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$S(\bar{n_i}, Y_i)$-Terms and Their Algebraic Properties

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Abstract A clone is a set of operations defined on a base set which is closed under composition and contains all projection operations. A special kind of clone satisfies the superassociative law is called a Menger algebra. In this paper, we introduce the new concept of $S(\bar{n_i}, Y_i)$-terms of type τ. The set of all $S(\bar{n_i}, Y_i)$-terms of type τ is closed under the superposition operation S^{n_i} and so forms a clone denoted by $clone_{S(\bar{n_i},Y_i)}(\tau)$. We show that the $clone_{S(\bar{n_i},Y_i)}(\tau)$ is a Menger algebra of rank n_i and study its algebraic properties. A connection between identities in $clone_{S(\bar{n_i},Y_i)}(\tau)$ and $S(\bar{n_i}, Y_i)$-hyperidentities is established.

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1. Introduction and Preliminaries

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as letters to X as an alphabet, and also refer to the set $X_n := \{x_1, x_2, \ldots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an *n_i-ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The sequence of the values of function τ, written as $(n_i)_{i\in I}$, is called a *type*. An *n-ary term of type* τ is defined inductively as follows:

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.
(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The set of all n-ary terms of type τ, closed under finite number of applications of (ii), is denoted by $W_\tau(X_n)$. We call the set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ.

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Now, we recall the concept of superposition operation of terms. For each natural numbers $m, n \geq 1$, the superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$$

defined by

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$, if $x_j \in X_n$,
(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra can be defined by

$$\text{clone } \tau := ((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{n,m\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}),$$

which is called the *clone of all terms of type* τ.

The set of all terms of type τ can be used as the universe of an algebra of type τ. For every $i \in I$, an n_i-ary operation $\bar{f}_i : W_\tau(X)^{n_i} \longrightarrow W_\tau(X)$ is defined by

$$\bar{f}_i\,(t_1, \ldots, t_{n_i}) := f_i\,(t_1, \ldots, t_{n_i})\,.$$

The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i\in I})$ is called the *absolutely free algebra* of type τ over the set X.

A hypersubstitution of type τ is a map $\sigma : \{f_i \,|\, i \in I\} \longrightarrow W_\tau(X)$ which maps each operation symbol f_i to an n_i-ary term $\sigma(f_i)$ of type τ. Any hypersubstitutions $\sigma : \{f_i \,|\, i \in I\} \longrightarrow W_\tau(X)$ can be uniquely extended to a map $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$; and
(ii) $\hat{\sigma}[t] := S^{n_i}\,(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the binary associative operation, denoted by $\circ_h$:

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

where $\circ$ is the usual composition of functions, together with the identity $\sigma_{id} : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ such that $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$.

Let τ_n be the sequence of operation symbols having the same arity n, i.e., $\tau_n = (n_i)_{i\in I}$ with $n_i = n$ for all $i \in I$. In 2004, Denecke and Jampachon [1], inductively defined n-ary *full terms* of type τ_n by

(i) If f_i is an n-ary operation symbol and $\alpha \in T_n$ where T_n is the set of all full transformation on $\{1, 2, \ldots, n\}$, then $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n.
(ii) If f_i is an n-ary operation symbol and $t_1, \ldots, t_n$ are n-ary full terms of type τ_n, then $f_i\,(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n.

Let $W^F_{\tau_n}(X_n)$ be the set of all n-ary full terms of type τ_n. The set $W^F_{\tau_n}(X_n)$ is closed under finite application of (ii). If T_n is replaced by a submonoid $\{1_n\}$, then $W^F_{\tau_n}(X_n)$ is denoted by $W^{SF}_{\tau_n}(X_n)$ called the set of all *strongly full terms* of type τ_n which was studied by Denecke and Freiberg [2]. In 2011, Phuapong and Leeratanavalee [3] were introduced the concept of generalized of full terms and strongly full terms of type τ. In 2015, Phuapong [4] studied some algebraic properties of generalized clone automorphisms.

In 2019, Wattanatripop and Changphas [5] studied the clone of $K^*(n, r)$-full terms. The result obtained the notion of $K^*(n, r)$-full closed identities. In this paper, we use idea of $K^*(n, r)$-full terms of type τ_n to define a new concept of n_i-ary $S(\bar{n_i}, Y_i)$-terms of type τ induced by transformations with invariant set. We study some algebraic properties of $clone_{S(\bar{n_i},Y_i)}(\tau)$ and $S(\bar{n_i}, Y_i)$-hypersubstitutions. The related $S(\bar{n_i}, Y_i)$-hyperidentities and $S(\bar{n_i}, Y_i)$-solid varieties is also mentioned.

2. The Menger Algebra of $S(\bar{n_i}, Y_i)$-Terms

The first of our main results is a new concept of a specific term, based on full transformation mappings and the original notions of terms.

Let X be a nonempty set and let $T(X)$ be the semigroup of full transformations from X into itself under composition of mappings. This semigroup has an important role in semigroup theory, combinatorics, many-valued logic, etc.

