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Admitting a Semihyperring with Zero of Certain Linear Transformation Subsemigroups of $L_R(V, W)$ (Part II)

S. Chaopraknoi, S. Hobuntud and S. Pianskool

Abstract : A *semihyperring with zero* is a triple $(A, +, \cdot)$ such that $(A, +)$ is a semihypergroup, $(A, \cdot)$ is a semigroup, $\cdot$ is distributive over $+$ and there exists $0 \in A$ (called a *zero*) such that $x+0 = 0+x = \{x\}$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in A$. For a semigroup S, let S^0 be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup S with a zero adjoined. We say that a semigroup S is said to *admit a semihyperring with zero* if there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a semihyperring with zero 0 where $\cdot$ is the operation on S^0 and 0 is the zero of S^0. Let V be a vector space over a division ring R, W a subspace of V and $L_R(V, W)$ the semigroup under composition of all linear transformations from V into W. For each $\alpha \in L_R(V, W)$, let $F(\alpha)$ consist of all elements in V fixed by α. Denote by $OM_R(V, W)$, $OE_R(V, W)$, $AI_R(\underline{V}, W)$ and $AI_R(V, \underline{W})$ the set of all linear transformations α in $L_R(V, W)$ where $\dim_R \operatorname{Ker} \alpha$ are infinite, the set of all linear transformations α in $L_R(V, W)$ where $\dim_R(W/\operatorname{Im}\alpha)$ are infinite, the set of all linear transformations α in $L_R(V, W)$ where $\dim_R(V/F(\alpha))$ are finite and the set of all linear transformations α in $L_R(V, W)$ where $\dim_R(W/F(\alpha))$ are finite, respectively. Moreover, let H and S be subsemigroups of $AI_R(\underline{V}, W)$ and $AI_R(V, \underline{W})$, respectively.

We show that $OM_R(V, W) \cup H$, $OE_R(V, W) \cup H$, $OM_R(V, W) \cup S$ and $OE_R(V, W) \cup S$ are semigroups. Furthermore, we determine whether or when they admit the structure of a semihyperring with zero.

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1 Introduction and Preliminaries

A *hyperoperation* on a nonempty set H is a map $\circ : H \times H \to P^*(H)$ where $P(H)$ is the power set of H and $P^*(H) = P(H) \setminus \{\emptyset\}$. For A, $B \subseteq H$, let $A \circ B$

be the union of all subsets $a \circ b$ of H where $a \in A$ and $b \in B$. A *semihypergroup* is a system $(H, \circ)$ where H is a nonemty set, $\circ$ is a hyperoperation on H and $(x \circ y) \circ z = x \circ (y \circ z)$ for all x, y, $z \in H$. A *hypergroup* is a semihypergroup $(H, \circ)$ such that $H \circ x = x \circ H = H$ for all $x \in H$. For x, y in a hypergroup $(H, \circ)$, x is called an *inverse* of y if there exists an identity e of $(H, \circ)$ such that $e \in (x \circ y) \cap (y \circ x)$. A hypergroup H is called *regular* if every element of H has an inverse in H. A regular hypergroup $(H, \circ)$ is said to be *reversible* if for $x, y, z \in H, x \in y \circ z$ implies $z \in u \circ x$ and $y \in x \circ v$ for some inverse u of y and some inverse v of z. A *canonical hypergroup* is a hypergroup $(H, \circ)$ such that

(i) $(H, \circ)$ is commutative,

(ii) $(H, \circ)$ has a scalar identity,

(iii) every element of H has a unique inverse in H and

(iv) $(H, \circ)$ is reversible.

By a *semihyperring* we mean a triple $(A, +, \cdot)$ such that

(i) $(A, +)$ is a semihypergroup,

(ii) $(A, \cdot)$ is a semigroup and

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all x, y, $z \in A$.

An element 0 of a semihyperring $(A, +, \cdot)$ is called a *zero of* $(A, +, \cdot)$ if $x + 0 = 0 + x = x (= \{x\}\, 0)$ and $x \circ 0 = 0 \circ x = 0$ for all $x \in A$. By the definition, every semiring with zero is a semihyperring with zero. A *Krasner hyperring* is a system $(A, +, \cdot)$ where

(i) $(A, +)$ is a canonical hypergroup,

(ii) $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and

(iii) the operation $\cdot$ is distributive over the hyperoperation $+$.

Then every (Krasner) hyperring is a semihyperring with zero. Consequently, semihyperrings with zero are a generalization of hyperrings. In [2], if A is a set whose cardinality is at least 3 and 0 is an element of A, then $(A, +, \cdot)$ with

$$\begin{aligned} x + 0 = 0 + x &= \{x\} && \text{for all } x \in A, \\ x + y &= A && \text{for all } x, y \in A \setminus \{0\}, \\ x \cdot y &= 0 && \text{for all } x, y \in A. \end{aligned}$$

is clearly a semihyperring with zero 0 but not a hyperring.

A semigroup S is said to *admit a ring[hyperring] structure* if $(S^0, +, \cdot)$ is a ring[hyperring] for some operation[hyperoperation] $+$ on S^0 where $\cdot$ is the operation on S^0. Similarly, S is said to *admit a semihyperring with zero* if there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a semihyperring with zero. Semigroups admitting ring structures have long been studied. For examples, see [3] and [6]. There were some studies of semigroups admitting hyperring structures. These can be seen from [4] and [5].

