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Some Best Proximity Coincidence Point Theorems for G-Proximal Geraghty Contraction Type M Mapping in Complete Metric Space endowed with Directed Graphs

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Abstract In this study, we proved the existence of best proximity coincidence point for G-proximal Geraghty type M mapping in a complete metric spaces endowed with directed graph. We also give some example to support our result. Moreover, we apply our main result to several consequences and for the case of metric spaces endowed with symmetric binary relations.

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1. Introduction

Let A and B be nonempty subsets of a metric space (X,d) and $T : A \to B$ a non-self mapping. Given the equation $Tx = x$, it is generally known that there might not be any solution to it, especially when A and B are disjoint. It is yet possible to determine approximate solutions to this problem for which the distance between x and Tx must be minimized. With the aid of the best proximity point theorems, the global minimization of $d(x,Tx)$ can be guaranteed under the condition that an approximate solution x must satisfy $d(x,Tx) = d(A,B)$. A family of such approximate solutions, x, is called the best proximity points of the mapping T. In the case when a mapping is a self-mapping, these best proximity points become fixed points according to the best proximity point theorems which sometimes can be accounted for natural generalizations of fixed points.

There have been several studies on the best proximity point theorems [1–11]. Their approaches differ depending on imposed conditions that assure the existence. One of the prominent generalizations of the Banach contraction principle for the existence of fixed

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points for self mappings on metric spaces is the theorem by Geraghty [12]. Here, he considered the class Θ as the class of mappings $\theta : [0, \infty) \to [0, 1)$ such that

$$\theta(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0.$$

Whilst Biligili, Karapinar, and Sadarangani [13] introduced the notion of Geraghty-contractions as well as considered the related the best proximity point for a pair (A, B) of subsets on a metric space X by applying the concept of the P-property defined by Raj [14]. Zhang and Su [15] subsequently gave the notion of the weak P-property which is more feasible than P-property. They also provided the concise proof of the non-self contraction case proposed by Geraghty [12].

In different aspect, Jachymski [16] utilized the concept of graph to the fixed point theorems on metric spaces. This consequently leads to the generalization of the Banach contraction principle to mappings on metric spaces endowed with a graph. Klanarong and Suantai [17] later extended it by introducing the notion of a G-proximal generalized contraction which is defined from self mappings to non-self mappings. Altogether, they proved the best proximity point theorems for such mappings in a complete metric space endowed with a directed graph.

Motivated by the work of Klanarong and Suantai [17], generalized theorems on the existence of the best proximity point for G-proximal Geraghty type M non-self mappings defined on a subset of a complete metric space are presented. Moreover, we also show that other theorems of the best proximity coincidence points for the case of metric spaces endowed with symmetric binary relations can be deduced from our main theorem.

2. Preliminaries and Definitions

Let (A, B) be a pair of nonempty subsets of a metric space (X, d). We adopt the following notations:

$$\begin{aligned} &d(A, B) := \inf\{d(a, b) : a \in A, b \in B\};\\ &A_0 := \{a \in A : \text{ there exists } b \in B \text{ such that } d(a, b) = d(A, B)\};\\ &B_0 := \{b \in B : \text{ there exists } a \in A \text{ such that } d(a, b) = d(A, B)\}. \end{aligned}$$

Definition 2.1 ([1, 11])**.** Let $T : A \to B$ and $g : A \to A$ be mappings.

(1) An element x^* is said to be **a best proximity point** of T if $d(x^*, Tx^*) = d(A, B)$;

(2) An element x^* is said to be **a best proximity coincidence point** of the pair (T, g) if $d(gx^*, Tx^*) = d(A, B)$.

Definition 2.2 ([15])**.** Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty. Then the pair (A, B) is said to have **the weak P-property** iff $d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Longrightarrow d(x_1, x_2) \le d(y_1, y_2)$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Suppose now that (X, d) is a metric space, and Δ denotes the diagonal of $X \times X$. We construct a directed graph $G = (V(G), E(G))$ from X such that the set of vertices $V(G)$ consists of all elements in X and the set of edges $E(G)$ contains the diagonal Δ of $X \times X$. Moreover, we assume further throughout this work that $E(G)$ contains no parallel edges. A metric space (X, d) is said to be **endowed with a directed graph** G if all of the above mentioned properties hold.