For a fixed nonempty subset Y of X, let

$$S(X, Y) := \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Then $S(X, Y)$ is a semigroup of full transformations on X which leave Y invariant. In 1966, Magill [6] introduced and studied this semigroup. In fact, if $Y = X$, then $S(X, Y) = T(X)$. So we may regard $S(X, Y)$ as a generalization of $T(X)$. The full transformation semigroup T_{n_i} consists of the set of all maps $\alpha_i : \{1, 2, \ldots, n_i\} \longrightarrow \{1, 2, \ldots, n_i\}$ and the usual composition of mappings. Indeed, T_{n_i} is a monoid and identity map 1_{n_i} acts as its identity. For a fixed nonempty subset Y_i of $\bar{n_i} := \{1, 2, \ldots, n_i\}$, it is well-known that the set

$$S(\bar{n_i}, Y_i) := \{\alpha_i \in T_{n_i} \mid Y_i\alpha_i \subseteq Y_i\}$$

is a submonoid of T_{n_i} where 1_{n_i} is an identity element.

Then we define the definition of n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ.

Definition 2.1. Let $\tau = (n_i)_{i\in I}$ and f_i be an n_i-ary operation symbol and $\alpha_i \in S(\bar{n_i}, Y_i)$. An n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ is inductively defined by the following steps :

(i) $f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})$ is an n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ;

(ii) if $t_1, \ldots, t_{n_i}$ are n_i-ary $S(\bar{n_i}, Y_i)$-terms of type τ, then $f_i(t_1, \ldots, t_{n_i})$ is an n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ.

Let $W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ be the set of all n_i-ary $S(\bar{n_i}, Y_i)$-terms of type τ.

Example 2.2. Let $I = \{1, 2\}$ and consider a type $\tau = (3, 4)$ with f_1, f_2 are operation symbols having arity $3, 4$ respectively.

For $i = 1$; we fixed subset $Y_1 = \{1, 2\} \subseteq \{1, 2, 3\}$ and $\alpha_1 : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$, and for $i = 2$; we fixed subset $Y_2 = \{3, 4\} \subseteq \{1, 2, 3, 4\}$ and $\alpha_2 : \{1, 2, 3, 4\} \longrightarrow \{1, 2, 3, 4\}$.

Then we have $f_1(x_1, x_2, x_3), f_1(x_1, x_1, x_2), f_1(x_1, x_2, x_2), f_1(x_2, x_2, x_3)$ are some examples of $W_\tau^{S(\bar{3},Y_1)}(X_3)$ and $f_2(x_1, x_2, x_3, x_4), f_2(x_1, x_1, x_3, x_3), f_2(x_3, x_4, x_4, x_3)$ are some examples of $W_\tau^{S(\bar{4},Y_2)}(X_4)$. ■

It can be seen that $\left(W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i}); (\bar{f_i})_{i\in I}\right)$ is a subalgebra of $\left(W_\tau(X); (\bar{f_i})_{i\in I}\right)$.

Then we define a superposition operation S^{n_i} on the set $W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ by the following steps:

Definition 2.3. Let $\alpha_i \in S(\bar{n_i}, Y_i)$ and $s_1, \ldots, s_{n_i}, t_1, \ldots, t_{n_i} \in W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$. Then

$$S^{n_i} : \left(W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})\right)^{n_i+1} \longrightarrow W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$$

is defined by

(i) $S^{n_i}\left(f_i\left(x_{\alpha_i(1)},\ldots,x_{\alpha_i(n_i)}\right),t_1,\ldots,t_{n_i}\right) := f_i\left(t_{\alpha_i(1)},\ldots,t_{\alpha_i(n_i)}\right)$;
(ii) $S^{n_i}\left(f_i\left(s_1,\ldots,s_{n_i}\right),t_1,\ldots,t_{n_i}\right)$
$:= f_i\left(S^{n_i}\left(s_1,t_1,\ldots,t_{n_i}\right),\ldots,S^{n_i}\left(s_{n_i},t_1,\ldots,t_{n_i}\right)\right)$.

Then we can form the algebra

$$clone_{S(\bar{n}_i,Y_i)}(\tau) := \left(W_\tau^{S(\bar{n}_i,Y_i)}(X_{n_i}),S^{n_i}\right)$$

which is called the *clone of all* $S(\bar{n}_i,Y_i)$*-terms of type* τ. The following theorem shows the fact that the algebra $\left(W_\tau^{S(\bar{n}_i,Y_i)}(X_{n_i}),S^{n_i}\right)$ satisfies the superassociative law (SASS) :

$$\tilde{S^{n_i}}(X_0,\tilde{S^{n_i}}(Y_1,X_1,\ldots,X_{n_i}),\ldots,\tilde{S^{n_i}}(Y_{n_i},X_1,\ldots,X_{n_i})) \approx \tilde{S^{n_i}}(\tilde{S^{n_i}}(X_0,Y_1,\ldots,Y_{n_i}),X_1,\ldots,X_{n_i}),$$

where $\tilde{S^{n_i}}$ is (n_i+1)-ary operation symbol and X_i,Y_j are variables.