Throughout this paper, let V be a vector space over a division ring R, W a subspace of V and $L_R(V,W)$ the semigroup under composition of all linear transformations from V into W. Then $L_R(V,W)$ admits a ring structure. For $\alpha \in L_R(V,W)$, let $F(\alpha)$ consist of all elements in V fixed by α. Then $F(\alpha)$ is a subspace of W so that it is also a subspace of V for all $\alpha \in L_R(V,W)$. Moreover, let

$$\begin{aligned} OM_R(V,W) &= \{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite}\}\,, \\ OE_R(V,W) &= \{\alpha \in L_R(V,W) \mid \dim_R(W/\operatorname{Im}\alpha) \text{ is infinite}\}\,, \\ AI_R(\underline{V},W) &= \{\alpha \in L_R(V,W) \mid \dim_R(V/F(\alpha)) \text{ is finite}\}\,, \\ AI_R(V,\underline{W}) &= \{\alpha \in L_R(V,W) \mid \dim_R(W/F(\alpha)) \text{ is finite}\}\,. \end{aligned}$$

It has been shown in [7] that $OM_R(V,W)$ and $OE_R(V,W)$ are subsemigroups of $L_R(V,W)$. This paper, first, shows that $OM_R(V,W) \cup H$, $OE_R(V,W) \cup H$, $OM_R(V,W) \cup S$ and $OE_R(V,W) \cup S$ are semigroups where H and S are subsemigroup of $AI_R(\underline{V},W)$ and $AI_R(V,\underline{W})$, respectively. The other purpose of this paper is showing that whether or when $OM_R(V,W)\cup H$, $OE_R(V,W)\cup H$, $OM_R(V,W)\cup S$ and $OE_R(V,W) \cup S$ admit the structure of a semihyperring with zero.

2 Main Results

In this paper, we assume that $\dim_R V$ is infinite because if $\dim_R V$ is finite, then $OM_R(V,W)$ and $OE_R(V,W)$ are empty sets. In order to study $OE_R(V,W)$, we must assume further that $\dim_R W$ is infinite otherwise $OE_R(V,W)$ is an empty set.

2.1 Subsemigroups of $L_R(V,W)$

Our aim of this subsection is to show that $OM_R(V,W) \cup H$, $OE_R(V,W) \cup H$, $OM_R(V,W) \cup S$ and $OE_R(V,W) \cup S$ are semigroups. In order to do so, we prove that all of them are subsemigroups of $L_R(V,W)$.

Proposition 2.1. *([7]) The following statements hold.*

(i) $OM_R(V,W)$ *is a right ideal of* $L_R(V,W)$.

(ii) $OE_R(V,W)$ *is a left ideal of* $L_R(V,W)$.

Note 2.1. $AI_R(\underline{V},W)$ *is a subset of* $AI_R(V,\underline{W})$ *because* $W/F(\alpha)$ *is a subspace of* $V/F(\alpha)$ *for any* $\alpha \in L_R(V,W)$.

Proposition 2.2. $AI_R(\underline{V},W)$ *and* $AI_R(V,\underline{W})$ *are subsemigroups of* $L_R(V,W)$.

Proof. Let $\alpha, \beta \in AI_R(\underline{V}, W)[AI_R(V, \underline{W})]$. Then $\dim_R(V/F(\alpha))[\dim_R(W/F(\alpha))]$ and $\dim_R(V/F(\beta))[\dim_R(W/F(\beta))]$ are finite. We claim that $\dim_R(V/F(\alpha\beta))$ $[\dim_R(W/F(\alpha\beta))]$ is finite. Since $F(\alpha) \cap F(\beta) \subseteq F(\alpha\beta)$, it suffices to show that $\dim_R(V/F(\alpha) \cap F(\beta))[\dim_R(W/(F(\alpha) \cap F(\beta))]$ is finite. Let B_1 be a basis of $F(\alpha) \cap F(\beta)$ and $B_2 \subseteq F(\alpha) \setminus B_1$ and $B_3 \subseteq F(\beta) \setminus B_1$ be such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $F(\alpha)$ and $F(\beta)$, respectively. We will show that $(B_1 \cup B_2) \cup B_3$ is linearly independent over R. Let $u_1, u_2, \ldots, u_k \in B_1 \cup B_2$, $v_1, v_2, \ldots, v_l \in B_3$ be distinct and $\sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{l} b_j v_j = 0$ where $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in R$. Then $\sum_{i=1}^{k} a_i u_i = -\sum_{j=1}^{l} b_j v_j \in F(\alpha) \cap F(\beta) = \langle B_1 \rangle$. Hence $\sum_{j=1}^{l} b_j v_j \in \langle B_1 \rangle \cap \langle B_3 \rangle = \{0\}$. Since B_3 is linearly independent, $b_j = 0$ for all $j = 1, 2, \ldots, l$, so that $\sum_{i=1}^{k} a_i u_i = 0$. This implies that $a_i = 0$ for all $i = 1, 2, \ldots, k$. Hence $(B_1 \cup B_2) \cup B_3$ is linearly independent over R. Let $B_4 \subseteq V \setminus (B_1 \cup B_2) \cup B_3 [W \setminus (B_1 \cup B_2) \cup B_3]$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of $V[W]$. Hence $\{v + F(\alpha) \mid v \in B_3 \cup B_4\}$ is a basis of $V/F(\alpha)[W/F(\alpha)]$ and $\{v + F(\alpha) \mid v \in B_2 \cup B_4\}$ is a basis of $V/F(\beta)[W/F(\beta)]$. But $\dim_R(V/F(\alpha))[\dim_R(W/F(\alpha))]$ and $\dim_R(V/F(\beta))[\dim_R(W/F(\beta))]$ are finite, so $B_3 \cup B_4$ and $B_2 \cup B_4$ are finite. Therefore $B_2 \cup B_3 \cup B_4$ is finite. Hence $\{v + (F(\alpha) \cap F(\beta))\}$ is a basis of $V/(F(\alpha) \cap F(\beta))[W/(F(\alpha) \cap F(\beta))]$. This implies that $\dim_R(V/F(\alpha) \cap F(\beta))[\dim_R(W/(F(\alpha) \cap F(\beta))]$ is finite. □