Definition 2.3 ([16])**.** Let (X,d) be a metric space endowed with a directed graph G. We say that the set of edges of G, $E(G)$, satisfies **the transitivity property** if for all $x,y,z\in X$,

$$(x,z),(z,y)\in E(G)\Rightarrow (x,y)\in E(G).$$

Definition 2.4. [17] Let (X,d) be a metric space and $G=(V(G),E(g))$ is a directed graph such that $V(G)=X$. A mapping $T:A\to B$ is said to be G**-proximal edge preserving** if $(x_1,x_2)\in E(G)$ and $d(u_1,Tx_1)=d(u_2,Tx_2)=d(A,B)\Longrightarrow (u_1,u_2)\in E(G)$ for all $x_1,x_2,u_1,u_2\in A$.

3. Main Results

Definition 3.1. Let (X,d) be a metric space and $G=(V(G),E(G))$ a directed graph such that $V(G)=X$ and assume that $T:A\to B$ and $g:A\to A$ are mappings. Then the pair (T,g) is said to be G**-proximal edge preserving** if $(gx_1,gx_2)\in E(G)$ and $d(gu_1,Tx_1)=d(gu_2,Tx_2)=d(A,B)\Longrightarrow (gu_1,gu_2)\in E(G)$ for all $x_1,x_2,u_1,u_2\in A$.

Remark 3.2. It is obvious to see that if T is a G-proximal edge preserving mapping, then (T,g) is G-proximal edge preserving mapping, where $g=I$ is the identity mapping.

Definition 3.3. Suppose that (X,d) is a metric space and $G=(V(G),E(G))$ is a directed graph such that $V(G)=X$. Let $T:A\to B$ and $g:A\to A$ be mappings. Then the pair (T,g) is said to be a G**-proximal Geraghty type M mapping** if the following hold;

(1) (T,g) is G-proximal edge preserving;
(2) there exists $\theta\in\Theta$ such that for all $x,y,u,v\in A$ if $d(gu,Tx)=d(gv,Ty)=d(A,B)$ and $(gx,gy)\in E(G)$, then

$$d(Tx,Ty)\le\theta(M(x,y,u,v))M(x,y,u,v), \tag{3.1}$$

where $M(x,y,u,v)=\max\left\{d(gx,gy),d(gx,gu),d(gy,gv),\dfrac{d(gx,gv)+d(gy,gu)}{2}\right\}$.

Theorem 3.4. *Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d) endowed with a directed graph $G=(V(G),E(G))$ in which $E(G)$ has the transitive property. Let $T:A\to B$ and $g:A\to A$ be mappings such that a pair (T,g) is a G-proximal Geraghty type M mapping. Assume that A_0 is nonempty such that $A_0\subseteq g(A_0)$ and $g(A_0)$ is a closed subset of X. If the following assertions hold;*

(i) $T(A_0)\subseteq B_0$ and the pair (A,B) satisfies the weak P-property;
(ii) There exist elements $x,y\in A_0$ such that $d(gx,Ty)=d(A,B)$ and $(gy,gx)\in E(G)$;
(iii) For any sequence $\{gx_n\}$ in A if $gx_n\to gx^*$, for some $gx^*\in A$ and $(gx_n,gx_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, then there is a subsequence $\{gx_{n_k}\}$ such that $(gx_{n_k},gx^*)\in E(G)$ for all $k\in\mathbb{N}$.

Consequently there is $x^*\in A$ such that $d(gx^*,Tx^*)=d(A,B)$. Moreover if $(gx^*,gy^*)\in E(G)$ for all best proximity coincidence points $x^*,y^*\in A$, then $gx^*=gy^*$.

Proof. By assumption (ii), we take $x_0,x_1\in A$

$$d(gx_1,Tx_0)=d(A,B)\text{ and }(gx_0,gx_1)\in E(G). \tag{3.2}$$

Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ and (T, g) is G-proximal edge preserving, there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B) \text{ and } (gx_1, gx_2) \in E(G). \tag{3.3}$$

By repeating this process, we obtain a sequence $\{x_n\} \subseteq A_0$ and $\{gx_n\} \subseteq g(A_0)$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B) \text{ and } (gx_{n-1}, gx_n) \in E(G) \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

By applying (3.4) together with the weak P-property of (A, B), it leads to

$$d(gx_n, gx_{n+1}) \leq d(Tx_{n-1}, Tx_n) \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