Theorem 2.4. *The algebra* $clone_{S(\bar{n}_i,Y_i)}(\tau)$ *satisfies the superassociative law* (*SASS*).

Proof. We give a proof by induction on the complexity of an n_i-ary $S(\bar{n}_i,Y_i)$-term t which is substituted for X_0.

If we substitute for X_0 by a $S(\bar{n}_i,Y_i)$-term $t = f_i(x_{\alpha_i(1)},\ldots,x_{\alpha_i(n_i)})$ where $\alpha_i \in S(\bar{n}_i,Y_i)$, then

$$\begin{aligned}
&S^{n_i}(f_i(x_{\alpha_i(1)},\ldots,x_{\alpha_i(n_i)}),S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i}))\\
&= f_i(S^{n_i}(x_{\alpha_i(1)},S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i})),\ldots,\\
&\quad S^{n_i}(x_{\alpha_i(n_i)},S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i})))\\
&= f_i(S^{n_i}(t_{\alpha_i(1)},s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{\alpha_i(n_i)},s_1,\ldots,s_{n_i}))\\
&= S^{n_i}(f_i(t_{\alpha_i(1)},\ldots,t_{\alpha_i(n_i)}),s_1,\ldots,s_{n_i})\\
&= S^{n_i}(S^{n_i}(f_i(x_{\alpha_i(1)},\ldots,x_{\alpha_i(n_i)}),t_1,\ldots,t_{n_i}),s_1,\ldots,s_{n_i}).
\end{aligned}$$

If we substitute for X_0 by a $S(\bar{n}_i,Y_i)$-term $t = f_i(r_1,\ldots,r_{n_i})$ where $r_1,\ldots,r_{n_i} \in W_\tau^{S(\bar{n}_i,Y_i)}(X_{n_i})$ and assume that

$$S^{n_i}(r_k,S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i})) = S^{n_i}\left(S^{n_i}(r_k,t_1,\ldots,t_{n_i}),s_1,\ldots,s_{n_i}\right)$$

for all $1 \le k \le n_i$, then

$$\begin{aligned}
&S^{n_i}(f_i(r_1,\ldots,r_{n_i}),S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i}))\\
&= f_i(S^{n_i}(r_1,S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i})),\ldots,\\
&\quad S^{n_i}(r_{n_i},S^{n_i}(t_1,s_1,\ldots,s_{n_i}),\ldots,S^{n_i}(t_{n_i},s_1,\ldots,s_{n_i})))\\
&= f_i\left(S^{n_i}\left(S^{n_i}(r_1,t_1,\ldots,t_{n_i}),s_1,\ldots,s_{n_i}\right),\ldots,(S^{n_i}(r_{n_i},t_1,\ldots,t_{n_i}),s_1,\ldots,s_{n_i})\right)\\
&= S^{n_i}(f_i(S^{n_i}(r_1,t_1,\ldots,t_{n_i}),\ldots,S^{n_i}(r_{n_i},t_1,\ldots,t_{n_i})),s_1,\ldots,s_{n_i})\\
&= S^{n_i}(S^{n_i}(f_i(r_1,\ldots,r_{n_i}),t_1,\ldots,t_{n_i}),s_1,\ldots,s_{n_i}).
\end{aligned}$$
■

An algebra $\mathcal{M} := (M,S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if $\mathcal{M}$ satisfies the condition (SASS) [7]. Then by Theorem 2.4, $clone_{S(\bar{n}_i,Y_i)}(\tau)$ is an example of a Menger algebra of rank n_i.

The freeness of this algebra will be presented in the next theorem. It is observed that $clone_{S(\bar{n}_i,Y_i)}(\tau)$ is generated by

$$F_{W_\tau^{S(\bar{n}_i,Y_i)}(X_{n_i})} := \left\{f_i\left(x_{\alpha_i(1)},\ldots,x_{\alpha_i(n_i)}\right) \mid i \in I,\alpha_i \in S(\bar{n}_i,Y_i)\right\}.$$

Let $V^{S(\bar{n_i},Y_i)}$ be the variety of type $\tau = (n_i + 1)$ generated by the superassociative law (SASS). Now, let $\mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$ be the free algebra with respect to $V^{S(\bar{n_i},Y_i)}$, freely generated by an alphabet $\{Y_l \mid l \in J\}$ where $J = \{(i, \alpha_i) \mid i \in I \ , \alpha_i \in S(\bar{n_i}, Y_i)\}$. The operation of $\mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$ is denoted by $\tilde{S}^{n_i}$. Then we have:

Theorem 2.5. *The algebra* $clone_{S(\bar{n_i},Y_i)}(\tau)$ *is isomorphic to* $\mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$ *and therefore free with respect to the variety* $V^{S(\bar{n_i},Y_i)}$*, and freely generated by the set* $\{f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}) \mid i \in I, \alpha_i \in S(\bar{n_i}, Y_i)\}$.

