Thai Journal of Mathematics
Volume 20 Number 1 (2022)
Pages 305–313
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Boundary Value Problems in Normed Fuzzy Spaces

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Abstract In this paper, we study fuzzy differential equations (FDE) in the setting of normed spaces by introducing new definitions and extending the H-differentiability to normed spaces. This concept stands on the expansion of the class of differentiable fuzzy mappings and, for this, the lateral Hukuhara derivatives should be propounded. We will see that both derivatives are different and they lead us to different solutions from a FDE. Some illustrative examples are also given and some comparisons with other methods for solving FDE are made.

MSC: 39A10

Keywords: normed space; fuzzy set; H-difference; fuzzy measurable mapping; initial fuzzy value problem; Banach's contraction

Submission date: 25.08.2019 / Acceptance date: 19.11.2021

1. Introduction

Fuzzy set theory is applied for modeling uncertainty data and for processing ambiguous or subjective information in mathematical models. Their main objective of development have been diverse and its applications appear in the very varied real problems, for instance, in the golden mean [1], particle systems [2], quantum optics and gravity [3], synchronize hyper chaotic systems [4], chaotic system [5–7], medicine [8, 9], and engineering problems [10]. Particularly, fuzzy differential equation is a very important topic from a theoretical point of view (see [11–17]) as well as of their applications, for example, in population models [18, 19], civil engineering [20] and in modeling hydraulic [21]. Initially, the derivative for fuzzy valued mappings was developed by Puri and Ralescu [16], that generalized

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Published by The Mathematical Association of Thailand.

and extended the concept of Hukuhara differentiability (H-derivative) for set-valued mappings to the class of fuzzy mappings. Subsequently, using the H-derivative, Kaleva [13] started to develop a theory for FDE. In the last few years, many works have been done by several authors in theoretical and applied fields (see [11, 13, 15, 17, 21–23]). In some cases this approach suffers certain disadvantages since the diameter $diam(x(t))$ of the solution is unbounded as time t increases [11, 12]. This problem demonstrates that this interpretation is not a good generalization of the associated crisp case and we assume that this problem is due to the fuzzification of the derivative utilized in the formulation of the FDE. In this direction, Bede and Gal in [24, 25] introduced a more general definition of derivative for fuzzy mappings enlarging the class of differentiable fuzzy mappings. Following this idea, in this paper we define the fuzzy lateral H-derivative for a fuzzy mapping $G : I = (a, b) \to F(X)$ where X is a normed space. We give some nice properties of it and we finally, by using Banach's contraction theorem, provide an existence result to the initial fuzzy value problem.

2. Basic Concepts

Let $K(X)$ denote the family of all nonempty compact subsets of a normed space X. If $A, B \in K(X)$, then we define the operations of addition and scalar multiplication as

$$A + B = \{a + b \mid a \in A, b \in B\}, \quad \lambda A = \{\lambda a \mid a \in A\}.$$

It is easy to verify that $K(X)$ satisfies all axioms of being a vector space except having additive inverse. Note that $A + \{\theta\} = A$, for each $A \in K(X)$ where θ denotes the zero vector of X.

If $A \in K(X)$ we define the $\varepsilon-$neighborhood of A as the set

$$N(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$ and $\|.\|$ the norm on X.

Remark 2.1. It is clear that $A \subset N(A, \varepsilon)$. If $\{x_n\}$ is also a sequence in $N(A, \varepsilon)$ with $x_n \to x$, then $d(x, A) \leq \varepsilon$. Hence, the closure of $N(A, \varepsilon)$ (denoted by $\overline{N(A, \varepsilon)}$) is a subset of $\{x \in X \mid d(x, A) \leq \varepsilon\}$. It is well known that the mapping $x \to d(x, A)$ is continuous and so the set $N(A, \varepsilon)$ is an open subset of X while the set $\{x \in X \mid d(x, A) \leq \varepsilon\}$ is closed. Hence, if A is a nonempty compact subset of X and $\theta \notin A$, then since X is connected (note that each normed space is connected), $N(A, \varepsilon) \subsetneqq \{x \in X \mid d(x, A) \leq \varepsilon\}$. Finally, it is obvious that, for each $\varepsilon > 0$,

$$X = \bigcup_{A \in K(X)} N(A, \varepsilon)$$

and for each $A \in K(X)$,

$$N(A, \varepsilon) = \bigcup_{n=1}^{\infty} N(A, r_n),$$

where $\{r_n\}_{n=1}^{\infty}$ is a dense subset of $[0, \varepsilon]$. Consequently, the family

$$\{N(A, \varepsilon) \mid A \in K(X), \varepsilon > 0\}$$

is a base for a topology on X.

