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Diffrential Subordination and Generalized Bessel Functions

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Abstract In this paper, differential subordination preserving properties for analytic functions in the open unit disk with an operator involving generalized Bessel functions are investigated. Some particular cases involving trigonometric functions of our main results are also derived.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{P}$ be the class of Carathéodory functions in U. For $n \in \mathbb{N} = \{1, 2, \dots\}$, and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

and let $\mathcal{H}_0 = \mathcal{H}[0, 1]$. We denote by A the class of functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, \tag{1.1}$$

which are analytic in the open unit disk U. For two functions f and F, analytic in U, we say that the function f is subordinate to F, and write $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \qquad z \in U,$$

such that $f(z) = F(w(z))$. Furthermore, if the function F is univalent in U, then we have the following equivalence (cf. [1]):

$$f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(U) \subset F(U).$$

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Let $f \in A$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1}, \qquad z \in U,$$

then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1} = (g * f)(z) \qquad z \in U.$$

Note that $f * g \in A$.

The generalized Bessel function of the first kind $w = w_{p,b,c}$ is defined as the particular solution of the second-order linear homogeneous differential equation [2, 3]

$$z^2 w''(z) + bzw'(z) + (cz^2 - p^2 + (1-b)p)w(z) = 0, \tag{1.2}$$

which is natural generalization of Bessel differential equation. This function has the representation

$$w(z) = w_{p,b,c}(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{n!\Gamma(p+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p}, \tag{1.3}$$

where $b, p, c, z \in \mathbb{C}$ and $c \neq 0$ and Γ stands for the Euler-Gamma function.

The series (1.3) permits the study of Bessel, modified Bessel and spherical Bessel functions in a unified manner. We note that,

(i) For $b = c = 1$ in (1.3), we have the familiar Bessel function of the first kind of order p defined by (see [4] and also [3])

$$J_p(z) = w_{p,1,1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p}, \qquad z \in \mathbb{C},$$

(ii) For $b = 1$ and $c = -1$ in (1.3), we obtain the modified Bessel function of the first kind of order p defined by (see [4] and also [3])

$$I_p(z) = w_{p,1,-1}(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p}, \qquad z \in \mathbb{C},$$

(iii) For $b = 2$ and $c = 1$ in (1.3), the function $w_{p,b,c}(z)$ reduces to $\sqrt{2} j_p(z)/\sqrt{\pi}$, where j_p is the spherical Bessel function of the first kind of order p defined by (see [3])

$$j_p(z) = w_{p,2,1} = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p}, \qquad z \in \mathbb{C},$$

Recently, Baricz et al. [5], Deniz et al. [6] and Deniz [7] (see also [2, 3, 8–14]) considered the function $u_{p,b,c}(z) : U \to \mathbb{C}$ defined, in terms of the generalized Bessel function $w_{p,b,c}(z)$, by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}} w_{p,b,c}(\sqrt{z}). \tag{1.4}$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(a)_n$ defined, for $a, n \in \mathbb{C}$ and in terms of the Euler Γ-function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, a \in \mathbb{C}\backslash\{0\}, \\ a(a+1)\dots(a+n-1), & n \neq 0, a \in \mathbb{C}, \end{cases}$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n c^n}{4^n (\kappa)_n} \frac{z^{n+1}}{n!}, \tag{1.5}$$

where $\kappa = p + \frac{b+1}{2} \neq 0, -1, -2, \dots$. For convenience, we write $u_{\kappa,c}(z) = u_{p,b,c}(z)$. This function is analytic on $\mathbb{C}$ and satisfies the second order linear differential equation

$$4z^2 u''(z) + 2(2p+b+1)zu'(z) + czu(z) = 0.$$

Now, we introduce a new operator $B_\kappa^c : A \to A$, which is defined by the Hadamard product

$$B_{\kappa,c}(f)(z) = u_{\kappa,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n (\kappa)_n} \frac{z^{n+1}}{n!}. \tag{1.6}$$

It is easy to verify from the definition (1.6) that

$$z(B_{\kappa+2,c}(f)(z))' = (\kappa+1)B_{\kappa+1,c}(f)(z) - \kappa B_{\kappa+2,c}(f)(z). \tag{1.7}$$

In fact, the function $B_{\kappa,c}(f)(z)$ is an elementary transform of the generalized hypergeometric function defined by (see [15–19]; also [20, 21])