Proof. We define the mapping $\varphi : W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i}) \longrightarrow \mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$ inductively as follows:

(i) $\varphi(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}) = y_{(i,\alpha_i)}$;

(ii) $\varphi(f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)})) = \tilde{S}^{n_i}(y_{(i,\alpha_i)}, \varphi(t_1), \ldots, \varphi(t_{n_i}))$.

Since φ maps the generating system of $clone_{S(\bar{n_i},Y_i)}(\tau)$ onto the generating system of $\mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$, it is surjective. We prove the homomorphism property

$$\varphi(S^{n_i}(t_0, t_1, \ldots, t_{n_i})) = \tilde{S}^{n_i}(\varphi(t_0), \varphi(t_1), \ldots, \varphi(t_{n_i}))$$

by induction on the complexity of an n_i-ary $S(\bar{n_i}, Y_i)$-term t_0.

If $t_0 = f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})$ where $\alpha_i \in S(\bar{n_i}, Y_i)$, then

$$\begin{aligned}\varphi(S^{n_i}(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), t_1, \ldots, t_{n_i}))\\ &= \varphi(f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}))\\ &= \tilde{S}^{n_i}(y_{(i,\alpha_i)}, \varphi(t_1), \ldots, \varphi(t_{n_i}))\\ &= \tilde{S}^{n_i}(\varphi(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})), \varphi(t_1), \ldots, \varphi(t_{n_i})).\end{aligned}$$

If $t_0 = f_i(r_1, \ldots, r_{n_i})$ and assume that

$$\varphi(S^{n_i}(r_k, t_1, \ldots, t_{n_i})) = \tilde{S}^{n_i}(\varphi(r_k), \varphi(t_1), \ldots, \varphi(t_{n_i}))$$

for all $1 \leq k \leq n_i$, then

$$\begin{aligned}\varphi(S^{n_i}(f_i(r_1, \ldots, r_{n_i}), t_1, \ldots, t_{n_i}))\\ &= \varphi(f_i(S^{n_i}(r_1, t_1, \ldots, t_{n_i}), \ldots, S^{n_i}(r_{n_i}, t_1, \ldots, t_{n_i})))\\ &= \tilde{S}^{n_i}(y_{(i,1_{n_i})}, \varphi(S^{n_i}(r_1, t_1, \ldots, t_{n_i})), \ldots, \varphi(S^{n_i}(r_{n_i}, t_1, \ldots, t_{n_i})))\\ &= \tilde{S}^{n_i}(y_{(i,1_{n_i})}, \tilde{S}^{n_i}(\varphi(r_1), \varphi(t_1), \ldots, \varphi(t_{n_i})), \ldots, \tilde{S}^{n_i}(\varphi(r_{n_i}), \varphi(t_1), \ldots, \varphi(t_{n_i})))\\ &= \tilde{S}^{n_i}(\tilde{S}(y_{(i,1_{n_i})}, \varphi(r_1), \ldots, \varphi(r_{n_i})), \varphi(t_1), \ldots, \varphi(t_{n_i}))\\ &= \tilde{S}^{n_i}(\varphi(f_i(r_1, \ldots, r_{n_i})), \varphi(t_1), \ldots, \varphi(t_{n_i})).\end{aligned}$$

Thus φ is a homomorphism. The mapping φ is bijective since $\{y_{(i,\alpha_i)} \mid i \in I, \ \alpha_i \in S(\bar{n_i}, Y_i)\}$ is free independent set. Therefore, we have

$$\begin{aligned}y_{(i,\alpha_i)} = y_{(j,\alpha_j)} &\Longrightarrow (i, \alpha_i) = (j, \alpha_j)\\ &\Longrightarrow i = j, \ \alpha_i = \alpha_j.\end{aligned}$$

So $f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}) = f_j(x_{\alpha_j(1)}, \ldots, x_{\alpha_j(n_j)})$. Thus φ is a bijection between the generating sets of $clone_{S(\bar{n_i},Y_i)}(\tau)$ and $\mathcal{F}_{V^{S(\bar{n_i},Y_i)}}(\{Y_l \mid l \in J\})$ and therefore φ is an isomorphism. ■

3. The Monoid of $S(\bar{n_i}, Y_i)$-Hypersubstitutions

The concept of a hypersubstitution is the main tool used to study hyperidentities and hypervarieties. In this section, the monoid of $S(\bar{n_i}, Y_i)$-hypersubstitutions will be studied. Next, we give the definition of a $S(\bar{n_i}, Y_i)$-hypersubstitution and introduce some properties of $S(\bar{n_i}, Y_i)$-hypersubstitutions.

Definition 3.1. A $S(\bar{n_i}, Y_i)$-hypersubstitution of type τ is a mapping σ from the set $\{f_i \mid i \in I\}$ of n_i-ary operation symbols of type τ to the set $W_\tau^{S(\bar{n_i},Y_i)}(X)$ of all n_i-ary $S(\bar{n_i}, Y_i)$-terms of type τ, i.e., $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau^{S(\bar{n_i},Y_i)}(X)$.