The Hausdorff separation $\rho(A,B)$ of $A,B \in K(X)$ is defined by

$$\rho(A,B) = \inf\{\varepsilon > 0 \mid A \subset N(\varepsilon, B)\}$$

and the Hausdorff metric on $K(X)$ is defined by

$$h(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

Remark that, for each $(A,B) \in K(X) \times K(X)$, we have $\rho(A,B) \in [0,\infty)$. Because otherwise, for all natural number n, there exists $x_n \in A \backslash N(n,B)$. Hence, $d(x_n, B) \geq n$ and since B is compact there exists $y_n \in B$ such that

$$d(x_n, B) = d(x_n, y_n) \geq n. \tag{2.1}$$

Now since $A \times B$ is compact there is $(x,y) \in A \times B$ such that

$$d(x_n, y_n) \to d(x,y)$$

which is contradicted by (2.1) (note that it follows from (2.1) that $\limsup_n d(x_n, y_n) \geq \limsup_n n = \infty$).

It is straightforward to see that $(K(X), h)$ is a complete metric space.

A fuzzy set u in an universe set X is a mapping $u : X \to [0,1]$ which $u(x)$ assigns the degree of membership of element x in the fuzzy set A, for each $x \in X$. If u is a fuzzy set in X, we define $[u]^\alpha = \{x \in X \mid u(x) \geq \alpha\}$ the α-level of u, with $0 < \alpha \leq 1$. For $\alpha = 0$ the support of u is defined

$$[u]^0 = \text{supp(u)} = \overline{\{x \in X \mid u(x) \geq 0\}}.$$

Proposition 2.2. *Let u be a fuzzy set in X, then the family $\{L_\alpha(u) = [u]^\alpha \mid \alpha \in [0,1]\}$ satisfies the following properties:*

i) $[u]^\beta \subseteq [u]^\alpha \subseteq [u]^0$, *for all* $0 \leq \alpha \leq \beta$.

ii) *If $\alpha_n \uparrow \alpha$, then $[u]^\alpha = \bigcap_{n=1}^\infty [u]^{\alpha_n}$ (i.e., the level-application is left-continuous).*

iii) *(Representation). Let $M \subseteq X$ and suppose that $\{B_\alpha \mid \alpha \in [0,1]\}$ is a family of subsets of M verifying (i) and (ii) and $B_0 = M$. Then, there exists a fuzzy set u in X such that $[u]^\alpha = B_\alpha$, for all $\alpha \in [0,1]$.*

Proof. (i) is obvious from the definition of the α-level set. To see (ii), it follows from $\alpha_n \uparrow \alpha$ and (i) that

$$[u]^\alpha \subset [u]^{\alpha_n}, \quad \forall n \in N (\text{the natural numbers}),$$

and so

$$[u]^\alpha \subset \bigcap_{n=1}^\infty [u]^{\alpha_n}.$$

Now let $x \in \bigcap_{n=1}^\infty [u]^{\alpha_n}$, hence

$$u(x) \geq \alpha_n, \quad \forall n \in N$$

and so

$$u(x) \geq \alpha_n, \quad \forall n \in N.$$

Then $u(x) \geq \lim_{n\to\infty} \alpha_n = \alpha$ which means that $x \in [u]^\alpha$. This completes the proof of (ii). To verify (iii), it is enough to define the fuzzy set u in X as

$$u(x) = \begin{cases} \sup\{\alpha \mid x \in B_\alpha\}, & if\ x \in M, \\ 0, & if\ x \notin M. \end{cases}$$

This completes the proof. ∎

Remark that the result of part (iii) in Proposition 2.2 still valid if we replace the assumption $\alpha_n \uparrow \alpha$ by the weaker condition $\alpha_n \alpha$, for all $n \in N$ and there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ with $\limsup \alpha_{n_k} = \alpha$ or $\liminf \alpha_{n_k} = \alpha$. The simple example

$$\alpha_n = \begin{cases} \frac{1}{3}, & if\ n = 2k, \\ 0, & if\ n = 2k-1. \end{cases}$$

and $\alpha = \frac{1}{2}$ does not satisfy (iii) of Proposition 2.2 while $\alpha_n \leq \alpha$, for all $n \in N$ and $\limsup(\alpha_{2k-1} = \frac{1}{2}) = \frac{1}{2}$.