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, \dots, (\alpha_q)_n}{(\beta_1)_n, \dots, (\beta_s)_n} \frac{z^n}{n!}$$

$$(\alpha_i \in \mathbb{C};\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ q \leq s+1;\ q, s \in \mathbb{N} \cup \{0\};\ i = 1, 2, \dots, q;\ j = 1, 2, \dots, s).$$

That is, we have

$$B_{\kappa,c}(f)(z) = z{}_0F_1\left(\kappa; -\frac{c}{4}z\right) * f(z).$$

We observe that, for suitable choices of the parameters b and c, we obtain some new operators:

(i) For $b = c = 1$ in (1.6), we have the operator $\mathcal{J}_p : A \to A$ related with Bessel function, defined by

$$\mathcal{J}_p f(z) = w_{p,1,1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2} J_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}.$$

(ii) For $b = 1$ and $c = -1$ in (1.6), we obtain the operator $\mathcal{I}_p : A \to A$ related with Bessel function, defined by

$$\mathcal{I}_p f(z) = w_{p,1,-1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2} I_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}.$$

(iii) For $b=2$ and $c=1$ in in (1.6), we get the operator $\mathcal{S}_p : A \to A$ related with Bessel function, defined by

$$\mathcal{S}_p f(z) = w_{p,2,1}(z) * f(z) = \left[\pi^{-\frac{1}{2}} 2^{p+\frac{1}{2}} \Gamma\left(p+\frac{3}{2}\right) z^{1-p/2} j_p(\sqrt{z})\right] * f(z)$$
$$= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+\frac{3}{2})_n} \frac{z^{n+1}}{n!}.$$

which satisfies the recurrence relation

$$z[\mathcal{S}_{p+1} f(z)]' = \left(p+\frac{3}{2}\right) \mathcal{S}_p f(z) - \left(p+\frac{1}{2}\right) \mathcal{S}_{p+1} f(z).$$

In the present paper, by making use of the differential subordination results of Miller and Mocanu [1], we determine certain subclasses of analytic functions and obtain some subordination of analytic functions associated with the $B_{\kappa,c}$-operator defined by (1.6). To prove our main results, we need the following definition and lemmas.

Definition 1.1. Let $0 \le \eta < 1$ and $p, b, c \in \mathbb{C}$ be such that $c \neq 0$, $\kappa = p + \frac{b+1}{2} \neq 0, -1, -2, \ldots$ and let $S^c(\kappa, \eta, h)$ be the class of functions $f \in A$ satisfying the condition

$$\frac{1}{1-\eta}\left(\frac{z(B_{\kappa,c}(f)(z))'}{B_{\kappa,c}(f)(z)} - \eta\right) \prec h(z), \qquad 0 \le \eta < 1, h \in \mathcal{P}.$$

For simplicity we write

$$S^c\left(\kappa, \eta, \frac{1+Az}{1+Bz}\right) = S^c(\kappa, \eta, A, B), \qquad -1 \le B < A \le 1.$$

Lemma 1.2. [22] *For $\beta, \gamma \in \mathbb{C}$ let h be convex univalent in U with $h(0) = 1$ and $\Re\,(\beta h(z) + \gamma) > 0$, if p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

implies that $p(z) \prec h(z)$.

Lemma 1.3. [23] *Let h be convex univalent in U and w be analytic in U with $\Re\, w(z) > 0$. If p is analytic in U and $p(0) = h(0)$, and*

$$p(z) + zw(z)p'(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Lemma 1.4. [24] *Let p be analytic in U with $p(0) = 1$ and $p(z) \neq 0$. Suppose that there exists a point $z_0 \in U$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha, \qquad |z| < |z_0|,$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha, \qquad 0 < \alpha \le 1,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = is\alpha,$$

where

$$s \geq \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \textit{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha,$$

$$s \leq -\frac{1}{2}\left(a + \frac{1}{a}\right), \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia, \quad a > 0.$$

Lemma 1.5. [25] *The function*

$$(1-z)^{\gamma} \equiv \exp(\gamma \log(1-z)), \qquad \gamma \neq 0,$$

is univalent if only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.