In the next step, we will prove that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0.$$

which is useful for verifying that the sequence $\{gx_n\}$ is a Cauchy sequence.
Since (T, g) is a G-proximal Geraghty type M mapping, (3.4) and (3.5), we obtain

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \theta(M(x_{n-1}, x_n, x_n, x_{n+1}))M(x_{n-1}, x_n, x_n, x_{n+1}) \\ &< M(x_{n-1}, x_n, x_n, x_{n+1}), \text{ for all } n \geq 1, \end{aligned} \tag{3.6}$$

where

$$M(x_{n-1}, x_n, x_n, x_{n+1}) = \max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{d(gx_{n-1}, gx_{n+1})}{2}\}.$$

We now consider $M(x_{n-1}, x_n, x_n, x_{n+1})$ in three cases:

Case 1. If $M(x_{n-1}, x_n, x_n, x_{n+1}) = d(gx_{n-1}, gx_n)$, then according to (3.6), we can write

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \theta(d(gx_{n-1}, gx_n))d(gx_{n-1}, gx_n) \\ &< d(gx_{n-1}, gx_n) \text{ for all } n \geq 1. \end{aligned} \tag{3.7}$$

This implies that $d(gx_{n-1}, gx_n)$ is decreasing, then there exists $r \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = r \geq 0.$$

Suppose that $r > 0$ and by taking $n \to \infty$ in (3.7), we derive

$$1 \leq \lim_{n\to\infty} \theta(d(gx_{n-1}, gx_n)) \leq 1. \tag{3.8}$$

We can conclude that $\lim_{n\to\infty} \theta(d(gx_{n-1}, gx_n)) = 1$. By the definition of θ, we have

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = r = 0,$$

which is a contradiction. Thus $\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0$.

Case 2. If $M(x_{n-1}, x_n, x_n, x_{n+1}) = d(gx_n, gx_{n+1})$, then from (3.6), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \theta(d(gx_n, gx_{n+1}))d(gx_n, gx_{n+1}) \\ &< d(gx_n, gx_{n+1}) \text{ for all } n \geq 1, \end{aligned} \tag{3.9}$$

which is a contradiction.

Case 3. When $M(x_{n-1}, x_n, x_n, x_{n+1}) = \dfrac{d(gx_{n-1}, gx_{n+1})}{2}$ and by applying (3.6), we find that

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \theta\left(\frac{d(gx_{n-1}, gx_{n+1})}{2}\right) \frac{d(gx_{n-1}, gx_{n+1})}{2} \\ &\leq \theta\left(\frac{d(gx_{n-1}, gx_{n+1})}{2}\right) \left[\frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2}\right] \\ &< \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2} \text{ for all } n \geq 1. \end{aligned} \tag{3.10}$$

From (3.10), we have

$$d(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n) \text{ for all } n \geq 1. \tag{3.11}$$

This means that $d(gx_{n-1}, gx_n)$ is decreasing, then there exists $r \in \mathbb{R}$ such that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = r \geq 0$$

which also implies

$$\lim_{n\to\infty} \frac{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}{2} = r \geq 0.$$

By taking $n \to \infty$ in (3.10) and assuming $r > 0$, we have

$$1 \leq \lim_{n\to\infty} \theta\left(\frac{d(gx_{n-1}, gx_{n+1})}{2}\right) \leq 1.$$

From the definition of θ, it follows that

$$\lim_{n\to\infty} \frac{d(gx_{n-1}, gx_{n+1})}{2} = 0.$$

By using (3.10), we can deduce that

$$d(gx_n, gx_{n+1}) < \frac{d(gx_{n-1}, gx_{n+1})}{2}. \tag{3.12}$$

By letting $n \to \infty$ in (3.12), we therefore obtain

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = r = 0$$

which is a contradiction. Hence, it can be concluded that

$$\lim_{n\to\infty} d(gx_n, gx_{n-1}) = 0. \tag{3.13}$$

Here we will show that $\{gx_n\}$ is a Cauchy sequence. Suppose that $\{gx_n\}$ is not a Cauchy sequence. There exist $\epsilon > 0$ and both $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ are subsequences of $\{gx_n\}$. For all $k \in \mathbb{N}$ with $m(k) > n(k) \geq k$, this implies that

$$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon \tag{3.14}$$

Additionally, we can choose the smallest $m(k)$ satisfying (3.14) for each $k \in \mathbb{N}$ so that

$$d(gx_{m(k)-1}, gx_{n(k)}) < \epsilon.$$

By applying (3.14), we have

$$\begin{aligned} \epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \\ &\leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)}) \\ &< d(gx_{m(k)}, gx_{m(k)-1}) + \epsilon. \end{aligned} \tag{3.15}$$

