Thai Journal of Mathematics
Volume 20 Number 1 (2022)
Pages 315–321

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Quick Look at the Stability of the New Generalized Linear Functional Equation

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Abstract In this paper, we show that the stability of the generalized linear functional equation introduced by Aiemsomboon and Sintunavarat [L. Aiemsomboon, W. Sintunavarat, Stability of the new generalized linear functional equation in normed spaces via the fixed point method in generalized metric spaces, Thai J. Math. 16 (2018) 113–124] follows easily from the well-known results of Găvruţa [P. Găvruţa, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mapping, J. Math. Anal. Appl. 184 (1994) 431–436] and Jung [S.M. Jung, On the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 204 (1996) 221–226, S.M. Jung, Stability of the quadratic equation of Pexider type, Abh. Math. Sem. Unniv. Hamburg. 70 (2000) 175–190]. Moreover, we show that the new upper bound of our estimate is not only better than the ones proposed by Aiemsomboon and Sintunavarat, but also sharp at least some particular functions.

MSC: 39B72; 47H09

Keywords: generalized linear functional equation; stability result

Submission date: 02.10.2019 / Acceptance date: 02.11.2021

1. Introduction

In 1940, Ulam [1] raised an interesting problem concerning the stability of homomorphisms between two groups. Some mathematicians tried to solve this problem for a normed space. Given two normed spaces X and Y, a function $f : X \to Y$ is said to satisfy the *additive functional equation* if

$$f(x+y) - f(x) - f(y) = 0 \quad \text{for all } x, y \in X. \tag{1.1}$$

Hyers [2] gave a partial answer to this problem on a Banach space. Moreover, Rassias [3], Aoki [4], and Gajda [5] generalized the results of Hyers. The technique used by Hyers' proof has been known as the *direct method*. The following stability result of the additive functional equation was given by Găvruţa [6] and Jung [7, 8] (see also [9, Corollary 2.19]).

Theorem GJ ([6–9]). *Let X be a normed space and Y a Banach space. Suppose that $\varphi : X \times X \to [0, \infty)$ is a function such that one of the following conditions is satisfied:*

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(1) $\widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$ *for all* $x, y \in X$;
(2) $\widetilde{\varphi}(x,y) := \sum_{k=1}^{\infty} 2^{k}\varphi(2^{-k} x, 2^{-k} y) < \infty$ *for all* $x, y \in X$.

Suppose that $g : X \to Y$ *is a function satisfying*

$$\|g(x+y) - g(x) - g(y)\| \leq \varphi(x,y) \quad \text{for all } x, y \in X.$$

Then there exists a unique function $G : X \to Y$ *satisfying the additive functional equation and*

$$\|G(x) - g(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,x) \quad \text{for all } x \in X.$$

Recently, Aiemsomboon and Sintunavarat [10] introduced the concept of the generalized linear functional equation (see Definition 1.1) on a normed space. By using the fixed point method, they also proved some stability results.

We first recall the following definition.

Definition 1.1 ([10, Definition 3.1]). Let X and Y be two normed spaces. We say that a function $f : X \to Y$ satisfies the *generalized linear functional equation* if

$$2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y) = 0 \quad \text{for all } x, y \in X. \tag{1.2}$$

Remark 1.2. If f satisfies (1.1), then it satisfies (1.2).

For convenience, we assume the following notations:

- X is a normed space and Y is a Banach space;
- $D_f(x,y) := 2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y)$ where $f : X \to Y$ is an arbitrary function and $x, y \in X$.

The following stability results of the generalized linear functional equation were proved by Aiemsomboon and Sintunavarat [10].

Theorem AS1 ([10, Theorem 3.18]). *Suppose that* $L \in (0,1)$ *and* $\rho : X \times X \to [0, \infty)$. *If* ρ *satisfies*

$$\rho(x,y) \leq 2L\rho\left(\frac{x}{2}, \frac{y}{2}\right) \quad \text{for all } x, y \in X$$

and $f : X \to Y$ *satisfies*

$$\|D_f(x,y)\| \leq \rho(x,y) \quad \text{for all } x, y \in X,$$

then there exists a unique function $F : X \to Y$ *satisfying the generalized linear functional equation and*

$$\|F(x) - f(x)\| \leq \frac{1}{2-2L}\rho(x,x) \quad \text{for all } x \in X. \tag{1.3}$$