Lemma 1.6. [26] *Let q be analytic in U and let $\Theta(w)$ and $\phi(w)$ be analytic in a domain $\mathbb{D}$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \; h(z) = \Theta(q(z)) + Q(z),$$

and suppose that

(1) *Q is starlike; either*
(2) *h is convex;*
(3) $\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\Theta'(q(z)}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0.$

If p is analytic in U with $p(0) = q(0)$ and $p(U) \subset \mathbb{D}$, and

$$\Theta(p(z)) + zp'(z)\phi(p(z)) \prec \Theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

2. Main Results

Theorem 2.1. *Let $f \in A$, $h \in \mathcal{P}$, $0 \leq \eta < 1$, $c \in \mathbb{C}$ with $c \neq 0$, $p, b \in \mathbb{R}$ be such that $\kappa > -1$ and suppose that*

$$\Re\big((1-\eta)h(z) + \eta + \kappa\big) > 0.$$

Then the subordination condition

$$\frac{1}{1-\eta}\left(\frac{z\left(B_{\kappa+1,c}(f)(z)\right)'}{B_{\kappa+1,c}(f)(z)} - \eta\right) \prec h(z),$$

implies that

$$\frac{1}{1-\eta}\left(\frac{z\left(B_{\kappa+2,c}(f)(z)\right)'}{B_{\kappa+2,c}(f)(z)} - \eta\right) \prec h(z),$$

where $B_{\kappa+2,c}(f)(z) \neq 0$ for $z \in U$.

Proof. Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z\left(B_{\kappa+2,c}(f)(z)\right)'}{B_{\kappa+2,c}(f)(z)} - \eta\right) \prec h(z),$$

where p is an analytic function with $p(0) = 1$. By using the equation

$$z\left(B_{\kappa+2,c}(f)(z)\right)' = (\kappa+1)B_{\kappa+1,c}(f)(z) - \kappa B_{\kappa+2,c}(f)(z), \quad z \in U. \tag{2.1}$$

we have

$$(1-\eta)p(z) + \eta + \kappa = \frac{(\kappa+1)B_{\kappa+1,c}(f)(z)}{B_{\kappa+2,c}(f)(z)}. \tag{2.2}$$

Differentiating logarithmically derivatives in both sides of (2.2) and using (2.1) we have

$$p(z) + \frac{zp'(z)}{(1-\eta)p(z)+\eta+\kappa} = \frac{1}{1-\eta}\left(\frac{z\left(B_{\kappa+1,c}(f)(z)\right)'}{B_{\kappa+1,c}(f)(z)} - \eta\right), \quad 0 \le \eta < 1.$$

Since $\Re\left((1-\eta)h(z)+\eta+\kappa\right) > 0$, applying Lemma 1.2, it follows that $p(z) \prec h(z)$, that is

$$\frac{1}{1-\eta}\left(\frac{z\left(B_{\kappa+2,c}(f)(z)\right)'}{B_{\kappa+2,c}(f)(z)} - \eta\right) \prec h(z).$$

This completes the proof. ∎

Theorem 2.2. *Let $0 < \rho < 1$, $\gamma \neq 1$, $c \in \mathbb{C}$ with $c \neq 0$, $p, b \in \mathbb{R}$ be such that $\kappa > -1$ satisfying either $|2(\kappa+1)\gamma\rho - 1| \le 1$ or $|2(\kappa+1)\gamma\rho + 1| \le 1$. If $f \in A$ satisfiees the condition*

$$\Re\left(1 + \frac{B_{\kappa+1,c}(f)(z)}{B_{\kappa+2,c}(f)(z)}\right) > 1-\rho, \quad z \in U, \tag{2.3}$$

then

$$\left(z^{\kappa}B_{\kappa+2,c}(f)(z)\right)^{\gamma} \prec q_1(z) = \frac{1}{(1-z)^{2(\kappa+1)\gamma\rho}},$$

where q_1 is the best dominant and $z^{\kappa}B_{\kappa+2,c}(f)(z) \neq 0$ for $z \in U$.