By taking $k \to +\infty$ in (3.15) and using (3.13), we have

$$\lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) = \epsilon. \tag{3.16}$$

From (3.4), we find that

$$d(gx_{n(k)+1}, Tx_{n(k)}) = d(A, B) \text{ and } d(gx_{m(k)+1}, Tx_{m(k)}) = d(A, B). \tag{3.17}$$

By using the weak P-property, we have

$$d(gx_{n(k)+1}, gx_{m(k)+1}) \leq d(Tx_{n(k)}, Tx_{m(k)}).$$

Also, since $(gx_{n(k)}, gx_{n(k)+1}) \in E(G)$ where $E(G)$ has transitive property, it follows that $(gx_{n(k)}, gx_{m(k)}) \in E(G)$.

Due to the fact that (T, g) is a G-proximal Geraghty type M mapping, we obtain

$$\begin{aligned} &d(gx_{n(k)+1}, gx_{m(k)+1}) \\ &\leq d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \theta(M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}))M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M(x_{n(k)}, x_{m(k)}, &x_{n(k)+1}, x_{m(k)+1}) \\ &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gx_{n(k)}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{m(k)+1}), \\ &\qquad \frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2}\}. \end{aligned}$$

Let us consider $M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1})$ in four separate cases:

Case 1. In the case when $M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}) = d(gx_{n(k)}, gx_{m(k)})$ and according to (3.18), we therefore have

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &\leq d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \theta(d(gx_{n(k)}, gx_{m(k)}))d(gx_{n(k)}, gx_{m(k)}) \\ &< d(gx_{n(k)}, gx_{m(k)}). \end{aligned} \tag{3.19}$$

By taking $k \to \infty$ in (3.19) and applying (3.16), we derive

$$1 \leq \lim_{n\to\infty} \theta(d(gx_{n(k)}, gx_{m(k)})) \leq 1. \tag{3.20}$$

This results in $\lim_{n\to\infty} \theta(d(gx_{n(k)}, gx_{m(k)})) = 1$. By the definition of θ, we have

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \epsilon = 0,$$

in which it is a contradiction.

Case 2. When $M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}) = d(gx_{n(k)}, gx_{n(k)+1})$ and by exploiting (3.18), it leads to

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &\leq d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \theta(d(gx_{n(k)}, gx_{n(k)+1}))d(gx_{n(k)}, gx_{n(k)+1}) \\ &< d(gx_{n(k)}, gx_{n(k)+1}). \end{aligned} \tag{3.21}$$

By taking $k \to \infty$ in the inequality above along with (3.13), we obtain

$$\lim_{k\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \epsilon = 0,$$

a contradiction.

Case 3. When $M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}) = d(gx_{m(k)}, gx_{m(k)+1})$ together with (3.18), we have

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &\leq d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \theta(d(gx_{m(k)}, gx_{m(k)+1}))d(gx_{m(k)}, gx_{m(k)+1}) \\ &< d(gx_{m(k)}, gx_{m(k)+1}). \end{aligned} \tag{3.22}$$

By taking $k \to \infty$ in the inequality above along with (3.13), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \epsilon = 0,$$

which is also a contradiction.

Case 4. If

$$M(x_{n(k)}, x_{m(k)}, x_{n(k)+1}, x_{m(k)+1}) = \frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2},$$

then from (3.18), we have

$$\begin{aligned} &d(gx_{n(k)+1}, gx_{m(k)+1}) \\ &\leq d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \theta\left(\tfrac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2}\right)\tfrac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2} \\ &< \frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2}. \end{aligned} \tag{3.23}$$

The triangular inequality is utilized so that

$$\frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2}$$
$$\leq \frac{d(gx_{n(k)},gx_{m(k)})+d(gx_{m(k)},gx_{m(k)+1})+d(gx_{m(k)},gx_{n(k)})+d(gx_{n(k)},gx_{n(k)+1})}{2}. \tag{3.24}$$

By (3.13), this results in

$$\lim_{n\to\infty} \frac{d(gx_{n(k)},gx_{m(k)})+d(gx_{m(k)},gx_{m(k)+1})+d(gx_{m(k)},gx_{n(k)})+d(gx_{n(k)},gx_{n(k)+1})}{2}$$
$$= \epsilon. \tag{3.25}$$