Theorem AS2 ([10, Theorem 3.19]). *Suppose that* $L \in (0,1)$ *and* $\rho : X \times X \to [0, \infty)$. *If* ρ *satisfies*

$$\rho(x,y) \leq \frac{L}{2}\rho(2x, 2y) \quad \text{for all } x, y \in X$$

and $f : X \to Y$ *satisfies*

$$\|D_f(x,y)\| \leq \rho(x,y) \quad \text{for all } x, y \in X,$$

then there exists a unique function $F : X \to Y$ *satisfying the generalized linear functional equation and*

$$\|F(x) - f(x)\| \leq \frac{L}{2-2L}\rho(x,x) \quad \text{for all } x \in X. \tag{1.4}$$

The purpose of this paper is to give a quick look at Theorems AS1 and AS2 via Theorem GJ. We not only obtain Theorems AS1 and AS2, but also improve the bounds of (1.3) and (1.4). Moreover, we show that our bounds are sharp at least for some particular functions.

2. Main Results

2.1. A General Solution of the Generalized Linear Functional Equation

We characterize a general solution of the generalized linear functional equation (1.2) on a normed space.

Proposition 2.1. *Suppose that $f : X \to Y$ is a function. Then the following statements are equivalent.*

(1) *f satisfies the generalized linear functional equation.*
(2) *$f - f(0)$ satisfies the additive functional equation.*

Proof. Define $g := f - f(0)$. We first note that $g(0) = 0$.

(1)⇒(2). Suppose that f satisfies the generalized linear functional equation. Then so does g, that is, $D_g(x, y) = 0$ for all $x, y \in X$. Put $y := 0$ and hence we have

$$2g(x) + g(x) + g(-x) - 2g(x) - 2g(0) = 0.$$

So $g(-x) = -g(x)$ for all $x \in X$. Let $x, y \in X$ be given. We see that

$$\begin{aligned}2g(x+y) - 2g(x) - 2g(y) &= 2(f(x+y) - f(0)) - 2(f(x) - f(0)) - 2(f(y) - f(0)) \\ &= 2f(x+y) - f(x) - 2f(y) + 2f(0) \\ &= 2f(x+y) + g(x-y) + g(y-x) - 2f(x) - 2f(y) + 2f(0) \\ &= D_f(x, y) = 0.\end{aligned}$$

This shows that g satisfies the additive functional equation.

(2)⇒(1). Suppose that g satisfies the additive functional equation. Let $x, y \in X$ be given. We have

$$\begin{aligned}D_f(x, y) &= 2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y) \\ &= 2g(x+y) + g(x-y) + g(y-x) - 2g(x) - 2g(y) \\ &= 2(g(x+y) - g(x) - g(y)) = 0.\end{aligned}$$

This completes the proof. ∎

Remark 2.2. The preceding result holds if Y is a normed space.

As a consequence of Proposition 2.1, we obtain the following result.

Corollary 2.3. *A function $f : X \to Y$ satisfies (1.2) if and only if f is of the form*

$$f = A + c$$

where $A : X \to Y$ satisfies the additive functional equation and $c \in Y$ is a constant.

2.2. Stability Results

We now state our two results obtained directly from Theorem GJ and our Proposition 2.1.

Theorem 2.4. *Suppose that* $\rho : X \times X \to [0, \infty)$ *satisfies* $\rho(0,0) = 0$ *and*

$$\widetilde{\rho}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\rho(2^k x, 2^k y) < \infty \quad \textit{for all} x, y \in X.$$

Suppose that $f : X \to Y$ *satisfies*

$$\|D_f(x,y)\| \le \rho(x,y) \quad \textit{for all } x, y \in X. \tag{2.1}$$

Then there exists a unique function $F : X \to Y$ *satisfying the generalized linear functional equation and*

$$\|F(x) - f(x)\| \le \frac{1}{4}\widetilde{\rho}(x,x) \quad \textit{for all } x \in X. \tag{2.2}$$