Lemma 2.3. *$AI_R(V, \underline{W}) OM_R(V, W) \subseteq OM_R(V, W)$.*

Proof. Let $\alpha \in AI_R(V, \underline{W})$ and $\beta \in OM_R(V, W)$. Let B_1 be a basis of $F(\alpha) \cap \operatorname{Ker} \beta$, $B_2 \subseteq \operatorname{Ker} \beta \setminus B_1$ such that $B_1 \cup B_2$ is a basis of $\operatorname{Ker} \beta \cap W$, $B_3 \subseteq \operatorname{Ker} \beta \setminus B_1 \cup B_2$ such that $B_1 \cup B_2 \cup B_3$ is a basis of $\operatorname{Ker} \beta$. Since $\beta \in OM_R(V, W)$, $B_1 \cup B_2 \cup B_3$ is infinite. Let $v_1, v_2, \ldots, v_n$ be distinct elements of B_2 and let $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i (v_i + F(\alpha)) = F(\alpha)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\alpha) \cap \operatorname{Ker} \beta$. But B_1 is a basis of $F(\alpha) \cap \operatorname{Ker} \beta$ and $B_1 \cup B_2$ is linearly independent over R, so $a_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. This shows that $\{v + F(\alpha) | v \in B_2\}$ is a linearly independent subset of the quotient space $W/F(\alpha)$ and $u + F(\alpha) \neq w + F(\alpha)$ for all distinct $u, w \in B_2$. Since $\dim_R W/F(\alpha) < \infty$, the set $\{v + F(\alpha) | v \in B_2\}$ is finite. But $\big|\{v + F(\alpha) | v \in B_2\}\big| = |B_2|$ so that B_2 is finite. Let $B_4 \subseteq W \setminus B_1 \cup B_2$ be such that $B_1 \cup B_2 \cup B_4$ is a basis of W and let $C = B_1 \cup B_2 \cup B_4$. Moreover, let $B_5 \subseteq V \setminus C \cup B_3$ be such that $C \cup B_3 \cup B_5$ is a basis of V and let $B = C \cup B_3 \cup B_5$.

Case 1. $B \setminus C$ is finite. Since $B_3 \subseteq B \setminus C$, $|B_3| \leq |B \setminus C|$. Thus B_3 is finite. Hence $B_2 \cup B_3$ is finite. This implies that B_1 is infinite. Since $B_1 \subseteq F(\alpha) \cap \operatorname{Ker} \beta$, we have $B_1 \alpha\beta = B_1 \beta = \{0\}$, so $B_1 \subseteq \operatorname{Ker} \alpha\beta$. Hence $\dim_R \operatorname{Ker} \alpha\beta$ is infinite. Thus $\alpha\beta \in OM_R(V, W)$.

Case 2. $B \setminus C$ is infinite. Claim that $\dim_R \operatorname{Ker} \alpha$ is infinite. Suppose that $\dim_R \operatorname{Ker} \alpha$ is finite. Let $E = \{v'_1, v'_2, \ldots, v'_k\}$ be a basis of $\operatorname{Ker} \alpha$ such that $E \subseteq B$.

Clearly, $B\backslash(C\cup E)$ is infinite. Next, we will show that there is $w\in B\backslash(C\cup E)$ such that $w\alpha=v\alpha$ for some $v\in V\setminus\langle E\cup\{w\}\rangle$. Suppose that for each $w\in B\setminus(C\cup E)$,

$$w\alpha\neq v\alpha \quad \text{for all } v\in V\setminus\langle E\cup\{w\}\rangle. \tag{1}$$

Hence

$$w_1\alpha\neq w_2\alpha \quad \text{for every } w_1\neq w_2\in B\backslash(C\cup E). \tag{2}$$

Hence $\{w\alpha\,|\,w\in B\setminus(C\cup E)\}$ consists of distinct elements. Since $B\setminus(C\cup E)$ is infinite, the set $\{w\alpha\,|\,w\in B\setminus(C\cup E)\}$ must be infinite. We will show that $\{w\alpha\,|\,w\in B\setminus(C\cup E)\}$ is linearly independent set. Assume that

$$a_1w_1\alpha+a_2w_2\alpha+\cdots+a_nw_n\alpha=0$$

where $a_1,a_2,\ldots,a_n\in R$ and $w_1,w_2,\ldots,w_n\in B\backslash(C\cup E)$. Hence

$$(a_1w_1+a_2w_2+\cdots+a_nw_n)\alpha=0.$$

Therefore $a_1w_1+a_2w_2+\cdots+a_nw_n\in\operatorname{Ker}\alpha$. Hence

$$a_1w_1+a_2w_2+\cdots+a_nw_n\in\langle E\rangle\cap\langle B\setminus(C\cup E)\rangle=\{0\}.$$

Consequently, $a_1w_1+a_2w_2+\cdots+a_nw_n=0$ so that $a_1=a_2=\cdots=a_n=0$. Hence $\{w\alpha\,|\,w\in B\setminus(C\cup E)\}$ is linearly independent. Let $w^*\in B\setminus(C\cup E)$. Suppose that $(w^*\alpha)\alpha=w^*\alpha$, so $w^*\alpha\in\langle E\cup\{w^*\}\rangle$ because $w^*\alpha\neq w^*$. Then there are $b,a_1,a_2,\ldots,a_k\in R$ such that $w^*\alpha=bw^*+\sum\limits_{i=1}^{k}a_iv_i'$. Thus

$$bw^*=w^*\alpha-\sum_{i=1}^{k}a_iv_i'\in\langle C\cup E\rangle.$$

Hence $bw^*\in\langle B\setminus(C\cup E)\rangle\cap\langle C\cup E\rangle=\{0\}$, we have $bw^*=0$. Thus

$$w^*\alpha=bw^*+\sum_{i=1}^{k}a_iv_i'=\sum_{i=1}^{k}a_iv_i'\in\operatorname{Ker}\alpha,$$

so $0=(w^*\alpha)\alpha=w^*\alpha$. Therefore $w^*\in\operatorname{Ker}\alpha$ which leads to a contradiction. Thus $(w^*\alpha)\alpha\neq w^*\alpha$. Hence $w\alpha\notin F(\alpha)$ for all $w\in B\setminus(C\cup E)$. Next, we will show that $\{w\alpha+F(\alpha)\,|\,w\in B\setminus(C\cup E)\}$ is a linearly independent subset of $W/F(\alpha)$. Assume that

$$\sum_{i=1}^{n}a_i(w_i\alpha+F(\alpha))=F(\alpha)$$

where $a_1,a_2,\ldots,a_n\in R$ and $w_1,w_2,\ldots,w_n\in B\backslash(C\cup E)$. Hence $\sum\limits_{i=1}^{n}a_iw_i\alpha\in F(\alpha)$. Therefore

$$\left(\sum_{i=1}^{n} a_i w_i \alpha\right)\alpha = \sum_{i=1}^{n} a_i w_i \alpha \in F(\alpha).$$