For a $S(\bar{n_i}, Y_i)$-term t, we need the $S(\bar{n_i}, Y_i)$-term t_{β_i} arising from t by replacement a variable $x_{\alpha_i(j)}$ in t by a variable $x_{\beta_i(\alpha_i(j))}$ for a mapping $\beta_i \in S(\bar{n_i}, Y_i)$. This can be defined as follows.

Definition 3.2. Let $\tau = (n_i)_{i \in I}$ and f_i be an n_i-ary operation symbol, $t, t_1, \ldots, t_{n_i} \in W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ and $\alpha_i, \beta_i \in S(\bar{n_i}, Y_i)$. Then we define the n_i-ary $S(\bar{n_i}, Y_i)$-term t_{β_i} by the following step:

(i) If $t = f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})$, then $t_{\beta_i} := f_i(x_{\beta_i(\alpha_i(1))}, \ldots, x_{\beta_i(\alpha_i(n_i))})$.
(ii) If $t = f_i(t_1, \ldots, t_{n_i})$, then $t_{\beta_i} := f_i((t_1)_{\beta_i}, \ldots, (t_{n_i})_{\beta_i})$.

It is observed that if t is an n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ, then t_{β_i} is an n_i-ary $S(\bar{n_i}, Y_i)$-term of type τ for all $\beta_i \in S(\bar{n_i}, Y_i)$. Any $S(\bar{n_i}, Y_i)$-hypersubstitution σ induces a mapping $\hat{\sigma}$ defined on the set $W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ of n_i-ary $S(\bar{n_i}, Y_i)$-terms of type τ.

Definition 3.3. Let σ be a $S(\bar{n_i}, Y_i)$-hypersubstitution of type τ and $\alpha_i \in S(\bar{n_i}, Y_i)$. Then σ induces a mapping

$$\hat{\sigma} : W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i}) \longrightarrow W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$$

by setting

(i) $\hat{\sigma}[f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})] := (\sigma(f_i))_{\alpha_i}$;
(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

The set of all $S(\bar{n_i}, Y_i)$-hypersubstitutions of type τ will be denoted by $Hyp^{S(\bar{n_i},Y_i)}(\tau)$. It is easy to see that $\left(Hyp^{S(\bar{n_i},Y_i)}(\tau); \circ_h, \sigma_{id}\right)$ is a submonoid of $(Hyp(\tau); \circ_h, \sigma_{id})$.

The following lemma shows the property of a term t_{α_i} and the extension $\hat{\sigma}$.

Lemma 3.4. *Let $t, t_1, \ldots, t_{n_i} \in W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$. Then*

$$S^{n_i}(t, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) = S^{n_i}(t_{\alpha_i}, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$$

for all $\alpha_i \in S(\bar{n_i}, Y_i)$.

Proof. If $t = f_i\left(x_{\beta_i(1)}, \ldots, x_{\beta_i(n_i)}\right)$ where $\beta_i \in S(\bar{n_i}, Y_i)$, then for all $\alpha_i \in S(\bar{n_i}, Y_i)$, we have

$$\begin{aligned} &S^{n_i}(t, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) \\ &= S^{n_i}(f_i(x_{\beta_i(1)}, \ldots, x_{\beta_i(n_i)}), \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) \\ &= f_i(\hat{\sigma}[t_{\alpha_i(\beta_i(1))}], \ldots, \hat{\sigma}[t_{\alpha_i(\beta_i(n_i))}]) \\ &= S^{n_i}(f_i(x_{\alpha_i(\beta_i(1))}, \ldots, x_{\alpha_i(\beta_i(n_i))}), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \\ &= S^{n_i}(t_{\alpha_i}, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]). \end{aligned}$$

If $t = f_i(s_1, \ldots, s_{n_i})$ and assume that

$$S^{n_i}(s_k, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) = S^{n_i}((s_k)_{\alpha_i}, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}])$$

for all $1 \leq k \leq n_i$ and $\alpha_i \in S(\bar{n_i}, Y_i)$, then

$$\begin{aligned} &S^{n_i}(t, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) \\ &= S^{n_i}(f_i(s_1, \ldots, s_{n_i}), \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]) \\ &= f_i(S^{n_i}(s_1, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}]), \ldots, S^{n_i}(s_{n_i}, \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}])) \\ &= f_i(S^{n_i}((s_1)_{\alpha_i}, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]), \ldots, S^{n_i}((s_{n_i})_{\alpha_i}, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])) \end{aligned}$$

$$= S^{n_i}(f_i((s_1)_{\alpha_i}, \ldots, (s_{n_i})_{\alpha_i}), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$$
$$= S^{n_i}(t_{\alpha_i}, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]).$$ ■

Theorem 3.5. *For $\sigma \in Hyp^{S(\bar{n_i}, Y_i)}(\tau)$, the extension*

$$\hat{\sigma} : W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i}) \longrightarrow W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})$$

is an endomorphism on the algebra $clone_{S(\bar{n_i}, Y_i)}(\tau)$.