Proof. Define $g : X \to Y$ by

$$g(x) := f(x) - f(0) \quad \text{for all } x, y \in X.$$

We note that $g(0) = 0$. It is easy to see that

$$\|D_g(x,y)\| \le \rho(x,y) \quad \text{for all } x, y \in X. \tag{2.3}$$

By letting $y = 0$ in (2.3), we have

$$\|g(x) + g(-x)\| \le \rho(x,0) \quad \text{for all } x \in X.$$

We define $\varphi : X \times X \to [0, \infty)$ by

$$\varphi(x,y) := \frac{1}{2}\left(\rho(x,y) + \rho(x-y,0)\right) \quad \text{for all } x, y \in X.$$

Let $x, y \in X$ be given. Then we have

$$\begin{aligned} \|2g(x+y) - 2g(x) - 2g(y)\| &\le \|D_g(x,y)\| + \|g(x-y) + g(y-x)\| \\ &\le \rho(x,y) + \rho(x-y,0) \\ &= 2\varphi(x,y). \end{aligned}$$

In particular,

$$\|g(x+y) - g(x) - g(y)\| \le \varphi(x,y) \quad \text{for all } x, y \in X.$$

We see that

$$\widetilde{\varphi}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) = \frac{1}{2}\left(\widetilde{\rho}(x,y) + \widetilde{\rho}(x-y,0)\right) \quad \text{for all } x, y \in X.$$

By Theorem GJ, there exists a unique function $G : X \to Y$ satisfying the additive functional equation and

$$\|G(x) - g(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x) = \frac{1}{4}\left(\widetilde{\rho}(x,x) + \widetilde{\rho}(0,0)\right) = \frac{1}{4}\widetilde{\rho}(x,x) \quad \text{for all } x \in X.$$

Define $F : X \to Y$ by $F := G + f(0)$. It follows from Proposition 2.1 that F satisfies the generalized linear functional equation and we see that

$$\|F(x) - f(x)\| = \|G(x) - g(x)\| \le \frac{1}{4}\widetilde{\rho}(x,x) \quad \text{for all } x \in X.$$

This completes the proof. ∎

The proof of the following result is quite similar to that of Theorem 2.4. Hence the proof is omitted.

Theorem 2.5. *Suppose that* $\rho : X \times X \to [0, \infty)$ *satisfies* $\rho(0,0) = 0$ *and*

$$\widetilde{\rho}(x, y) := \sum_{k=1}^{\infty} 2^k \rho(2^{-k}x, 2^{-k}y) < \infty \quad \textit{for all } x, y \in X.$$

Suppose that $f : X \to Y$ *satisfies*

$$\|D_f(x, y)\| \leq \rho(x, y) \quad \textit{for all } x, y \in X. \tag{2.4}$$

Then there exists a unique function $F : X \to Y$ *satisfying the generalized linear functional equation and*

$$\|F(x) - f(x)\| \leq \frac{1}{4}\widetilde{\rho}(x, x) \quad \textit{for all } x \in X. \tag{2.5}$$

As a consequence of our main theorems, we obtain the following stability results related to Theorems AS1 and AS2.

Corollary 2.6. *Suppose that* $L \in (0, 1)$ *and* $\rho : X \times X \to [0, \infty)$ *satisfies*

$$\rho(x, y) \leq 2L\rho\left(\frac{x}{2}, \frac{y}{2}\right) \quad \textit{for all } x, y \in X.$$

Suppose that $f : X \to Y$ *satisfies*

$$\|D_f(x, y)\| \leq \rho(x, y) \quad \textit{for all } x, y \in X.$$

Then there exists a unique function $F : X \to Y$ *satisfying the generalized linear functional equation and*

$$\|F(x) - f(x)\| \leq \frac{1}{4 - 4L}\rho(x, x) \quad \textit{for all } x \in X.$$

Proof. We define $\rho_* : X \times X \to [0, \infty)$ by

$$\rho_*(x, y) := \begin{cases} \rho(x, y) & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For each pair $x, y \in X$, we note that

- $\rho_*(x, y) \leq \rho(x, y)$;
- $\rho_*(x, y) \leq 2L\rho_*(\frac{x}{2}, \frac{y}{2})$;
- $\|D_f(x, y)\| \leq \rho_*(x, y)$.