Thus $\left(\sum_{i=1}^{n} a_i w_i \alpha - \sum_{i=1}^{n} a_i w_i\right)\alpha = 0$. Hence $\sum_{i=1}^{n} a_i w_i \alpha - \sum_{i=1}^{n} a_i w_i \in \operatorname{Ker}\alpha$. It follows that

$$\sum_{i=1}^{n} a_i w_i \alpha - \sum_{i=1}^{n} a_i w_i = \sum_{j=1}^{k} b_j v_j'.$$

Thus

$$\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} a_i w_i \alpha - \sum_{j=1}^{k} b_j v_j' \in \langle C \cup E\rangle.$$

This implies that $\sum_{i=1}^{n} a_i w_i \in \langle B \setminus (C \cup E)\rangle \cap \langle C \cup E\rangle = \{0\}$. Since $\{w\alpha \mid w \in B \setminus (C \cup E)\}$ is linearly independent, $a_1 = a_2 = \cdots = a_n = 0$. Hence $\{w\alpha + F(\alpha) \mid w \in B \setminus (C \cup E)\}$ is a linearly independent subset of $W/F(\alpha)$.

We will show that for all v, $w \in B \setminus (C \cup E)$, if $v\alpha \neq w\alpha$, then

$$v\alpha + F(\alpha) \neq w\alpha + F(\alpha).$$

Let v, $w \in B \setminus (C \cup E)$. Assume that $v\alpha \neq w\alpha$. Suppose that $v\alpha + F(\alpha) = w\alpha + F(\alpha)$. We see that $v\alpha - w\alpha \in F(\alpha)$. Hence $(v\alpha - w\alpha)\alpha = v\alpha - w\alpha$. Thus $(v\alpha - w\alpha)\alpha + w\alpha = v\alpha$. Therefore

$$(v\alpha - w\alpha + w)\alpha = v\alpha. \tag{3}$$

If $v\alpha - w\alpha + w \in \langle E \cup \{v\}\rangle$, then there are $b, a_1, a_2, \ldots, a_k \in R$ such that $v\alpha - w\alpha + w = bv + \sum_{i=1}^{k} a_i v_i'$. Clearly, $bv - w = v\alpha - w\alpha - \sum_{i=1}^{k} a_i v_i' \in \langle C \cup E\rangle$. Therefore $bv - w \in \langle B \setminus (C \cup E)\rangle \cap \langle C \cup E\rangle = \{0\}$. This leads to a contradiction because of $bv = w$. Hence $v\alpha - w\alpha + w \notin \langle E \cup \{v\}\rangle$. It follows from (1) that $(v\alpha - w\alpha + w)\alpha \neq v\alpha$ contradicting (3). Thus $|\{w\alpha + F(\alpha) \mid w \in B \setminus (C \cup E)\}| = |\{w\alpha \mid w \in B \setminus (C \cup E)\}|$. Since $\{w\alpha + F(\alpha) \mid w \in B \setminus (C \cup E)\}$ is a linearly independent subset of $W/F(\alpha)$ and $\{w\alpha \mid w \in B \setminus (C \cup E)\}$ is infinite, $\dim_R W/F(\alpha)$ is infinite. A contradiction occurs. Thus there is a $w \in B \setminus (C \cup E)$ such that $w\alpha = v\alpha$ for some $v \in V \setminus \langle E \cup \{w\}\rangle$. Since $v \in V$, there are $v_1, v_2, \ldots, v_m \in B$ and $b_1, b_2, \ldots, b_m \in R$ such that $v = b_1v_1 + b_2v_2 + \cdots + b_mv_m$. It is clear that there is $v_i \notin E$ for some $i \in \{1, 2, \ldots, m\}$ because $v \notin \operatorname{Ker}\alpha$ and if $w = v_j$ for some $j \in \{1, 2, \ldots, m\}$, there is $v_k \notin E \cup \{w\}$ for some $k \in \{1, 2, \ldots, m\}$. Without loss of generality, $v = b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m$ where

$v_{l+1}, v_{l+2}, \ldots, v_m \in E$. Let $w' = b_1v_1 + b_2v_2 + \cdots + b_lv_l$. Note that

$$\begin{aligned} w\alpha &= v\alpha \\ &= (b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m)\alpha \\ &= (b_1v_1 + b_2v_2 + \cdots + b_lv_l)\alpha \\ &= w'\alpha. \end{aligned}$$

Hence $w\alpha = w'\alpha = (b_1v_1+b_2v_2+\cdots+b_lv_l)\alpha$ so $(w - b_1v_1 - b_2v_2 - \cdots - b_lv_l)\alpha = 0$. It follows that $w - b_1v_1 - b_2v_2 - \cdots - b_lv_l \in \operatorname{Ker}\alpha$. Thus

$$w - b_1v_1 - b_2v_2 - \cdots - b_lv_l = c_1v_1' + c_2v_2' + \cdots + c_kv_k'.$$

Therefore

$$w = b_1v_1 + b_2v_2 + \cdots + b_lv_l + c_1v_1' + c_2v_2' + \cdots + c_kv_k'.$$

Subcase 2.1 $w \neq v_j$ for all $j \in \{1, 2, \ldots, l\}$. Hence w can be written in a linear combination of $B \backslash \{w\}$ which is a contradiction.