Proof. It is clear that $\hat{\sigma} : W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i}) \longrightarrow W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})$ is well defined. Let $t_0, t_1, \ldots, t_{n_i} \in W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})$. We will show by induction on the complexity of t_0 that

$$\hat{\sigma}[S^{n_i}(t_0, t_1, \ldots, t_{n_i})] = S^{n_i}(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]).$$

If $t_0 = f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)})$ where $\alpha_i \in S(\bar{n_i}, Y_i)$, then

$$\begin{aligned}\hat{\sigma}[S^{n_i}(t_0, t_1, \ldots, t_{n_i})] &= \hat{\sigma}[S^{n_i}(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), t_1, \ldots, t_{n_i})]\\ &= \hat{\sigma}[f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)})]\\ &= S^{n_i}(\sigma(f_i), \hat{\sigma}[t_{\alpha_i(1)}], \ldots, \hat{\sigma}[t_{\alpha_i(n_i)}])\\ &= S^{n_i}(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]).\end{aligned}$$

If $t_0 = f_i(r_1, \ldots, r_{n_i})$ and we assume that

$$\hat{\sigma}[S^{n_i}(r_k, t_1, \ldots, t_{n_i})] = S^{n_i}(\hat{\sigma}[r_k], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$$

for all $1 \leq k \leq n_i$, then

$$\begin{aligned}&\hat{\sigma}[S^{n_i}(t_0, t_1, \ldots, t_{n_i})]\\ &= \hat{\sigma}[S^{n_i}(f_i(r_1, \ldots, r_{n_i}), t_1, \ldots, t_{n_i})]\\ &= \hat{\sigma}[f_i(S^{n_i}(r_1, t_1, \ldots, t_{n_i}), \ldots, S^{n_i}(r_{n_i}, t_1, \ldots, t_{n_i}))]\\ &= S^{n_i}(\sigma(f_i), \hat{\sigma}[S^{n_i}(r_1, t_1, \ldots, t_{n_i})], \ldots, \hat{\sigma}[S^{n_i}(r_{n_i}, t_1, \ldots, t_{n_i})])\\ &= S^{n_i}(\sigma(f_i), S^{n_i}(\hat{\sigma}[r_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]), \ldots, S^{n_i}(\hat{\sigma}[r_{n_i}], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]))\\ &= S^{n_i}(S^{n_i}(\sigma(f_i), \hat{\sigma}[r_1], \ldots, \hat{\sigma}[r_{n_i}]), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])\\ &= S^{n_i}(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]).\end{aligned}$$ ■

We complete this section by studying the connection between $S(\bar{n_i}, Y_i)$-terms and the extension of a mapping which maps fundamental term to any $S(\bar{n_i}, Y_i)$-terms.

As mentioned, the algebra $clone_{S(\bar{n_i}, Y_i)}(\tau)$ is generated by the set

$$F_{W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})} := \left\{ f_i\left(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}\right) \mid i \in I, \alpha_i \in S(\bar{n_i}, Y_i) \right\}.$$

Therefore, any mapping

$$\eta : F_{W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})} \longrightarrow W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})$$

called $S(\bar{n_i}, Y_i)$-*clone substitution*, can be uniquely extended to an endomorphism

$$\bar{\eta} : W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i}) \longrightarrow W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i}).$$

Let $Subst_{S(\bar{n_i}, Y_i)}(\tau)$ be the set of all $S(\bar{n_i}, Y_i)$-clone substitutions. Together with a binary composition operation $\odot$ defined by; for all $\eta_1, \eta_2 \in Subst_{S(\bar{n_i}, Y_i)}(\tau)$,

$$\eta_1 \odot \eta_2 := \bar{\eta_1} \circ \eta_2$$

where $\circ$ is usual composition of functions and with the identity mapping $id_{F_{W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})}}$ on $F_{W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})}$. Then $\left(Subst_{S(\bar{n_i}, Y_i)}(\tau); \odot, id_{F_{W_\tau^{S(\bar{n_i}, Y_i)}(X_{n_i})}} \right)$ forms a monoid.

Consider $\sigma \in Hyp^{S(\bar{n_i},Y_i)}(\tau)$ and by Theorem 3.5, we have $\hat{\sigma} : W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i}) \longrightarrow W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ is an endomorphism. Since $F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}$ generates $clone_{S(\bar{n_i},Y_i)}(\tau)$, we have $\hat{\sigma}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}}$ is an $S(\bar{n_i},Y_i)$-clone substitution with

$$\overline{\hat{\sigma}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}}} = \hat{\sigma}.$$

Define a mapping $\psi : Hyp^{S(\bar{n_i},Y_i)}(\tau) \longrightarrow Subst_{S(\bar{n_i},Y_i)}(\tau)$ by

$$\psi(\sigma) = \hat{\sigma}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}}.$$

We have that ψ is a homomorphism. In fact : Let $\sigma_1, \sigma_2 \in Hyp^{S(\bar{n_i},Y_i)}(\tau)$. Then

$$\begin{aligned}\psi(\sigma_1 \circ_h \sigma_2) &= (\sigma_1 \circ_h \sigma_2)^{\wedge}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}} \\ &= (\hat{\sigma_1} \circ \hat{\sigma_2})|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}} \\ &= \overline{\hat{\sigma_1}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}}} \circ \hat{\sigma_2}|_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}} \\ &= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\ &= \psi(\sigma_1) \odot \psi(\sigma_2).\end{aligned}$$

Clearly, ψ is an injection. Hence we have the following corollary.