We say that a fuzzy set u is compact if $[u]^\alpha \in K(X)$, for all $\alpha \in [0,1]$ and u is called convex if $[u]^\alpha$ is a convex subset of X, for all $\alpha \in [0,1]$.

As an extension of $K(X)$ we define the space $F(X)$ the space of all fuzzy sets $u : [0,1] \times X$ with the following properties:

i) u is normal, i.e. $\{x \in X \mid u(x) = 1\} \neq \emptyset$,

ii) u is fuzzy-convex, i.e., for all $x, y \in X$ and $\lambda \in [0,1]$ we have

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\},$$

iii) u is upper semicontinuous,

iv) $L_0 u$ is compact.

We will denote by *F(X)* the space of all compact and convex fuzzy sets on X. If we take $X = \mathbb{R}$, then it is easy to check that the α-level sets of $u \in F(\mathbb{R})$ are compact intervals of the real line, for all $\alpha \in [0,1]$. The sum and the scalar multiplication operations on *F(X)* are defined as $(u+v)(x) = \sup\{\inf\{u(y), v(x-y)\} : y \in X\}$ and

$$(\lambda.u)(x) = \begin{cases} u(\frac{x}{\lambda}), & if\ \lambda \neq 0, \\ \chi_{\{\theta\}}(x), & if\ \lambda = 0. \end{cases}$$

where θ is the zero vector of X and $\chi_{\{\theta\}}$ is the characteristic function of $\{\theta\}$.

It is easy to prove that $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda.u]^\alpha = \lambda[u]^\alpha$, for all $u, v \in F(X)$ and $\lambda \in \mathbb{R}$. We can also extend the Hausdorff metric h to $F(X)$ by means

$$D(u,v) = \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha), \ \forall u, v \in F(X).$$

One can show that $(F(X), h)$ is a complete metric space but it is not separable (note that there is an uncountable subset A of X and so if we consider the family $\{\chi_{\{a\}}\}_{a \in A}$, then $D(\chi_{\{a\}}, \chi_{\{b\}}) = \|a-b\|$). Also, the measure of $u \in F(X)$ is defined by $\|u\| = D(u, \chi_{\{\theta\}}) = \sup_{\alpha \in [0,1]} \|L_\alpha u\|$. It is obvious that $A \in K(X)$ then $\chi_{\{a\}} \in F(x)$ and $\|\chi_{\{a\}}\| = D(\chi_A, \chi_{\{\theta\}}) = \sup_{\alpha \in [0,1]} h([\chi_A]^\alpha, [\chi_{\{\theta\}}]^\alpha) = \sup\{\|x\| : x \in A\}$. It is also clear that if $A \in K(X)$ then $\chi_A \in F(x)$ and $\|\chi_A\| = \sup\{\|x\| : x \in A\}$. Moreover, the mapping $x \in X \to \chi_{\{x\}} \in F(X)$ is an isometric embedding. Finally, if $u \in F(X)$, then we say that $u \in X$ if $u = \chi_{\{x\}}$, for some $x \in X$.

The support function of $u \in F(X)$ is defined as $S_u : [0,1] \times X^* \longrightarrow \Re$,

$$S_u(\alpha, x^*) = \sup\{x^*(x) : x \in [u]^\alpha\}$$

where X^* is the dual of X.

In the next proposition, we give some interesting properties of the support function.

Proposition 2.3. *If $u, v \in F(x)$, $\lambda \geq 0$ and $(\alpha, x^*, y^*) \in [0,1] \times X^* \times X^*$, then*

(i) $S_{u+v}(\alpha, x^*) = S_u(\alpha, x^*) + S_v(\alpha, x^*)$,

(ii) $S_u(\alpha, x^* + y^*) = S_u(\alpha, x^*) + S_u(\alpha, y^*)$,
(iii) $S_{\lambda u}(\alpha, x^*) = S_u(\alpha, \lambda x^*) = \lambda S_u(\alpha, x^*)$,
(iv) *If* $0 \leq \alpha \leq \beta \leq 1$, *then* $S_u(\beta, x^*) \leq S_u(\alpha, x^*)$,
(v) *The mapping* $x^* \longrightarrow S_u(\alpha, x^*)$ *is lower semicontinuous and* $\alpha \longrightarrow S_u(\alpha, x^*)$ *is continuous.*