Subcase 2.2 $w = v_j$ for some $j \in \{1, 2, \ldots, l\}$. Without loss of generality, assume that $w = v_1$. Hence

$$w = b_1v_1 + b_2v_2 + \cdots + b_lv_l + c_1v_1' + c_2v_2' + \cdots + c_kv_k'.$$

Thus $0 = (b_1 - 1)w + b_2v_2 + \cdots + b_lv_l + c_1v_1' + c_2v_2' + \cdots + c_kv_k'$. This implies that

$$b_1 - 1 = b_2 = \cdots = b_l = c_1 = \cdots = c_k = 0$$

We obtain that $b_1 = 1$, $w' = b_1v_1 = w$. Thus

$$\begin{aligned} v &= b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &= w' + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &= w + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &\in \langle C \cup E \rangle, \end{aligned}$$

again, a contradiction occurs. Hence $\operatorname{Ker}\alpha$ is infinite. Since $\operatorname{Ker}\alpha \subseteq \operatorname{Ker}\alpha\beta$, $\operatorname{Ker}\alpha\beta$ is infinite. Therefore $\alpha\beta \in OM_R(V, W)$. □

Proposition 2.4. *If S is a subsemigroup of $AI_R(V, \underline{W})$, then $OM_R(V, W) \cup S$ is a subsemigroup of $L_R(V, W)$.*

Proof. This follows from the fact that $OM_R(V, W)$ and S are subsemigroups of $L_R(V, W)$, Proposition 2.1(i) and Lemma 2.3. □

Lemma 2.5. *$AI_R(\underline{V}, W)OM_R(V, W) \subseteq OM_R(V, W)$.*

Proof. The result follows the fact that $AI_R(\underline{V}, W) \subseteq AI_R(V, \underline{W})$. □

Proposition 2.6. *If H is subsemigroup of $AI_R(\underline{V}, W)$, then $OM_R(V, W) \cup H$ is a subsemigroup of $L_R(V, W)$.*

Proof. Proposition 2.1(i), Lemma 2.5 and the truth that both $OM_R(V,W)$ and H are susemigroups of $L_R(V,W)$ provide this result. $\square$

Lemma 2.7. *For every* $\alpha \in AI_R(V,\underline{W})$, $\dim_R \operatorname{Ker} \alpha|_W < \infty$.

Proof. Let $\alpha \in AI_R(V,\underline{W})$ and B a basis of $\operatorname{Ker} \alpha|_W$. Moreover, let $v_1, v_2, \ldots, v_n \in B$ be distinct and $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\alpha)) = F(\alpha)$. Then $\sum_{i=1}^{n} a_i v_i = F(\alpha)$ which implies that $\left(\sum_{i=1}^{n} a_i v_i\right)\alpha = \sum_{i=1}^{n} a_i v_i$. But $v_1, v_2, \ldots, v_n \in \operatorname{Ker} \alpha|_W$ so that $\left(\sum_{i=1}^{n} a_i v_i\right)\alpha = 0$. Thus $\sum_{i=1}^{n} a_i v_i = 0$. Since $v_1, v_2, \ldots, v_n$ are linearly independent over R, it follows that $a_i = 0$ for every $i \in \{1, 2, \ldots, n\}$. This proves that $\{v + F(\alpha) | v \in B\}$ is a linearly independent subset of $W/F(\alpha)$ and $v + F(\alpha) \neq w + F(\alpha)$ for all distinct $v, w \in B$. Since $\dim_R(W/F(\alpha))$ is finite, $\{v + F(\alpha) | v \in B\}$ is finite. Since $|\{v + F(\alpha) | v \in B\}| = |B|$, we have $\dim_R \operatorname{Ker} \alpha|_W < \infty$. $\square$

Proposition 2.8. $OE_R(V,W)AI_R(V,\underline{W}) \subseteq OE_R(V,W)$.

Proof. Let $\alpha \in OE_R(V,W)$ and $\beta \in AI_R(V,\underline{W})$. Define $\varphi : W/\operatorname{Im}\alpha \to \operatorname{Im}\beta|_W/\operatorname{Im}\alpha\beta$ by

$$(w + \operatorname{Im}\alpha)\varphi = w\beta + \operatorname{Im}\alpha\beta \text{ for all } w \in W.$$

Then φ is an epimorphism. Hence

$$(W/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi \cong \operatorname{Im}\beta|_W/\operatorname{Im}\alpha\beta.$$

We claim that $\dim_R(W/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi$ is infinite. To show this, let $C \subseteq W$ be such that $\{v + \operatorname{Im}\alpha | v \in C\}$ is a basis of $\operatorname{Ker}\varphi$ and $v + \operatorname{Im}\alpha \neq w + \operatorname{Im}\alpha$ for all distinct $v, w \in C$. For every $v \in C$, $v\beta + \operatorname{Im}\alpha\beta = (v + \operatorname{Im}\alpha)\varphi = \operatorname{Im}\alpha\beta$. Thus $v\beta \in \operatorname{Im}\alpha\beta = (\operatorname{Im}\alpha)\beta$ for all $v \in C$. As a result, there exists an element $w_v \in \operatorname{Im}\alpha$ such that $v\beta = w_v\beta$. Consequently,$\{v - w_v | v \in B\} \subseteq \operatorname{Ker}\beta|_W$. If $v_1, v_2, \ldots, v_n \in B$ are all distinct and $\sum_{i=1}^{n} a_i(v_i - w_{v_i}) = 0$ where $a_1, a_2, \ldots, a_n \in R$, then $\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i w_{v_i} \in \operatorname{Im}\alpha$, and hence $\sum_{i=1}^{n} a_i(v_i + \operatorname{Im}\alpha) = \operatorname{Im}\alpha$ in $W/\operatorname{Im}\alpha$. Thus $a_i = 0$ for every $i \in \{1, 2, \ldots, n\}$. This shows that $\{v - w_v | v \in B\}$ is linearly independent over R and $v - w_v \neq u - w_u$ for all distinct $u, v \in B$. It follows that $|B| = |\{v + \operatorname{Im}\alpha | v \in C\}| = |\{v - w_v | v \in B\}| \leq \dim_R \operatorname{Ker}\beta|_W$. Since $\dim_R \operatorname{Ker}\beta|_W < \infty$, it follows from Lemma 2.7 that B is finite. Thus $\dim_R \operatorname{Ker}\varphi < \infty$. However, $\dim_R(W/\operatorname{Im}\alpha)$ is infinite and $\dim_R(W/\operatorname{Im}\alpha) = \dim_R((W/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi) + \dim_R \operatorname{Ker}\varphi$, so we can condlude that $\dim_R((W/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi)$ is infinite. Then $\dim_R \operatorname{Im}\beta/\operatorname{Im}\alpha\beta$ is infinite. Consequently, $\dim_R(W/\operatorname{Im}\alpha\beta)$ is infinite, so $\alpha\beta \in OE_R(V,W)$. $\square$

