was presented in [31] in Example 4.2, and the NEM Algorithm (1.8) in Example 4.3. The numerical experiments are written in Matlab R2015b and performed on a Laptop with AMD Dual Core R3-2200U CPU @ 2.50GHz and RAM 4.00 GB. In Examples 4.1, 4.2 and 4.3, for each considered matrix, the $\|\cdot\|$ means the spectral norm.

Example 4.1. Let $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}^m$ be two real Hilbert spaces with the Euclidean norm. We consider a classical form of the bifunction which is given by the Cournot-Nash models, see [32],

$$\tilde{f}(x,y) = \langle P_1 x + q_1^n(y+x), y-x\rangle, \quad \forall x,y \in \mathbb{R}^n,$$

where $P_1 = \begin{pmatrix} 0 & q_1 & q_1 & \cdots & q_1 \\ q_1 & 0 & q_1 & \cdots & q_1 \\ q_1 & q_1 & 0 & \cdots & q_1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q_1 & q_1 & \cdot & \cdots & 0 \end{pmatrix}_{n\times n}$ is matrix with the positive real number q_1.

We know that the bifunction $\tilde{f}$ is pseudomonotone and it is not monotone, see [33].

Next, we consider the bifunction $\tilde{g}$ which is generated from Nash-Cournot oligopolistic equilibrium models of electricity markets, see [27, 34],

$$\tilde{g}(u,v) = \langle Mu + Nv, v-u\rangle, \quad \forall u,v \in \mathbb{R}^m,$$

where M, $N \in \mathbb{R}^{m\times m}$ are matrices such that N is symmetric positive semidefinite and $N-M$ is negative semidefinite. We see that $\tilde{g}(u,v)+\tilde{g}(v,u) = (u-v)^t(N-M)(u-v), \forall u,v \in \mathbb{R}^m$. Thus, by the property of $N-M$, we have $\tilde{g}$ is a monotone bifunction.

To be considered here are the bifunctions f and g, which are defined by

$$f(x,y) = \begin{cases} \tilde{f}(x,y), & \text{if } (x,y) \in C\times C, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$g(u,v) = \begin{cases} \tilde{g}(u,v), & \text{if } (u,v) \in Q\times Q, \\ 0, & \text{otherwise}, \end{cases}$$

where $C = \prod_{i=1}^n[-5,5]$ and $Q = \prod_{j=1}^m[-20,20]$ are the constrained boxes. We observe that f and g satisfy Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\|P_1\|$ and $d_1 = d_2 = \frac{1}{2}\|M-N\|$, respectively, see [15, 35].

On the other hand, for the functions $h_1 : \mathbb{R}^n \to \mathbb{R}$ and $h_2 : \mathbb{R}^m \to \mathbb{R}$, which are given by $h_1(x) = \frac{1}{2}x^tUx$, where $U \in \mathbb{R}^{n\times n}$ is invertible symmetric positive semidefinite matrix and $h_2(x) = \|x\|$, respectively, we consider the proximal mappings $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^m \to \mathbb{R}^m$ associated with the functions h_1 and h_2, respectively, which are defined by

$$T(x) = (I_n + U)^{-1}(x),$$

and

$$S(x) = \begin{cases} \left(1 - \frac{1}{\|x\|}\right)x, & \text{if } \|x\| \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$

where I_n is the identity matrix. We know that the proximal mappings T and S are nonexpansive and $F(T) = \arg\min h_1$ and $F(S) = \arg\min h_2$, see [36]. In this case, the fixed point problems can be converted to the minimization problems.

Here, the following setting is taken from Kim and Dinh [31]. Let $A_1 = \begin{pmatrix} I_n \\ -I_n \end{pmatrix}$ be the $2n \times n$ matrix and $b_1 = (5, 5, \ldots, 5)^t \in \mathbb{R}^{2n}$. The sets $\tilde{A}_k$ and $\tilde{b}_k$ are constructed by

$$\tilde{A}_k = \begin{pmatrix} (s_k - t_k)^t \\ (w_k - s_k)^t \end{pmatrix}, \text{ and } \tilde{b}_k = \begin{pmatrix} \frac{1}{2}(\|s_k\|^2 - \|t_k\|^2) \\ \frac{1}{2}(\|w_k\|^2 - \|s_k\|^2) \end{pmatrix}.$$

Therefore, we can compute the sets C_{k+1} as follows:

$$C_{k+1} = \{x \in \mathbb{R}^n : A_{k+1}x \leq b_{k+1}\},$$

where $A_{k+1} = \begin{pmatrix} A_k \\ \tilde{A}_k \end{pmatrix}$, and $b_{k+1} = \begin{pmatrix} b_k \\ \tilde{b}_k \end{pmatrix}$.