Proof. is a direct result of the definition of the support function and the relation $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$. (ii) and (iii) are obvious from the definition of the support function. (iv) follows from the fact that if $0 \leq \alpha\beta \leq 1$; then $[u]^\beta \subseteq [u]^\alpha$. The first part of (v) can be supposed it as the supremum of a family of continuous functions. In fact,

$$S_u(\alpha, x^*) = \sup\{\widehat{x}(x^*) : x \in [u]^\alpha\}$$

where, $\widehat{x} : X^* \longrightarrow \Re$ is defined by $\widehat{x}(x^*) = x^*(x)$, for all $x \in X$. The second part of (v) follows from being It is easy the function $\alpha \longrightarrow S_u(\alpha, x^*)$ decreasing and $S_u(\lambda\alpha, x^*) = \lambda S_u(\alpha, x^*)$, for all $\lambda \in [0, 1]$. ∎

3. The Fuzzy Derivative

It is well-known that the H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings was initially introduced by Puri and Ralescu [16] in the setting of finite dimensional Euclidean space $\Re^n$. In this section we are going to extend it to an infinite dimensional normed space. To do it we need interpret the meaning of the difference of two fuzzy sets.

Definition 3.1. Let $u, v \in F(X)$. If there exists $w \in F(X)$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $u - v$.

Remark that $u - u = \chi_{\{\}}$ and $u + \chi_{\{\theta\}} = u$, for each $u \in F(X)$.

Definition 3.2. Let $I = (a, b)$ and consider a fuzzy mapping $H : I \to F(X)$. We say that H is differentiable at $t_0 \in I$ if there exists an element $H^{'}(t_0) \in F(X)$ such that the limits

$$\lim_{h\to 0^+} \frac{H(t_0 + h) - H(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{H(t_0) - H(t_0 - h)}{h}$$

exist and are equal to $H^{'}(t_0)$. Here the limit is taken in the metric space $(F(X), D)$.

Remark 3.3. As seen in the above, one can consider X as a subset of $F(X)$, in fact X has been isometrically embedded X into $F(X)$ by the mapping $x \in \chi_{\{x\}}$. Hence if $f : I =]a, b[\to X$ is differentiable at $t_0 \in I$ in the usual sense then the limits

$$\lim_{h\to 0^+} \frac{f(t_0 + h) - f(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(t_0) - f(t_0 - h)}{h}$$

exist and are equal to the element $f^{'}(t_0) \in X$. Equivalently one can verify that the limits

$$\lim_{h\to 0^+} \frac{\chi_{(\{t_0+h\})} - \chi_{\{t_0\}}}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{\chi_{\{t_0\}} - \chi_{(\{t_0-h\})}}{h}$$

exist and equal to $\chi_{f'(t_0)}$. This means that Definition 3.2 is a generalization of the usual definition of differentiability. Although this definition of the differentiability is an extension of the usual definition but it is very restrictive; for instance, if we take $H(t) = c.g(t)$, where c is a fuzzy number and $g : [a, b] \to [0, \infty)$ is a function with $g^{'}(t) < 0$, then H is not differentiable. To avoid this difficulty, we define a more general definition of derivative for fuzzy mappings enlarging the class of differentiable fuzzy mappings by considering a lateral type of $H-$derivatives.

In this paper we consider the following definition in the setting of normed spaces:

Definition 3.4. Let $H : I =]a, b[\to F(X)$ and $t_0 \in I$. We say that H is differentiable at t_0 if: there exists an element $H^{'}(t_0) \in F(X)$ such that, for all $h > 0$ sufficiently near to 0, there are $H(t_0 + h) - H(t_0)$, $H(t_0) - H(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{H(t_0 + h) - H(t_0)}{h} = \lim_{h\to 0^+} \frac{H(t_0) - H(t_0 - h)}{h}) = H^{'}(t_0) \tag{3.1}$$

or

there exists an element $H^{'}(t_0) \in F(X)$ such that, for all $h < 0$ sufficiently near to 0, there are $H(t_0 + h) - H(t_0)$, $H(t_0) - H(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0^-} \frac{H(t_0 + h) - H(t_0)}{h} = \lim_{h\to 0^-} \frac{H(t_0) - H(t_0 - h)}{h} = H^{'}(t_0) \tag{3.2}$$