Proposition 2.9. *If S is subsemigroup of $AI_R(V, \underline{W})$, then $OE_R(V, W) \cup S$ is a subsemigroup of $L_R(V, W)$.*

Proof. This result is obtained by appliying the fact that $OE_R(V, W)$ and S are subsemigroups of $L_R(V, W)$, Proposition 2.1(ii) and Proposition 2.8. □

In the similar manner as Lemma 2.5 and Proposition 2.6, we overcome the two following facts.

Lemma 2.10. $OE_R(V, W)AI_R(\underline{V}, W) \subseteq OE_R(V, W)$.

Proposition 2.11. *If H is subsemigroup of $AI_R(\underline{V}, W)$, then $OE_R(V, W) \cup H$ is a subsemigroup of $L_R(V, W)$.*

2.2 Subsemigroups admitting the structure of semihyperring with zero

We know from the previous section that all $OM_R(V, W) \cup S$, $OE_R(V, W) \cup S$, $OM_R(V, W) \cup H$ and $OE_R(V, W) \cup H$ are semigroups. Thus, it is reasonable to consider whether they admit the structure of a semihyperrings with zero. Fortunately, we can characterize when $OM_R(V, W) \cup S$ and $OM_R(V, W) \cup H$ admit the structure of a semihyperrings with zero. However, the semigroups $OE_R(V, W) \cup S$ and $OE_R(V, W) \cup H$ are found that they cannot admit the structure of a semihyperrings with zero.

Theorem 2.12. *$OM_R(V, W) \cup S$ does not admit the structure of a semihyperring with zero if and only if $\dim_R V = \dim_R W$.*

Proof. Let S be a subsemigroup of $AI_R(V, \underline{W})$. First, we assume that $\dim_R V \neq \dim_R W$. Since $OM_R(V, W) \subseteq OM_R(V, W) \cup S \subseteq L_R(V, W)$, it follows that $L_R(V, W) = OM_R(V, W) \cup S$. Thus $OM_R(V, W) \cup S$ admits the structure of a ring with zero. Therefore $OM_R(V, W) \cup S$ admits the structure of a semihyperring with zero.

On the other hand, we assume that $\dim_R V = dim_R W$. Let B be a basis of V and C a basis of W such that $C \subseteq B$.

Case 1.$B = C$. We see that $OM_R(V, W) = OM_R(V)$ and $AI_R(V, \underline{W}) = AI_R(V)$. By [1], $OM_R(V, W) \cup S$ does not admit the structure of a semihyperring with zero.

Case 2.$B \neq C$. Suppose that there exist a hyperoperation $\oplus$ such that the structure $(OM_R(V, W) \cup S, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $OM_R(V, W) \cup S$. Then $B \setminus C \neq \emptyset$ since $B \neq C$. Let $D = B \setminus C$ and D_1, D_2 be subsets of D such that $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = D$. Since $|B| = |C|$, C is infinite and there are subsets C_1, C_2 of C such that $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2 = C$ and $|C_1| = |C_2| = |C| = |B|$. Since $C_2 \subseteq C_1 \cup D_1 \subseteq B$, $|C_2| = |C_1 \cup D_1|$, similarly $|C_1| = |C_2 \cup D_2|$ and clearly that $B = D_1 \cup D_2 \cup C_1 \cup C_2$. Since $|C_1 \cup D_1| = |C_2|$

and $|C_2 \cup D_2| = |C_1|$, there are bijections $\varphi : C_1 \cup D_1 \to C_2$ and $\gamma : C_2 \cup D_2 \to C_1$, respectively. Define $\alpha,\ \beta \in L_R(V, W)$ by

$$\alpha = \begin{pmatrix} C_2 \cup D_2 & v \\ 0 & v\varphi \end{pmatrix}_{v \in C_1 \cup D_1} \qquad \beta = \begin{pmatrix} C_1 \cup D_1 & v \\ 0 & v\gamma \end{pmatrix}_{v \in C_2 \cup D_2}$$

Hence $\operatorname{Ker}\alpha = \langle C_2 \cup D_2\rangle$ and $\operatorname{Ker}\beta = \langle C_1 \cup D_1\rangle$. Thus $\alpha,\ \beta \in OM_R(V, W) \subseteq OM_R(V, W) \cup H$. Clearly, $\alpha^2 = \beta^2 = 0$. Hence

$$\begin{aligned} \alpha(\alpha \oplus \beta) = \alpha^2 \oplus \alpha\beta = 0 \oplus \alpha\beta = \{\alpha\beta\} = \alpha\beta \oplus 0 = \alpha\beta \oplus \beta^2 = (\alpha \oplus \beta)\beta \\ \beta(\alpha \oplus \beta) = \beta\alpha \oplus \beta^2 = \beta\alpha \oplus 0 = \{\beta\alpha\} = 0 \oplus \beta\alpha = \alpha^2 \oplus \beta\alpha = (\alpha \oplus \beta)\alpha \end{aligned} \tag{1}$$