The numerical experiment is considered under the following setting: the matrices M, N, and U are randomly chosen from the interval $[-5, 5]$ such that they satisfy above required properties and the positive real number q_1 is randomly chosen from the interval $(1, 1.5)$. In addition, the linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix, which is randomly chosen from the interval $[-2, 2]$. Note that the solution set Ω is nonempty because of $0 \in \Omega$. We will concern with these control parameters: $\eta = \frac{1}{2\|A\|^2}$, $\alpha_k = \frac{1}{k+2}$, $\lambda_1 = \mu_1 = 1$, and $\beta = \gamma = 0.9$. The following five cases of the parameter θ_k are considered:

Case 1. $\theta_k = 0$.

Case 2. $\theta_k = 0.59 - \frac{1}{k+1}$.

Case 3. $\theta_k = 0.99 - \frac{1}{k+1}$.

Case 4. $\theta_k = -\left(0.59 - \frac{1}{k+1}\right)$.

Case 5. $\theta_k = -\left(0.99 - \frac{1}{k+1}\right)$.

We use the function *quadprog* in Matlab Optimization Toolbox to solve vectors y_k, z_k, u_k, and v_k. The starting points $x_0 = x_1 \in \mathbb{R}^n$ are randomly chosen from the interval $[-5, 5]$. The SIEM Algorithm was tested along with the SEPM Algorithm (1.7) by using the stopping criteria $\|x_{k+1} - x_k\| < 10^{-4}$. We randomly 10 starting points and presented results are in average, where $n = 5$ and $m = 10$.

Table 1. The numerical results for the split equilibrium and fixed point problems in Example 4.1

Cases	Average CPU times (sec)		Average iterations	
	SIEM	SEPM	SIEM	SEPM
1	2.6109		50.2	
2	1.7125		35.1	
3	0.4500	4.6031	12.8	77.8
4	1.9563		36.9	
5	0.4594		15.0	

Table 1 shows that the parameter $\theta_k = 0.99 - \frac{1}{k+1}$ yields better both the CPU times and the number of iterations than other cases. Moreover, we see that the CPU times and the number of iterations of the SIEM Algorithm are better than those of the SEPM Algorithm in all considered cases.

Example 4.2. We consider the split equilibrium and fixed point problems (3.1), when $T = I_{H_1}$ and $S = I_{H_2}$ are identity mappings on H_1 and H_2, respectively. It follows that the split equilibrium and fixed point problems (3.1) become the split equilibrium problems. In this case, we compare the SIEM Algorithm with the following algorithm (4.1), which was presented by Kim and Dinh [31], when the bifunctions f and g are pseudomonotone and satisfy Lipschitz-type continuous with positive constants p_1 and p_2:

$$\begin{cases} x_0 \in C_0 = C, \\ y_k = \arg\min\{\lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ z_k = \arg\min\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ u_k = \arg\min\{\mu_k g(Az_k, u) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q\}, \\ v_k = \arg\min\{\mu_k g(u_k, u) + \frac{1}{2}\|u - Az_k\|^2 : u \in Q\}, \\ t_k = P_C(z_k + \eta A^*(v_k - Az_k)), \\ C_{k+1} = \{x \in C_k : \|x - t_k\| \le \|x - z_k\| \le \|x - x_k\|\}, \\ x_{k+1} = P_{C_{k+1}}(x_0), \end{cases} \tag{4.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*, $\eta \in \left(0, \frac{1}{\|A\|^2}\right)$, and $\{\lambda_k\}$, $\{\mu_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \le \overline{\rho} < \min\left\{\frac{1}{2p_1}, \frac{1}{2p_2}\right\}$. They proved that the sequence $\{x_k\}$ generated by (4.1) converges strongly to a solution of the split equilibrium problems. Here, the algorithm (4.1) will be called SEM Algorithm.

This numerical experiment is considered under the problem setting and the control parameters as in Example 4.1, but the bifunction $\tilde{g}$ is given by

$$\tilde{g}(u, v) = \langle P_2 u + q_2^m (v + u), v - u\rangle, \quad \forall u, v \in \mathbb{R}^m,$$

where $P_2 = \begin{pmatrix} 0 & q_2 & q_2 & \cdots & q_2 \\ q_2 & 0 & q_2 & \cdots & q_2 \\ q_2 & q_2 & 0 & \cdots & q_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q_2 & q_2 & \cdot & \cdots & 0 \end{pmatrix}_{m\times m}$ is matrix with the positive real number q_2.