By taking $k \to \infty$ in (3.23) and exploiting (3.24) and (3.25), we acquire

$$1 \leq \lim_{k\to\infty} \theta(\frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2}) \leq 1. \tag{3.26}$$

According to the property of θ, this leads to

$$\lim_{k\to\infty} \frac{d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)}, gx_{n(k)+1})}{2} = 0. \tag{3.27}$$

By taking $k \to \infty$ in (3.23) and using (3.16) and (3.27), we obtain

$$\lim_{k\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \epsilon = 0,$$

which is a contradiction.

Thus, the sequence $\{gx_n\}$ is a Cauchy sequence in the closed subset $g(A_0)$ of the complete metric space (X, d). Then there exists $x^* \in A_0$ such that

$$\lim_{k\to\infty} gx_n = gx^*. \tag{3.28}$$

Since $A_0 \subseteq g(A_0)$ and $T(A_0) \subseteq B_0$, it follows that there exists $a \in A_0$ such that

$$d(ga, Tx^*) = d(A, B). \tag{3.29}$$

By applying (3.4) and (3.28) along with the assumption (iii), there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $(gx_{n(k)}, gx^*) \in E(G)$ for all $k \in \mathbb{N}$. From (3.4), we have

$$d(gx_{n(k)+1}, Tx_{n(k)}) = d(A, B) \text{ for all } k \in \mathbb{N}. \tag{3.30}$$

By using (3.29) and (3.30) and (A, B) have the weak P-property, this implies that

$$d(gx_{n(k)+1}, ga) \leq d(Tx_{n(k)}, Tx^*). \tag{3.31}$$

Since $(gx_{n(k)}, gx^*) \in E(G)$, (3.29), (3.30), (3.31) and (T, g) is a G-proximal Geraghty type M mapping, we have

$$\begin{aligned} d(gx_{n(k)+1}, ga) &\leq d(Tx_{n(k)}, Tx^*) \\ &\leq \theta(M(x_{n(k)}, x^*, x_{n(k)+1}, a))M(x_{n(k)}, x^*, x_{n(k)+1}, a) \\ &< M(x_{n(k)}, x^*, x_{n(k)+1}, a), \text{ for all } k \geq 1, \end{aligned} \tag{3.32}$$

where

$$M(x_{n(k)}, x^*, x_{n(k)+1}, a) = \max\{d(gx_{n(k)}, gx^*), d(gx_{n(k)}, gx_{n(k)+1}), d(gx^*, ga), \tfrac{d(gx_{n(k)},ga)+d(gx^*,gx_{n(k)+1})}{2}\}. \tag{3.33}$$

By using (3.13) and (3.28) , it is rather straightforward to see that

$$\lim_{k\to\infty} M(x_{n(k)}, x^*, x_{n(k)+1}, a) = d(gx^*, ga) \geq 0. \tag{3.34}$$

We now suppose $d(gx^*, ga) > 0$. By letting $k \to \infty$ in (3.32), we derive

$$1 \leq \lim_{k\to\infty} \theta(M(x_{n(k)}, x^*, x_{n(k)+1}, a)) \leq 1. \tag{3.35}$$

This implies that

$$\lim_{k\to\infty} \theta(M(x_{n(k)}, x^*, x_{n(k)+1}, a)) = 1.$$

By property of θ, we have

$$\lim_{k\to\infty} M(x_{n(k)}, x^*, x_{n(k)+1}, a) = d(gx^*, ga) = 0$$

which is a contradiction and this results in $gx^* = ga$. Therefore, from (3.29) there exists $x^* \in A$ such that

$$d(gx^*, Tx^*) = d(A, B). \tag{3.36}$$

In the next step, x^* and y^* are supposed to be the best proximity coincidence points of (T, g) such that $(gx^*, gy^*) \in E(G)$. Also, the condition $gx^* \neq gy^*$ is presumed. Consequently, this makes

$$d(gx^*, gy^*) > 0 \text{ and } d(gx^*, Tx^*) = d(gy^*, Ty^*) = d(A, B).$$

By using the weak P- property, it can be concluded that

$$0 < d(gx^*, gy^*) \leq d(Tx^*, Ty^*).$$

Since (T, g) is a G-proximal Geraghty type M mapping and $M(x^*, y^*, x^*, y^*) = d(gx^*, gy^*)$, we obtain that