Let $\lambda \in \alpha \oplus \beta$. It follows from (1) that $\alpha\lambda = \alpha\beta = \lambda\beta$ and $\beta\lambda = \beta\alpha = \lambda\alpha$. For $v \in C_1 \cup D_1$, $v\lambda \in \langle C\rangle$ so there are distinct $w_1, w_2, \ldots, w_n \in C_1$ and $w'_1, w'_2, \ldots, w'_m \in C_2$ such that

$$v\lambda = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m$$

where $a_i, b_j \in R$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Note that

$$\begin{aligned} 0 = 0\alpha = (v\beta)\alpha = v(\beta\alpha) & \\ &= v(\lambda\alpha) \\ &= (v\lambda)\alpha \\ &= (a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m)\alpha \\ &= \sum_{i=1}^{n} a_i(w_i\alpha) + \sum_{j=1}^{m} b_j(w'_j\alpha) \\ &= \sum_{i=1}^{n} a_i(w_i\varphi) \end{aligned}$$

Since φ is one to one, $w_i\varphi$ are all distinct in C_2. Hence $a_i = 0$ for all i. Thus $v\lambda \in \langle C_2\rangle$. Consider $v\lambda\beta = v\alpha\beta = (v\alpha)\beta = (v\varphi)\beta$. Since $\beta|_{C_2}$ is one to one, $\beta|_{\langle C_2\rangle}$ is also one to one. Thus $v\lambda = v\varphi$ so that $\lambda|_{C_1\cup D_1} = \varphi$. Similarly, for $v \in C_2 \cup D_2$, $\lambda|_{C_2\cup D_2} = \gamma$. Hence

$$\lambda = \begin{pmatrix} v & w \\ v\varphi & w\gamma \end{pmatrix}_{v \in C_1\cup D_1, w \in C_2 \cup D_2}$$

Thus λ is a one to one linear transformation from V onto W and then $\dim_R \operatorname{Ker}\lambda = 0 < \infty$. Thus $\lambda \notin OM_R(V, W)$.

Next, we claim that $\dim_R(W/F(\lambda))$ is infinite. Let $v_1, v_2, \ldots, v_n \in C_1$ be all distinct and $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\lambda)) = F(\lambda)$. Then $\sum_{i=1}^{n} a_iv_i \in F(\lambda)$, so $\left(\sum_{i=1}^{n} a_iv_i\right)\lambda = \sum_{i=1}^{n} a_iv_i$. However, $\left(\sum_{i=1}^{n} a_iv_i\right)\lambda = \sum_{i=1}^{n} a_i(v_i\lambda) \in$

$\langle C_2\rangle$. Hence $\sum_{i=1}^{n} a_i v_i \in \langle C_1 \langle\cap\rangle C_2\rangle$ implying that $a_i = 0$ for all i. This shows that $\{v + F(\lambda) | v \in C_1\}$ is a linearly independent subset of $W/F(\lambda)$ and $v+F(\lambda) \neq w+F(\lambda)$ for all distinct $v, w \in C_1$. Hence $\dim_R W/F(\lambda) \geq C_1$. Then $\dim_R W/F(\lambda)$ is infinite since C_1 is infinite. Therefore $\lambda \notin S$. Thus $\lambda \notin OM_R(V,W) \cup S$ leading to a contradiction. □

Corollary 2.13. *$OM_R(V,W) \cup S$ does not admit hyperring[ring] structure if and only if $\dim_R V = \dim_R W$.*

Corollary 2.14. *$OM_R(V,W) \cup H$ does not admit the structure of a semihyperring with zero if and only if $\dim_R V = \dim_R W$.*

Proof. Let H be a subsemigroup of $AI_R(\underline{V},W)$. It is clear that H is a subsemigroup of $AI_R(V,\underline{W})$. Applying Theorem 2.12, we obtain that $OM_R(V,W) \cup H$ does not admit the structure of a semihyperring with zero if and only if $\dim_R V = \dim_R W$. □

Corollary 2.15. *$OM_R(V,W) \cup H$ does not admit hyperring[ring] structure if and only if $\dim_R V = \dim_R W$.*

Theorem 2.16. *$OE_R(V,W) \cup S$ does not admit the structure of a semihyperring with zero.*

Proof. Let B be a basis of V, C a basis of W such that $C \subseteq B$ and S a subsemigroup of $AI_R(V,\underline{W})$.

Case 1.$B = C$. Note that $OE_R(V,W) = OE_R(V)$ and $AI_R(V,\underline{W}) = AI_R(V)$. By [1], $OE_R(V,W) \cup S$ does not admit the structure of a semihyperring with zero.

Case 2.$B \neq C$. Suppose that there exists a hyperoperation $\oplus$ such that $(OE_R(V,W) \cup S, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $OE_R(V,W) \cup S$. Since $\dim_R W$ is infinite, C is infinite. There are subsets C_1, C_2 of C such that $C_1 \cup C_2 = C$, $C_1 \cap C_2 = \emptyset$ and $|C_1| = |C_2| = |C|$. As a result, there is a bijection $\varphi : C_1 \to C_2$. Let $C_3 = B \setminus C$ Then $C_3 \neq \emptyset$. Define α, $\beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} C_2 \cup C_3 & v \\ 0 & v\varphi \end{pmatrix}_{v \in C_1} \qquad \beta = \begin{pmatrix} C_1 \cup C_3 & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in C_2} \tag{1}$$

$\dim_R(W/Im\alpha) = |C \setminus C_2| = |C_1|$, $\dim_R(W/Im\beta) = |C \setminus C_1| = |C_2|$. Hence α, $\beta \in OE_R(V,W) \subset OE_R(V,W) \cup S$. Since (1), $\alpha^2 = 0$, $\beta^2 = 0$. Hence

$$\begin{aligned} \alpha(\alpha \oplus \beta) = \alpha^2 \oplus \alpha\beta = 0 \oplus \alpha\beta = \{\alpha\beta\} = \alpha\beta \oplus 0 = \alpha\beta \oplus \beta^2 = (\alpha \oplus \beta)\beta \\ \beta(\alpha \oplus \beta) = \beta\alpha \oplus \beta^2 = \beta\alpha \oplus 0 = \{\beta\alpha\} = 0 \oplus \beta\alpha = \alpha^2 \oplus \beta\alpha = (\alpha \oplus \beta)\alpha. \end{aligned} \tag{2}$$