Corollary 3.6. *The monoid* $\left(Hyp^{S(\bar{n_i},Y_i)}(\tau); \circ_h, \sigma_{id}\right)$ *can be embedded into the monoid* $(Subst_{S(\bar{n_i},Y_i)}(\tau); \odot, id_{F_{W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})}})$.

4. $S(\bar{n_i}, Y_i)$-Hyperidentities and Clone Identities

In this section, we examine the relationship between a variety V of type τ and the identity in the $clone_{S(\bar{n_i},Y_i)}(\tau)$.

Let V be a variety of type τ and let IdV be the set of all identities of V. Let $Id^{S(\bar{n_i},Y_i)}V$ be the set of all $s \approx t$ of V such that s and t are both $S(\bar{n_i},Y_i)$-terms of type τ, that is,

$$Id^{S(\bar{n_i},Y_i)}V := \left(W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})\right)^2 \cap IdV.$$

It is well-known that IdV is a congruence on the free algebra $\mathcal{F}_\tau(X)$. However, in general this is not true for $Id^{S(\bar{n_i},Y_i)}V$. The following theorem shows that $Id^{S(\bar{n_i},Y_i)}V$ is a congruence on $clone_{S(\bar{n_i},Y_i)}(\tau)$.

Theorem 4.1. *Let V be a variety of type τ. Then $Id^{S(\bar{n_i},Y_i)}V$ is a congruence on the $clone_{S(\bar{n_i},Y_i)}(\tau)$.*

Proof. We will prove that from $r \approx t$, $r_k \approx t_k \in Id^{S(\bar{n_i},Y_i)}V, k = 1, 2, \ldots, n_i$, it follows that $S^{n_i}(r, r_1, \ldots, r_{n_i}) \approx S^{n_i}(t, t_1, \ldots, t_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V$.

Firstly, we prove by induction on the complexity of a term $t \in W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ that for every $i \in I$ from $t_k \approx r_k \in Id^{S(\bar{n_i},Y_i)}V, k = 1, 2, \ldots, n_i$, it follows that $S^{n_i}(t, t_1, \ldots, t_{n_i}) \approx S^{n_i}(t, r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V$.

If $t = f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), i \in I, \alpha_i \in S(\bar{n_i}, Y_i)$, then

$$\begin{aligned}&S^{n_i}(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), t_1, \ldots, t_{n_i}) \\ &\quad = f_i(t_{\alpha_i(1)}, \ldots, t_{\alpha_i(n_i)}) \\ &\quad \approx f_i(r_{\alpha_i(1)}, \ldots, r_{\alpha_i(n_i)})\end{aligned}$$

$$= S^{n_i}(f_i(x_{\alpha_i(1)}, \ldots, x_{\alpha_i(n_i)}), r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V$$

since IdV is compatible with the operation $\overline{f_i}$ of the absolutely free algebra $\mathcal{F}_\tau(X)$ and by the definition of n_i-ary $S(\bar{n_i}, Y_i)$-terms.

If $t = f_i(l_1, \ldots, l_{n_i}) \in W_\tau^{S(\bar{n_i},Y_i)}(X_{n_i})$ and assume that

$$S^{n_i}(l_k, t_1, \ldots, t_{n_i}) \approx S^{n_i}(l_k, r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V$$

for all $1 \leq k \leq n_i$, then

$$\begin{aligned} S^{n_i}(f_i(l_1, \ldots, l_{n_i}), t_1, \ldots, t_{n_i}) &= f_i(S^{n_i}(l_1, t_1, \ldots, t_{n_i}), \ldots, S^{n_i}(l_{n_i}, t_1, \ldots, t_{n_i})) \\ &\approx f_i(S^{n_i}(l_1, r_1, \ldots, r_{n_i}), \ldots, S^{n_i}(l_{n_i}, r_1, \ldots, r_{n_i})) \\ &= S^{n_i}(f_i(l_1, \ldots, l_{n_i}), r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V. \end{aligned}$$

Now, we prove the implication

$$t \approx r \in Id^{S(\bar{n_i},Y_i)}V \Rightarrow S^{n_i}(t, r_1, \ldots, r_{n_i}) \approx S^{n_i}(r, r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V.$$

This is a consequence of the fully invariant of IdV as a congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$ and the definition of an n_i-ary $S(\bar{n_i}, Y_i)$-term. Assume now that $t \approx r, t_k \approx r_k \in Id^{S(\bar{n_i},Y_i)}V$. Then

$$S^{n_i}(t, t_1, \ldots, t_{n_i}) \approx S^{n_i}(r, t_1, \ldots, t_{n_i}) \approx S^{n_i}(r, r_1, \ldots, r_{n_i}) \in Id^{S(\bar{n_i},Y_i)}V. \quad \blacksquare$$

By using the concept of a $S(\bar{n_i}, Y_i)$-hypersubstitution as we presented in Section 3. We will define $S(\bar{n_i}, Y_i)$-hyperidentities in a variety of type τ.