$$\begin{aligned} d(gx^*, gy^*) \leq d(Tx^*, Ty^*) &\leq \theta(M(x^*, y^*, x^*, y^*))M(x^*, y^*, x^*, y^*) \\ &= \theta(d(gx^*, gy^*))d(gx^*, gy^*) \\ &< d(gx^*, gy^*) \end{aligned}$$

which is a contradiction. This gives $gx^* = gy^*$. ■

Example 3.5. Let $X = \mathbb{R}^3$ equipped with the metric d given by

$$d((x, y, z), (u, v, w)) = \sqrt{(x-u)^2 + (y-v)^2 + (z-w)^2}.$$

Let

$$A = \{(1, y, 2) : 0 \leq y \leq 6\}$$

and

$$B = \{(1, y, -2) : 0 \leq y \leq 3\}.$$

It is simple to see that the pair (A, B) satisfies the weak P-property when $d(A, B) = 4$,

$$A_0 = \{(1, y, 2) : 0 \leq y \leq 3\} \text{ and } B_0 = \{(1, y, -2) : 0 \leq y \leq 3\}.$$

Let $T : A \to B$ be a mapping defined by

$$T(1, y, 2) = (1, \frac{y^2}{12}, -2), \text{ for all } (1, y, 2) \in A,$$

and $g : A \to A$ such that

$$g(1, y, 2) = \begin{cases} (1, \frac{7y}{6}, 2) & \text{if } 0 \leq y \leq \frac{36}{7} \\ \\ (1, 0, 2) & \text{if } \frac{36}{7} < y \leq 6 \end{cases}$$

We have

$$A_0 = \{(1, y, 2) : 0 \leq y \leq 3\} \subseteq g(A_0) = \{(1, y, 2) : 0 \leq y \leq \frac{7}{2}\}.$$

Define a directed graph $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{((x, y, z), (u, v, w)) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \leq u, y \geq v \text{ and } z \leq w\}.$$

Hence $E(G)$ is transitive and

$$T(A_0) = \{(1, y, -2) : 0 \leq y \leq \frac{9}{12}\} \subseteq B_0 = \{(1, y, -2) : 0 \leq y \leq 3\}.$$

We will show the pair (T, g) is G-proximal Geraghty type M mapping. First justifying that (T, g) is G-proximal edge preserving.

Let $(1, x, 2), (1, y, 2), (1, u, 2), (1, v, 2) \in A$ such that

$$(g(1, x, 2), g(1, y, 2)) = ((1, \frac{7x}{6}, 2), (1, \frac{7y}{6}, 2)) \in E(G)$$

and

$$d(g(1, u, 2), T(1, x, 2)) = d(A, B) = d(g(1, v, 2), T(1, y, 2)).$$

Then

$$x \geq y \text{ and } d((1, \frac{7u}{6}, 2), (1, \frac{x^2}{12}, -2)) = d(A, B) = d((1, \frac{7v}{6}, 2), (1, \frac{y^2}{12}, -2)).$$

This implies that

$$\frac{7u}{6} = \frac{x^2}{12} \text{ and } \frac{7v}{6} = \frac{y^2}{12}.$$

Since $x \geq y$ and $x, y \in [0, 6]$, we have

$$\frac{7u}{6} \geq \frac{7v}{6}.$$

Thus

$$(g(1, u, 2), g(1, v, 2)) = ((1, \frac{7u}{6}, 2), (1, \frac{7v}{6}, 2)) \in E(G).$$

This means that (T, g) is G-proximal edge preserving.

Subsequently we also note that there is $\theta \in \Theta$ such that $\theta(t) = \dfrac{6}{7}$ for all $t \in [0, \infty)$. We then have

$$\begin{aligned}
&d(T(1,x,2), T(1,y,2)) \\
&= d((1, \frac{x^2}{12}, -2), (1, \frac{y^2}{12}, -2)) \\
&= \left| \frac{x^2}{12} - \frac{y^2}{12} \right| \\
&= \frac{6}{7} \frac{|x+y|}{12} \left| \frac{7x}{6} - \frac{7y}{6} \right| \\
&\leq \frac{6}{7} \left| \frac{7x}{6} - \frac{7y}{6} \right| \\
&= \theta(M((1,x,2),(1,y,2),(1,u,2),(1,v,2)))d((1, \frac{7x}{6}, 2), (1, \frac{7y}{6}, 2)) \\
&\leq \theta(M((1,x,2),(1,y,2),(1,u,2),(1,v,2)))M((1,x,2),(1,y,2),(1,u,2),(1,v,2)).
\end{aligned}$$

which leads to the conclusion that the pair (T, g) is G-proximal Geraghty type M mapping. We now show that the assumption (iii) in Theorem 3.4 holds.