It follows that

$$\widetilde{\rho_*}(x, y) := \sum_{k=0}^{\infty} 2^{-k}\rho_*(2^k x, 2^k y) \leq \frac{1}{1 - L}\rho_*(x, y) \quad \text{for all } x, y \in X.$$

By Theorem 2.4, there exists a unique function $F : X \to Y$ satisfying the generalized linear functional equation such that

$$\|F(x) - f(x)\| \leq \frac{1}{4}\widetilde{\rho_*}(x, x) \leq \frac{1}{4 - 4L}\rho(x, x) \quad \text{for all } x \in X.$$

This completes the proof. ∎

Corollary 2.7. *Suppose that $L \in (0,1)$ and $\rho : X \times X \to [0,\infty)$ satisfies*

$$\rho(x,y) \leq \frac{L}{2}\rho(2x,2y) \quad \textit{for all } x,y \in X.$$

Suppose that $f : X \to Y$ satisfies

$$\|D_f(x,y)\| \leq \rho(x,y) \quad \textit{for all } x,y \in X.$$

Then there exists a unique function $F : X \to Y$ satisfying the generalized linear functional equation and

$$\|F(x) - f(x)\| \leq \frac{L}{4-4L}\rho(x,x) \quad \textit{for all } x \in X.$$

Remark 2.8. Our Corollary 2.6 improves Theorem AS1 because

$$\frac{1}{4-4L}\rho(x,x) \leq \frac{1}{2-2L}\rho(x,x).$$

Similarly, our Corollary 2.7 improves Theorem AS2 because

$$\frac{L}{4-4L}\rho(x,x) \leq \frac{L}{2-2L}\rho(x,x).$$

2.3. Illustrative Examples

The upper bounds in (2.2) and (2.5) are *optimal* at least for some particular functions as shown in the following two examples.

Example 2.9 (The bound in (2.2) is optimal)**.** Let $X = Y := \mathbb{R}$ and $f : X \to Y$ be a function defined by

$$f(x) := \sqrt[3]{x} \quad \text{for all } x \in X.$$

It is easy to see that

$$|2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y)| = \rho(x,y) \quad \text{for all } x,y \in X,$$

where $\rho : X \times X \to [0,\infty)$ is defined by

$$\rho(x,y) := 2|\sqrt[3]{x+y} - \sqrt[3]{x} - \sqrt[3]{y}| \quad \text{for all } x,y \in X.$$

For each $x,y \in X$, we see that

$$\widetilde{\rho}(x,y) := \sum_{k=0}^{\infty} 2^{-k}\rho(2^k x, 2^k y) = \left(\sum_{k=0}^{\infty} 2^{-2k/3}\right)\rho(x,y) = \left(\frac{2}{2-\sqrt[3]{2}}\right)\rho(x,y).$$

In particular, for $x \in X$,

$$\widetilde{\rho}(x,x) = \left(\frac{2}{2-\sqrt[3]{2}}\right)\rho(x,x) = 4|\sqrt[3]{x}|.$$

We note that $F(x) = 0$ for all $x \in X$ and

$$|F(x) - f(x)| = |\sqrt[3]{x}| = \frac{1}{4}\widetilde{\rho}(x,x) \quad \text{for all } x \in X.$$

Example 2.10 (The bound in (2.5) is optimal)**.** Let $X = Y := \mathbb{R}$ and $f : X \to Y$ be a function defined by

$$f(x) := x^2 \quad \text{for all } x \in X.$$

We see that

$$|2f(x+y) + f(x-y) + f(y-x) - 2f(x) - 2f(y)| = \rho(x,y) \quad \text{for all } x, y \in X,$$

where $\rho : X \times X \to [0, \infty)$ is defined by

$$\rho(x,y) := 2(x^2 + y^2) \quad \text{for all } x, y \in X.$$

For each $x, y \in X$, it is easy to see that

$$\widetilde{\rho}(x,y) := \sum_{k=1}^{\infty} 2^k \rho(2^{-k}x, 2^{-k}y) = \left(\sum_{k=1}^{\infty} 2^{-k}\right) \rho(x,y) = \rho(x,y).$$

So we have

$$\widetilde{\rho}(x,x) = \rho(x,x) = 4x^2 \quad \text{for all } x \in X.$$

We note that $F(x) = 0$ for all $x \in X$. It follows that

$$|F(x) - f(x)| = x^2 = \frac{1}{4}\widetilde{\rho}(x,x) \quad \text{for all } x \in X.$$

Acknowledgements

JS would like to thank the Development and Promotion for Science and Technology talents project (DPST) for the financial support to this paper. SS is supported by Department of Mathematics, Faculty of Science, Khon Kaen University, Fiscal Year 2022.

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