Remark 3.5. Note that if H is differentiable in the sense (3.1) and (3.2) simultaneously, then for $h > 0$ sufficiently small we have $H(t_0 + h) = H(t_0) + u_1$, $H(t_0) = H(t_0 - h) + u_2$, $H(t_0 + h) = H(t_0) + v_1$ and $H(t_0) = (t_0 + h) + v_2$, with $u_1, u_2, v_1, v_2 \in F(X)$. Thus, $H(t_0) = H(t_0) + (u_2 + v_1)$, i.e., $u_2 + v_1 = \chi_{\{\theta\}}$, it implies two possibilities: $u_2 = v_1 = \chi_{\{\theta\}}$ if $H(t_0) = \chi_{\{\theta\}}$; or $u_2 = \chi_{\{a\}} = -v_1$, with $a \in F(X)$, if $H^{'}(t_0)$. Therefore, if there exists $H^{'}(t_0)$ in the first form (second form) with $H^{'}(t_0) \in F(X)$ then does not exist $H^{'}(t_0)$ in the second form (first form, respectively).

The following example shows that the first form and the second form of the differentiability in Definition 3.4 are different.

Example 3.6. Let $X = \Re^n$ and $H(t) = c \cdot g(t)$ be a fuzzy mapping where c is a fuzzy number and $g :]a, b[\to [0, \infty)$ is a differentiable function on $]a, b[$. In this case, if $g^{'}(t_0) > 0$, then H is differentiable in the first form (3.1) and we have $H^{'}(t_0) = c.g^{'}(t_0)$. But, from the previous remark H is not differentiable in the second form (3.2). Analogously, if $g^{'}(t_0) < 0$, then H is differentiable in the second form (3.2) and $H^{'}(t_0) = c.g^{'}(t_0)$, but H is not differentiable in the first form.

Theorem 3.7. *Let X be a normed space and $H :]a, b[\to \mathrm{F}(X)$ a fuzzy mapping. If H is a differentiable mapping of the second form in the sense of Definition 3.4 then H is continuous.*

Proof. The result follows directly by the definition of the second form of Definition 3.4 and the rules $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda.u]^\alpha = \lambda[u]^\alpha$, for all $u, v \in F(X)$, $\lambda \in \mathbb{R}$. ∎

Theorem 3.8. *Let X be a normed space and $G, H :]a, b[\to \mathrm{F}(X)$ two fuzzy mappings. If G, H are differentiable in the second form (3.2) at the point $t \in]a, b[$ and $\lambda \in \Re$, then $(H + G)^{'}(t) = H^{'}(t) + G^{'}(t)$ and $(\lambda H)^{'}(t) = \lambda H^{'}(t)$.*

Proof. The result follows by the definition of the second form of Definition 3.4, the relations $[u+v]^\alpha = [u]^\alpha+[v]^\alpha$, $[\lambda.u]^\alpha = \lambda[u]^\alpha$, for all $u, v \in F(X)$, $\lambda \in \mathbb{R}$, and the uniqueness of the limit. ∎

Theorem 3.9. *Let $H : I \to \mathrm{F}(X)$ be a fuzzy mapping and let $x_t, y_t \in [H(t)]^\alpha$, and $\alpha \in [0, 1]$. Then*

(i) *If H is differentiable in the first form* (3.1)*, then x_t and y_t are differentiable functions and* $[x_t^{'}, y_t^{'}] \subset [H^{'}(t)]^\alpha$.
(ii) *If H is differentiable in the second form* (3.2)*, then x_t and y_t are differentiable functions and* $[y_t^{'}, x_t^{'}] \subset [H^{'}(t)]^\alpha$.

Proof. (i) If $h > 0$ and $\alpha \in [0, 1]$, then we have

$$[x_{t+h} - x_t, y_{t+h} - y_t] \subset [H(t + h) - H(t)]^\alpha$$

and, multiplying by $\frac{1}{h}$ we have

$$\frac{1}{h}[x_{t+h} - x_t, y_{t+h} - y_t] = [\frac{x_{t+h} - x_t}{h}, \frac{y_{t+h} - y_t}{h}] \subset \frac{[H(t + h) - H(t)]^\alpha}{h}$$

Passing to the limit we have

$$[x_t^{'}, y_t^{'}] \subset [H^{'}(t)]^\alpha.$$

The proof of (ii) is similar to the proof of (i). ∎

It is worth noting when X is an infinite dimensional normed space then the interior of a compact and convex subset of X is empty. Hence, for each $\alpha \in [0, 1]$ the interior of $[H(t)]^\alpha$ is empty.