Let V be a variety of type τ. An identity $s \approx t \in Id^{S(\bar{n_i},Y_i)}V$ is called a $S(\bar{n_i}, Y_i)$-*hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all $\sigma \in Hyp^{S(\bar{n_i},Y_i)}(\tau)$. Moreover, the variety V is called $S(\bar{n_i}, Y_i)$-*solid* if the following hold:

$$\forall s \approx t \in Id^{S(\bar{n_i},Y_i)}V, \forall \sigma \in Hyp^{S(\bar{n_i},Y_i)}(\tau), \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV.$$

The following lemma give a sufficient condition for the $S(\bar{n_i}, Y_i)$-*hyperidentity* of a variety V.

Lemma 4.2. *Let V be a variety of type τ. If $Id^{S(\bar{n_i},Y_i)}V$ is a fully invariant congruence on $clone_{S(\bar{n_i},Y_i)}(\tau)$, then V is $S(\bar{n_i}, Y_i)$-solid.*

Proof. Assume that $Id^{S(\bar{n_i},Y_i)}V$ is a fully invariant congruence on $clone_{S(\bar{n_i},Y_i)}(\tau)$. Let $s \approx t \in Id^{S(\bar{n_i},Y_i)}V$ and $\sigma \in Hyp^{S(\bar{n_i},Y_i)}(\tau)$. By Theorem 3.5, $\hat{\sigma}$ is an endomorphism of $clone_{S(\bar{n_i},Y_i)}(\tau)$. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{S(\bar{n_i},Y_i)}V$, that is, V is $S(\bar{n_i}, Y_i)$-solid.

For a variety V of type τ, $Id^{S(\bar{n_i},Y_i)}V$ is a congruence on $clone_{S(\bar{n_i},Y_i)}(\tau)$ by Theorem 4.1. We can form the quotient algebra

$$clone_{S(\bar{n_i},Y_i)}(V) := clone_{S(\bar{n_i},Y_i)}(\tau)/Id^{S(\bar{n_i},Y_i)}V.$$

This quotient algebra belongs to the class of a Menger algebra of rank n_i. Note that we have a natural homomorphism

$$nat_{Id^{S(\bar{n_i},Y_i)}V} : clone_{S(\bar{n_i},Y_i)}(\tau) \longrightarrow clone_{S(\bar{n_i},Y_i)}(V)$$

such that

$$nat_{Id^{S(\bar{n_i},Y_i)}V}(t) = [t]_{Id^{S(\bar{n_i},Y_i)}V}. \quad \blacksquare$$

Finally, we prove the following connection between $S(\bar{n_i}, Y_i)$-*hyperidentities* of a variety V and clone identities.

Theorem 4.3. *Let V be a variety of type τ. If $s \approx t \in Id^{S(\bar{n_i},Y_i)}V$ is an identity in $clone_{S(\bar{n_i},Y_i)}(V)$, then $s \approx t$ is a $S(\bar{n_i},Y_i)$-hyperidentity of V.*

Proof. Assume that $s \approx t \in Id^{S(\bar{n_i},Y_i)}V$ is an identity in $clone_{S(\bar{n_i},Y_i)}(V)$. Let $\sigma \in Hyp^{S(\bar{n_i},Y_i)}(\tau)$. Then $\hat{\sigma} : clone_{S(\bar{n_i},Y_i)}(\tau) \longrightarrow clone_{S(\bar{n_i},Y_i)}(\tau)$ is an endomorphism by Theorem 3.5. Thus

$$nat_{Id^{S(\bar{n_i},Y_i)}V} \circ \hat{\sigma} : clone_{S(\bar{n_i},Y_i)}(\tau) \longrightarrow clone_{S(\bar{n_i},Y_i)}(V)$$

is a homomorphism. By assumption,

$$\left(nat_{Id^{S(\bar{n_i},Y_i)}V} \circ \hat{\sigma}\right)(s) = \left(nat_{Id^{S(\bar{n_i},Y_i)}V} \circ \hat{\sigma}\right)(t).$$

That is

$$nat_{Id^{S(\bar{n_i},Y_i)}V}(\hat{\sigma}[s]) = nat_{Id^{S(\bar{n_i},Y_i)}V}(\hat{\sigma}[t]).$$

Thus

$$[\hat{\sigma}[s]]_{Id^{S(\bar{n_i},Y_i)}V} = [\hat{\sigma}[t]]_{Id^{S(\bar{n_i},Y_i)}V},$$

and hence

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{S(\bar{n_i},Y_i)}V.$$

Therefore, $s \approx t$ is a $S(\bar{n_i},Y_i)$-hyperidentity of V. ∎

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