Proof. Denoting $p(z) = (z^{\kappa}B_{\kappa+2,c}(f)(z))^{\gamma}$. It follows that

$$\frac{zp'(z)}{p(z)} = \gamma(\kappa+1)\frac{B_{\kappa+1,c}(f)(z)}{B_{\kappa+2,c}(f)(z)}. \tag{2.4}$$

Combing (2.3) and (2.4), we find that

$$1 + \frac{zp'(z)}{(\kappa+1)\gamma p(z)} \prec \frac{1+(2\rho-1)z}{1-z}, \tag{2.5}$$

if we set $\Theta(w) = 1$, $\phi(w) = \frac{1}{\gamma(\kappa+1)w}$, and $q_1(z) = \frac{1}{(1-z)^{2(\kappa+1)\gamma\rho}}$, then by the assumption of the theorem and making use of Lemma 1.6, we know that q_1 is univalent in U. It follows that

$$Q(z) = zq_1'(z)\phi(q_1(z)) = \frac{2\rho z}{1-z},$$

and

$$h(z) = \Theta(q_1(z)) + Q(z) = \frac{1+(2\rho-1)z}{1-z}.$$

If we consider D such that

$$q(U) = \left\{w : |w^{\frac{1}{\xi}} - 1| < |w^{\frac{1}{\xi}}|, \quad \xi = 2\gamma(\kappa+1)\rho\right\} \subset D,$$

then it is easy to check that the conditions (i) and (ii) of Lemma 1.6 hold true. Thus, the desired result of Theorem 2.2 follows from (2.5). ∎

Taking into account the above result, we have the following particular case. Choosing $f(z) = \frac{z}{1-z}$, $\rho = \frac{1}{2}$, $p = -\frac{3}{2}$, $b = c = 1$ $(\kappa = -\frac{1}{2})$ and $p = -\frac{1}{2}$, $b = c = 1$ $(\kappa = \frac{1}{2})$, in the above theorem we obtain for $\kappa > -1$ and $z \in U$ the following results

$$\Re\left(1 + \sqrt{z}\cot\sqrt{z}\right) > \frac{1}{2} \Rightarrow \left(\frac{\sin\sqrt{z}}{z}\right)^{\gamma} \prec \frac{1}{(1-z)^{\frac{\gamma}{2}}},$$

and

$$\Re\left(1 + \frac{z\sin\sqrt{z}}{3(\sin\sqrt{z} - \sqrt{z}\cos\sqrt{z})}\right) > \frac{1}{2} \Rightarrow \left(3\frac{\sin\sqrt{z} - \sqrt{z}\cos\sqrt{z}}{z}\right)^{\gamma} \prec \frac{1}{(1-z)^{\frac{3}{2}\gamma}}.$$

Here we used the relations

$$u_{-\frac{1}{2},1,1} = \sqrt{\frac{\pi}{2}}z^{\frac{1}{4}}J_{-\frac{1}{2}}(\sqrt{z}) = \cos\sqrt{z}, \quad u_{\frac{1}{2},1,1} = \sqrt{\frac{\pi}{2}}z^{-\frac{1}{4}}J_{\frac{1}{2}}(\sqrt{z}) = \frac{\sin\sqrt{z}}{\sqrt{z}},$$

and

$$u_{\frac{3}{2},1,1} = 3\sqrt{\frac{\pi}{2}}z^{-\frac{31}{4}}J_{\frac{3}{2}}(\sqrt{z}) = 3\left(\frac{\sin\sqrt{z}}{z\sqrt{z}} - \frac{\cos\sqrt{z}}{z}\right).$$

Theorem 2.3. *Let h be convex univalent function in U and*

$$\Re\big(\mu + \eta + (1-\eta)h(z)\big) > 0, \qquad z \in U.$$

If $f \in A$ satisfies the condition

$$\frac{1}{1-\eta}\left(\frac{z(B_{\kappa+1,c}(f)(z))'}{B_{\kappa+1,c}(f)(z)} - \eta\right) \prec h(z), \qquad 0 \le \eta < 1,$$

then

$$\frac{1}{1-\eta}\left(\frac{z(B_{\kappa+1,c}(F_\mu(f))(z))'}{B_{\kappa+1,c}(F_\mu(f))(z)} - \eta\right) \prec h(z), \qquad 0 \le \eta < 1,$$

where F_μ is the Bernadi integral operator defined by

$$F_\mu(f)(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt. \tag{2.6}$$

Proof. From (2.6), we have

$$z(B_{\kappa+1,c}(F_\mu(f))(z))' = (\mu+1)B_{\kappa+1,c}(f)(z) - \mu B_{\kappa+1,c}(F_\mu(f))(z). \tag{2.7}$$

Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(B_{\kappa+1,c}(F_\mu(f))(z))'}{B_{\kappa+1,c}(F_\mu(f))(z)} - \eta\right), \tag{2.8}$$

where p is analytic function with $p(0) = 1$. Then, using (2.7) we get

$$\mu + \eta + (1-\eta)p(z) = (\mu+1)\frac{B_{\kappa+1,c}(f)(z)}{B_{\kappa+1,c}(F_\mu(f))(z)}. \tag{2.9}$$

Differentiating logarithmically in both sides of (2.9) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{\mu + \eta + (1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(B_{\kappa+1,c}(f)(z))'}{B_{\kappa+1,c}(f)(z)} - \eta\right).$$

Since $\Re\big(\mu + \eta + (1-\eta)p(z)\big) > 0$ thus by Lemma 1.2, we have

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(B_{\kappa+1,c}(F_\mu(f))(z))'}{B_{\kappa+1,c}(F_\mu(f))(z)} - \eta\right) \prec h(z).$$

This completes the proof. ∎

Theorem 2.4. *Let $f \in A$, $0 < \delta \leq 1$, $0 \leq \gamma < 1$, $c \in \mathbb{C}$ with $c \neq 0$, $p, b \in \mathbb{R}$ be such that $\kappa > -1$. If*

$$\left|\arg\left(\frac{z(B_{\kappa+1,c}(f)(z))'}{B_{\kappa+1,c}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

for some $g \in S^c(\kappa+1, \eta, A, B)$. Then

$$\left|\arg\left(\frac{z(B_{\kappa+2,c}(f)(z))'}{B_{\kappa+1,c}(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha.$$

where α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\arctan\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{(1-\gamma)(1+A)}{1+B}+\gamma+\kappa+\alpha\sin\frac{\pi}{2}t_1}, & B \neq -1, \\ \\ \alpha, & B = -1, \end{cases} \tag{2.10}$$

where

$$t_1 = \frac{2}{\pi}\arcsin\left(\frac{(1-\gamma)(A-B)}{(1-\gamma)(1-AB)+(\gamma+\kappa)(1-B^2)}\right). \tag{2.11}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(B_{\kappa+2,c}(f)(z))'}{B_{\kappa+1,c}(g)(z)} - \gamma\right).$$

Using (1.7), it is easy to see that

$$\big((1-\gamma)p(z)+\gamma\big)B_{\kappa+2,c}(g)(z) = (\kappa+1)B_{\kappa+1,c}(f)(z) - \kappa B_{\kappa+2,c}(f)(z). \tag{2.12}$$

Differentiating (2.12) and multiplying by z, we obtain

$$\begin{aligned}(1-\gamma)zp'(z)B_{\kappa+2,c}(g)(z) + \big((1-\gamma)p(z)+\gamma\big)z(B_{\kappa+2,c}(g)(z))' \\ = (\kappa+1)z(B_{\kappa+1,c}(f)(z))' - \kappa z(B_{\kappa+2,c}(f)(z))'\end{aligned} \tag{2.13}$$

Since $g \in S^c(\kappa+1, \eta, A, B)$, by Theorem 2.1, we have $g \in S^c(\kappa+2, \eta, A, B)$. Let

$$q(z) = \frac{1}{1-\gamma}\left(\frac{z(B_{\kappa+2,c}(g)(z))'}{B_{\kappa+1,c}(g)(z)} - \gamma\right).$$

Then by using (1.7) once again, we have

$$q(z)(1-\gamma)+\gamma+\kappa = (\kappa+1)\frac{B_{\kappa+1,c}(g)(z)}{B_{\kappa+2,c}(g)(z)}. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\frac{zp'(z)}{q(z)(1-\gamma)+\gamma+\kappa} + p(z) = \frac{1}{1-\gamma}\left(\frac{z(B_{\kappa+1,c}(f)(z))'}{B_{\kappa+1,c}(g)(z)} - \gamma\right).$$

Since $q(z) \prec \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, we have

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad, z \in U,\ B \neq -1, \tag{2.15}$$

and

$$\frac{1-A}{2} \leq \Re\, q(z) \qquad, z \in U,\ B = -1. \tag{2.16}$$

Therefore, from (2.15) and (2.16), we obtain

$$\left| q(z)(1-\gamma)+\gamma+\kappa-\frac{(1-\gamma)(1-AB)}{1-B^2}-\gamma-\kappa \right| < \frac{(1-\eta)(A-B)}{1-B^2}, \qquad B \neq -1.$$