Definition 3.10. Let X be a normed space and (X, Ω, μ), where ω is the smallest $\sigma-$ algebra contains all the open subsets of X and μ is a measure on it, is a measure space. We say that the fuzzy mapping $H : I =]a, b[\to F(X)$ is measurable,if for each $\alpha \in [0, 1]$ the set valued mapping $H_\alpha : I \to F(X)$ defined by $H_\alpha(t) = [H(t)]^\alpha$ is a measurable set as a subset of X. We define the integral of the fuzzy mapping $H : I \to F(X)$, denoted $\int_{t\in I} H(t)$, as

$$\left[\int_{t\in I} H(t)\right]^\alpha = \left\{\int_{t\in I} f(t)dt : f : I \to X \text{ is a measurable selection for } H_\alpha\right\},$$

for all $\alpha \in [0, 1]$.

A measurable fuzzy mapping $H : I \to X$ is said to be integrable over I if $\int Ht)dt \in X$.

Definition 3.11. A fuzzy mapping $H : I \to F(X)$ is called integrable bounded, if there exists an integrable mapping $h : I \to [0, \infty)$ such that

$$D(H(t), \chi_{\{\theta\}}) \leq h(t), \quad \forall t \in I.$$

Remark that the previous definition is an extension of the usual definition of integrable bounded for single real valued mappings. Also, if $H : I \to F(X)$ is a continuous fuzzy mapping then H is a measurable mapping.

By using the definitions of the integrability and differentiability, we deduce the following two theorems which are useful in the sequel and omit their proofs.

Theorem 3.12. *Let H be continuous in $I = [a, b]$. Then the mapping $G(t) = \int_a^t H$ is differentiable in the second form and $G^{'}(t) = H(t), \forall t \in I$.*

Theorem 3.13. *Let H be differentiable in $I = [a, b]$ with respect to the second form and assume that the derivative $H^{'}$ is integrable over I. Then for each $s \in I$, we have*

$$H(s) = H(a) + \int_a^s H.$$

Now we are ready to study the following fuzzy initial value problem in the setting of normed spaces.

$$x^{'}(t) = H(t, x(t)), \quad x(0) = x_0,$$

where $H : [0, a] \times \mathrm{F}(X) \to F(X)$ is a fuzzy mapping and x_0 is a fuzzy number.

If we take $X = \Re$ then the initial value problem collapses to the problem studied by Kaleva [13].

The following theorem provides an existence result to the initial fuzzy value problem.

Theorem 3.14. *Let $H : [0, a] \to \mathrm{F}(X)$ be continuous fuzzy mapping and assume that there exists a $0 < k < 1$ such that*

$$D(H(t, u), H(t, v)) \leq D(u, v), \quad \forall (u, v) \in \mathrm{F}(X) \times \mathrm{F}(X), \forall t \in I$$

Then the initial value problem

$$x^{'}(t) = H(t, x(t)), \quad x(0) = x_0,$$

has two unique solutions on $[0, a]$.

Proof. Set $M = \{f : f : [0, a] \to F(X)\}$. It is straightforward to see that the set M is a complete metric space with the metric $d(f, g) = \sup_{t\in[0,a]} D(f(t), g(t))$. Now we define $G : M \to M$ by

$$G(f)(s) = x_0 + \int_0^s H(t, x(t))dt, \quad \forall (f, s) \in M \times [0, a].$$

It follows from our assumptions and the Banach's contraction theorem that G has a fixed point such as $x \in M$ so that

$$x(s) = G(x)(s) = x_0 + \int_0^s H(t, x(t))dt, \quad \forall \in [0, a].$$

Hence, by using Theorem 3.12 and by applying the two forms of Definition 3.4 we get

$$x^{'}(s) = H(s, x(s)), \quad \forall s \in [0, a].$$

Further $x(0) = x_0 + \int_0^0 H(t, x(t))dt = x_0$. This completes the proof. ∎

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