Thus, the bifunction g is pseudomonotone and it is not monotone. Moreover, the bifunction g satisfies Lipschitz-type continuous with constants $d_1 = d_2 = \frac{1}{2}\|P_2\|$. The following four cases of the parameter θ_k are considered:

Case 1. $\theta_k = 0.59 - \frac{1}{k+1}$.

Case 2. $\theta_k = 0.99 - \frac{1}{k+1}$.

Case 3. $\theta_k = -\left(0.59 - \frac{1}{k+1}\right)$.

Case 4. $\theta_k = -\left(0.99 - \frac{1}{k+1}\right)$.

The function *quadprog* in Matlab Optimization Toolbox was used to solve vectors y_k, z_k, u_k, and v_k. The positive real number q_2 is randomly chosen from the interval $(2, 2.5)$. Notice that the solution set Ω is nonempty because of $0 \in \Omega$. The starting point $x_0 = x_1 \in \mathbb{R}^n$ are randomly chosen from the interval $[-5, 5]$. The SIEM Algorithm was tested along with the SEM Algorithm (4.1) by using the stopping criteria $\|x_{k+1} - x_k\| < 10^{-4}$.

We randomly 10 starting points and presented results are in average, where $n = 5$ and $m = 10$.

Table 2. The numerical results for the split equilibrium problems in Example 4.2

	Average CPU times (sec)		Average iterations	
Cases	SIEM	SEM	SIEM	SEM
1	1.7344		54.5	
2	0.2375	8.4406	10.3	175.7
3	2.0344		64.8	
4	0.2891		12.9	

From Table 2, we may suggest that the parameter $\theta_k = 0.99 - \frac{1}{k+1}$ yields better both the CPU times and the number of iterations than other cases. Besides, the CPU times and the number of iterations of the SIEM Algorithm are better than those of the SEM Algorithm in all considered cases.

Example 4.3. For this numerical experiment, we consider under the following setting: the mappings and the control parameters as in Example 4.1, and the bifunctions as in Example 4.2. The following five cases of the parameter θ_k are considered:

Case 1. $\theta_k = 0$.

Case 2. $\theta_k = 0.59 - \frac{1}{k+1}$.

Case 3. $\theta_k = 0.99 - \frac{1}{k+1}$.

Case 4. $\theta_k = -\left(0.59 - \frac{1}{k+1}\right)$.

Case 5. $\theta_k = -\left(0.99 - \frac{1}{k+1}\right)$.

We use the function *quadprog* in Matlab Optimization Toolbox to solve vectors y_k, z_k, u_k, and v_k. Notice that the solution set Ω is nonempty because of $0 \in \Omega$. The starting point $x_0 = x_1 \in \mathbb{R}^n$ are randomly chosen from the interval $[-5, 5]$. The SIEM Algorithm was tested along with the NEM Algorithm (1.8) by using the stopping criteria $\|x_{k+1} - x_k\| < 10^{-4}$. We randomly 10 starting points and presented results are in average, where $n = 5$ and $m = 10$.

Table 3. The numerical results for the split equilibrium and fixed point problems in Example 4.3

	Average CPU times (sec)		Average iterations	
Cases	SIEM	NEM	SIEM	NEM
1	1.4031		44.1	
2	1.0813		37.2	
3	0.4031	0.4203	17.4	21.1
4	0.8047		28.5	
5	0.2688		11.8	

Table 3 shows that the parameter $\theta_k = -\left(0.99 - \frac{1}{k+1}\right)$ yields better both the CPU times and the number of iterations than other cases. We notice that, in the cases of the parameter $\theta_k = 0.99 - \frac{1}{k+1}$ and $\theta_k = -(0.99 - \frac{1}{k+1})$, the CPU times and the number of iterations of the SIEM Algorithm are better than those of the NEM Algorithm. However, we would like to remind that the SIEM Algorithm is constructed without prior knowledge of the Lipschitz-type constants of the bifunctions. Meanwhile, the Lipschitz-type constants of the bifunctions are approximated for the input parameters of the NEM Algorithm.

5. Conclusion

We present an algorithm for finding a solution of the split equilibrium and fixed point problems for nonexpansive mappings and pseudomonotone bifunctions which satisfy Lipschitz-type continuous in real Hilbert spaces. We consider both inertial and extragradient methods together with the shrinking method for establishing sequence without the prior knowledge of the Lipschitz-type constants which is strongly convergent to a solution of the split equilibrium and fixed point problems. Some experiments are reported to illustrate the numerical behavior of the introduced algorithm in comparison with other algorithms. These numerical results are also confirmed that the algorithm with inertial effects seems to work better than those without inertial effects.

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