Let $\{g(1, x_n, 2)\} = \{(1, \frac{7x_n}{6}, 2)\}$ be a sequence in A and $(1, \frac{7x^*}{6}, 2) \in A$ such that

$$\lim_{n\to\infty} (1, \frac{7x_n}{6}, 2) = (1, \frac{7x^*}{6}, 2) \text{ and } ((1, \frac{7x_n}{6}, 2), (1, \frac{7x_{n+1}}{6}, 2)) \in E(G).$$

Hence $\dfrac{7x_n}{6} \geq \dfrac{7x_{n+1}}{6}$ which implies that $\{x_n\}$ is non-increasing and

$$\lim_{n\to\infty} x_n = x^* = \inf\{x_n : n \in \mathbb{N}\}.$$

Thus $x_n \geq x^*$ for all $n \in \mathbb{N}$, then there is a subsequence $\{g(1, x_{n(k)}, 2)\}$ with

$$(g(1, x_{n(k)}, 2), g(1, x^*, 2)) = ((1, \frac{7x_{n(k)}}{6}, 2), (1, \frac{7x^*}{6}, 2)) \in E(G)$$

for all $k \in \mathbb{N}$.

By Theorem 3.4 we can conclude that there is $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$ and $(1, 0, 2)$ is the best proximity coincidence point of the pair (T, g).

4. Consequence

This section presents several outcomes deriving from our main result in the previous section.

Definition 4.1. Suppose that (X, d) is a metric space and $G = (V(G), E(G))$ is a directed graph such that $V(G) = X$. Let $T : A \to B$ and $g : A \to A$ be mappings. Then the pair (T, g) is said to be a G**-proximal mapping** if the following hold;

(1) (T, g) is G-proximal edge preserving;
(2) there exists $k \in [0, 1)$ such that for all $x, y, u, v \in A$ if $d(gu, Tx) = d(gv, Ty) = d(A, B)$ and $(gx, gy) \in E(G)$, then

$$d(Tx, Ty) \leq kM(x, y, u, v), \tag{4.1}$$

where $M(x,y,u,v) = \max\left\{d(gx,gy), d(gx,gu), d(gy,gv), \dfrac{d(gx,gv)+d(gy,gu)}{2}\right\}$.

By putting $\theta(t) = k$, where $k \in [0,1)$ in Theorem 3.4, the corollary given as follows.

Corollary 4.2. *Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d) endowed with a directed graph $G=(V(G),E(G))$ in which $E(G)$ has the transitive property. Let $T : A \to B$ and $g : A \to A$ be mappings such that a pair (T,g) is a G-proximal mapping. Assume that A_0 is nonempty such that $A_0 \subseteq g(A_0)$ and $g(A_0)$ is a closed subset of X. If the following assertions hold;*

(i) $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the weak P-property;
(ii) There exist elements $x,y \in A_0$ such that $d(gx,Ty) = d(A,B)$ and $(gy,gx) \in E(G)$;
(iii) For any sequence $\{gx_n\}$ in A if $gx_n \to gx^*$, for some $gx^* \in A$ and $(gx_n, gx_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{gx_{n_k}\}$ such that $(gx_{n_k}, gx^*) \in E(G)$ for all $k \in \mathbb{N}$.

Consequently there is $x^* \in A$ such that $d(gx^*,Tx^*) = d(A,B)$. Moreover if $(gx^*,gy^*) \in E(G)$ for all best proximity coincidence points $x^*,y^* \in A$, then $gx^* = gy^*$.

Definition 4.3. Suppose that (X,d) is a metric space and $G=(V(G),E(G))$ is a directed graph such that $V(G) = X$. Let $T : A \to B$ and $g : A \to A$ be mappings. Then the pair (T,g) is said to be a ***G*-proximal type R mapping** if the following hold;

(1) (T,g) is G-proximal edge preserving;
(2) for all $x,y,u,v \in A$ if $d(gu,Tx) = d(gv,Ty) = d(A,B)$ and $(gx,gy) \in E(G)$, then

$$d(Tx,Ty) \leq \frac{M(x,y,u,v)}{M(x,y,u,v)+1}, \tag{4.2}$$

where $M(x,y,u,v) = \max\left\{d(gx,gy), d(gx,gu), d(gy,gv), \dfrac{d(gx,gv)+d(gy,gu)}{2}\right\}$.