For $B=-1$, we have

$$\Re\big(q(z)(1-\gamma)+\gamma+\kappa\big) > \frac{(1-\gamma)(1-A)}{2}+\gamma+\kappa.$$

Let $q(z)(1-\gamma)+\gamma+\kappa = r\exp(i\frac{\Phi}{2})$, where

$$\frac{(1-\gamma)(1-A)}{1-B}+\gamma+\kappa < r < \frac{(1-\gamma)(1+A)}{1+B}+\gamma+\kappa, \qquad B\neq -1,\ -t_1<\Phi<t_1,$$

and t_1 is given by (2.11), and

$$\frac{(1-\gamma)(1-A)}{2}+\gamma+\kappa<r<\infty, \qquad B=-1,\ -t_1<\Phi<t_1.$$

We note that p is analytic in U with $p(0)=1$, so by applying the assumption and Lemma 1.3 with

$$w(z)=\frac{1}{q(z)(1-\gamma)+\gamma+\kappa},$$

we have $\Re\ w(z)>0$. Set

$$Q(z)=\frac{1}{1-\gamma}\left(\frac{z(B_{\kappa+1,c}(f)(z))'}{B_{\kappa+1,c}(g)(z)}-\gamma\right), \qquad 0\leq\gamma<1.$$

At first, suppose that $p(z_0)^{\frac{1}{\alpha}}=ia(a>0)$. For $B\neq -1$ we have

$$\begin{aligned}
\arg(Q(z_0)) &= \arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\gamma)+\gamma+\kappa}+p(z_0)\right)\\
&= \frac{\pi}{2}\alpha+\arg\left(1+\frac{is\alpha}{r}\left(\exp\left(\frac{-i\pi}{2}\Phi\right)\right)\right)\\
&\geq \frac{\pi}{2}\alpha+\arctan\left(\frac{s\alpha\sin\dfrac{\pi}{2}(1-\Phi)}{r+k\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right)\\
&\geq \frac{\pi}{2}\alpha+\arctan\left(\frac{\alpha\cos\dfrac{\pi}{2}t_1}{\dfrac{(1-\gamma)(1+A)}{1+B}+\gamma+\kappa+\alpha\sin\dfrac{\pi}{2}t_1}\right)\\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (2.10) and (2.11), respectively.

Similarly, for the case $B=-1$, we have

$$\arg(Q(z))=\arg\left(\frac{z_0p'(z_0)}{q(z_0)(1-\gamma)+\gamma+\kappa}+p(z_0)\right)\geq\frac{\pi}{2}\alpha.$$

These results obviously contradict the assumption.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ $(a > 0)$, $B \neq -1$ and $z_0 \in U$. Applying the same method we have

$$\begin{aligned}
\arg(Q(z_0)) &= \arg\left(\frac{z_0 p'(z_0)}{q(z_0)(1-\gamma)+\gamma+\kappa} + p(z_0)\right) \\
&= \frac{-\pi}{2}\alpha + \arg\left(1 - is\alpha\left(r\exp\left(\frac{i\pi}{2}\Phi\right)\right)^{-1}\right) \\
&\leq \frac{-\pi}{2}\alpha - \arctan\left(\frac{s\alpha\sin\frac{\pi}{2}(1-\Phi)}{r + s\alpha\cos\frac{i\pi}{2}(1-\Phi)}\right) \\
&\leq \frac{-\pi}{2}\alpha - \arctan\left(\frac{\alpha\cos\frac{i\pi}{2}t_1}{\frac{(1-\gamma)(1+A)}{1+B} + \gamma + \kappa + \alpha\sin\frac{\pi}{2}t_1}\right) \\
&= \frac{-\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (2.10) and (2.11) respectively.

Similarly, for the case $B = -1$, we have

$$\arg(Q(z)) = \arg\left(\frac{z_0 p'(z_0)}{q(z_0)(1-\gamma)+\gamma+\kappa} + p(z_0)\right) \leq \frac{-\pi}{2}\alpha,$$

which contradicts the assumption of Theorem 2.4. Therefore, the proof of Theorem 2.4 is completed. ∎

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