Let $\lambda \in \alpha \oplus \beta$. We can see from (2) that $\alpha\lambda = \alpha\beta = \lambda\beta$ and $\beta\lambda = \beta\alpha = \lambda\alpha$. For $v \in C_1$, $v\lambda = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w_1' + b_2w_2' + \cdots + b_mw_m'$ where

$w_1, w_2, \ldots, w_n \in C_1$, $w'_1, w'_2, \ldots, w'_m \in C_2$ are all distinct and a_i, $b_j \in R$ for all i and j. Then

$$\begin{aligned}
0 = v\beta\alpha = v(\beta\alpha) &= v(\lambda\alpha) \\
&= (v\lambda)\alpha \\
&= (a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m)\alpha \\
&= \sum_{i=1}^{n} a_i(w_i\alpha) + \sum_{j=1}^{m} b_j(w'_j\alpha) \\
&= \sum_{i=1}^{n} a_i(w_i\alpha) \\
&= \sum_{i=1}^{n} a_i(w_i\varphi).
\end{aligned}$$

Since φ is one to one, $w_i\varphi$ are all distinct in C_2. Hence $a_i = 0$ for all i. Hence $v\lambda \in \langle C_2\rangle$. Consider $v\lambda\beta = v\alpha\beta = (v\alpha)\beta = (v\varphi)\beta$. Since $\beta|_{C_2}$ is one to one, $\beta|_{\langle C_2\rangle}$ is also one to one. Thus $v\lambda = v\varphi$. Therefore $\lambda|_{C_1} = \varphi$. Similarly, $\lambda|_{C_2} = \varphi^{-1}$ so $v\lambda = v\varphi^{-1}$ for $v \in C_2$. For $v \in C_3$, we can write $v\lambda = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m$ where $w_1, w_2, \ldots, w_n \in C_1$, $w'_1, w'_2, \ldots, w'_m \in C_2$ are all distinct and a_i, $b_j \in R$ for all i and j. Thus

$$\begin{aligned}
0 = v\beta\alpha = v(\beta\alpha) &= v(\lambda\alpha) \\
&= (v\lambda)\alpha \\
&= (a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m)\alpha \\
&= \sum_{i=1}^{n} a_i(w_i\alpha) + \sum_{j=1}^{m} b_j(w'_j\alpha) \\
&= \sum_{i=1}^{n} a_i(w_i\alpha) \\
&= \sum_{i=1}^{n} a_i(w_i\varphi).
\end{aligned}$$

Since φ is one to one, $w_i\varphi$ are all distinct in C_2. Hence $a_i = 0$ for all i. Hence $v\lambda \in \langle C_2\rangle$. Similarly,

$$\begin{aligned}
0 = v\alpha\beta = v(\alpha\beta) &= v(\lambda\beta) = (v\lambda)\beta \\
&= (a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m)\beta \\
&= \sum_{i=1}^{n} a_i(w_i\beta) + \sum_{j=1}^{m} b_j(w'_j\beta)
\end{aligned}$$

$$= \sum_{j=1}^{m} b_j (w'_j \beta)$$
$$= \sum_{j=1}^{m} b_j (w'_j \varphi^{-1}).$$

Since φ^{-1} is one to one, $w'_j \varphi$ are all distinct in C_1. Hence $b_j = 0$ for all j. Thus $v\lambda \in \langle C_1 \rangle$ and then $v\lambda \in \langle C_1 \rangle \cap \langle C_2 \rangle = \{0\}$. Hence

$$\lambda = \begin{pmatrix} C_3 & v \\ 0 & v \end{pmatrix}_{v \in C}$$

Since $\dim_R(W/\operatorname{Im}\lambda) = |C \setminus C| = |\emptyset| = 0 < \infty$, we have $\lambda \notin OE_R(V,W)$. Next, we will show that $\dim_R(W/F(\lambda))$ is infinite. Let $v_1, v_2, \ldots, v_n \in C_1$ be all distinct and $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\lambda)) = F(\lambda)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\lambda)$, so $\left(\sum_{i=1}^{n} a_i v_i\right)\lambda = \sum_{i=1}^{n} a_i v_i$. But $\left(\sum_{i=1}^{n} a_i v_i\right)\lambda = \sum_{i=1}^{n} a_i (v_i \lambda) \in \langle C_2 \rangle$. Hence $\sum_{i=1}^{n} a_i v_i \in \langle C_1 \langle \cap \rangle C_2 \rangle$ implying that $a_i = 0$ for all i. This shows that $\{v + F(\lambda) | v \in C_1\}$ is a linearly independent subset of $W/F(\lambda)$ and $v + F(\lambda) \neq w + F(\lambda)$ for all distinct $v, w \in C_1$. Hence $\dim_R(W/F(\lambda)) \geq C_1$. Since C_1 is infinite, $dim_R W/F(\lambda)$ must be infinite. Therefore $\lambda \notin S$. Consequently, $\lambda \notin OM_R(V,W) \cup S$ leading to a contradiction. □

Corollary 2.17. *$OE_R(V,W) \cup S$ does not admit hyperring[ring] structure.*

Corollary 2.18. *$OE_R(V,W) \cup H$ does not admit the structure of a semihyperring with zero.*

Proof. Let H be a subsemigroup of $AI_R(\underline{V}, W)$. Clearly, H is a subsemigroup of $AI_R(V, \underline{W})$. By Theorem 2.16, it follows that $OE_R(V,W) \cup H$ does not admit the structure of a semihyperring with zero □

Corollary 2.19. *$OE_R(V,W) \cup H$ does not admit hyperring[ring] structure.*

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S. Chaopraknoi, S. Hobuntud and S. Painskool
Department of Mathematics,
Faculty of Science,
Chulalongkorn University,
Bangkok 10330, Thailand
e-mail : samkhanhobuntud180@hotmail.com