Applying $\theta(t) = \dfrac{1}{t+1}$ in Theorem 3.4, we obtain the second corollary as follows.

Corollary 4.4. *Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d) endowed with a directed graph $G=(V(G),E(G))$ in which $E(G)$ has the transitive property. Let $T : A \to B$ and $g : A \to A$ be mappings such that a pair (T,g) is a G-proximal type R mapping. Assume that A_0 is nonempty such that $A_0 \subseteq g(A_0)$ and $g(A_0)$ is a closed subset of X. If the following assertions hold;*

(i) $T(A_0) \subseteq B_0$ and the pair (A,B) satisfies the weak P-property;
(ii) There exist elements $x,y \in A_0$ such that $d(gx,Ty) = d(A,B)$ and $(gy,gx) \in E(G)$;
(iii) For any sequence $\{gx_n\}$ in A if $gx_n \to gx^*$, for some $gx^* \in A$ and $(gx_n, gx_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{gx_{n_k}\}$ such that $(gx_{n_k}, gx^*) \in E(G)$ for all $k \in \mathbb{N}$.

Consequently there is $x^* \in A$ such that $d(gx^*,Tx^*) = d(A,B)$. Moreover if $(gx^*,gy^*) \in E(G)$ for all best proximity coincidence points $x^*,y^* \in A$, then $gx^* = gy^*$.

5. Applications

We will apply our result on the best proximity coincidence point on a metric space endowed with a symmetric binary relation R. Some properties are given below:

Definition 5.1. Let $T : A \to B$ and $g : A \to A$ be mappings. The pair (T, g) is said to be **a proximal comparative mapping** if $gxRgy$ and $d(gu_1, Tx) = d(gu_2, Ty) = d(A, B)$ $\Longrightarrow gu_1 R gu_2$ for all $x, y, u_1, u_2 \in A$.

Definition 5.2. Let $T : A \to B$ and $g : A \to A$ be mappings. The pair (T, g) is said to be **proximal comparative Geraghty type M mapping** if the following hold;

(1) The pair (T, g) is a proximal comparative mapping;
(2) There exists $\theta \in \Theta$ such that for all $x, y, u, v \in A$ if $d(gu, Tx) = d(gv, Ty) = d(A, B)$ and $gxRgy$, then

$$d(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v), \tag{5.1}$$

where $M(x, y, u, v) = \max\left\{d(gx, gy), d(gx, gu), d(gy, gv), \frac{d(gx, gv) + d(gy, gu)}{2}\right\}$.

Corollary 5.3. *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) endowed with R be a symmetric binary relation over X in which R has the transitive property. let $T : A \to B$ and $g : A \to A$ be mapping such that the pair (T, g) is a proximal comparative Geraghty type M mapping. Assume that A_0 is nonempty such that $A_0 \subseteq g(A_0)$ and $g(A_0)$ is a closed subset of X. If the following assertions hold;*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property;
(ii) There exist elements $x, y \in A_0$ such that $d(gx, Ty) = d(A, B)$ and $gyRgx$;
(iii) $\{gx_n\}$ is a sequence in A, if $gx_n \to gx^*$, for some $gx^* \in A$ and $gx_n R gx_{n+1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{gx_{n_k}\}$ with $gx_{n_k} R gx^*$ for all $k \in \mathbb{N}$.

Consequently there is $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$. Moreover if $gx^* R gy^*$ for all best proximity coincidence points $x^*, y^* \in A$, then $gx^* = gy^*$.

Proof. Let $T : A \to B$ and $g : A \to A$ be mappings.

Define a directed graph $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : gxRgy\}.$$

It is apparent that the hypotheses of Theorem 3.4 hold which implies that there is $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$.

What follows is we let $x^*, y^* \in A$ be two best proximity coincidence points of the pair (T, g). This gives $gx^* R gy^*$ which implies that $(gx^*, gy^*) \in E(G)$ according to Theorem 3.4. Ultimately, it leads to $gx^* = gy^*$.

∎

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