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A Study of Neutrosophic Cubic Ideals in Semigroups with Application

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Abstract The semigroup has many applications in finite state machine, transformation etc. So the abstract concept of neutrosophic cubic sets was required to establish these ideas. It was a motivation for the authors to present the idea of neutrosophic cubic semigroups. Operational properties of neutrosophic cubic sets are investigated. The notion of neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are introduced and several properties are investigated. Relations between neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are discussed. Characterizations of neutrosophic cubic left (resp. right) ideals are considered, and how the images or inverse images of neutrosophic cubic subsemigroups and cubic left (resp. right) ideals become neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals, respectively, are studied. An application on neutrosophic cubic ideals is discussed.

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1. Introduction

An institution decides to evaluate its students. The panel consists of both internal and external evaluators. What would be the best aggregated value? And why neutrosophic cubic sets are used? The answer to first question is the ideals in semigroups may be considered. To answer the second question we would suggest the review of following literature. Fuzzy sets are initiated by Zadeh [23]. In [24], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent the expert's degree of certainty in different statements, numbers from the interval [0, 1] are used. It is often difficult for an expert to exactly quantify his or her certainty, therefore, instead of a real

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number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications, for example, Sambuc [12] in Medical diagnosis in thyroidian pathology, Kohout [11] also in Medicine, in a system CLINAID, Gorzalczany [12] in Approximate reasoning, Turksen [12, 16] in Interval-valued logic, in preferences modelling [17], etc. These works and others show the importance of these sets. Fuzzy sets deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [3] introduced a new notion, called a cubic set, and investigated several properties. Cubic set theory is applied to BCK/BCI-algebras (see [5–8]), LA-semihypergroups [19] and Γ-semihypergroups [24]. Jun et al. [4] introduced the concept of cubic ideals in semigroups. They investigated operational properties of cubic sets, the notion of cubic subsemigroups and cubic left (resp. right) ideals, and investigated several properties. The concept of neutrosophic set (NS) developed by Smarandache [13], is a more general concept which extends the concepts of the classic set and fuzzy set [14]. Neutrosophic set theory has diverese applications in a number of different aspects (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm). Jun et al. [9] introduced the notion of neutrosophic cubic set extending the concept of cubic sets to the neutrosophic sets. Some applications of neutrosophic cubic sets can be found in [1, 20, 25].

In this paper, we introduce the concept of neutrosophic cubic ideals in semigroup. We investigate some operational properties of neutrosophic cubic sets. The notion of neutrosophic cubic subsemigroups and neutrosophic cubic left (resp.right) ideals are introduced, and several properties are investigated. Relations between neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are discussed. Characterizations of neutrosophic cubic left (resp. right) ideals are considered, and how the images or inverse images of neutrosophic cubic subsemigroups and cubic left (resp. right) ideals become neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals, respectively, are discussed. In the last section, an application of aggregation used to neutrosophic cubic ideals is provided.

2. Preliminaries

In this section we recall some definitions.

Definition 2.1. A non-empty set S together with an associative binary operation " $\cdot$ " is called a semigroup.

Definition 2.2. A non-empty subset A of a semigroup S is called a subsemigroup if $AA\subseteq A$.

Definition 2.3. A non-empty subset A of S is left(resp., right) ideal of S if $SA\subseteq A$ (resp., $AS\subseteq A$) .

Definition 2.4. A fuzzy set in X is a function $\mu : X \to [0,1]$.

Definition 2.5. An interval valued fuzzy set (briefly, IVF-set) $\widetilde{\mu}_A$ on X is defined as $\widetilde{\mu}_A = \left\{\left\langle x, [\mu_A^-(x), \mu_A^+(x)]\right\rangle : x \in X\right\}$, where $\mu_A^-(x) \leqslant \mu_A^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^- : X \to [0,1]$ and $\mu_A^+ : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\widetilde{\mu}$, respectively. Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$. Then $A = \{< x, \widetilde{\mu}_A(x) >: x \in X\}$, where $\tilde{\mu}_A : X \longrightarrow D[0,1]$.

Definition 2.6. [24] Let X be a non-empty set. A cubic set in X is a structure of the form: $C = \{(x, \widetilde{\mu}(x), \lambda(x))|x \in X\}$ where $\widetilde{\mu}$ is an interval-valued fuzzy set in X and λ is a fuzzy set in X.

Definition 2.7. [15] A neutrosophic set (NS) in X is a structure of the form: $\lambda = \{\langle \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle |x \in X\}$, where $\lambda_T : X \to [0,1]$ is a truth membership function, $\lambda_I : X \to [0,1]$ is an indeterminate membership function, and $\lambda_F : X \to [0,1]$ is a false membership function.

Definition 2.8. [22] Let X be a non-empty set. An interval neutrosophic set (INS) in X is a structure of the form: $\widetilde{\mu} = \{\langle \widetilde{\mu}_T(x), \widetilde{\mu}_I(x), \widetilde{\mu}_F(x)\rangle |x \in X\}$ where $\widetilde{\mu}_T, \widetilde{\mu}_I$ and $\widetilde{\mu}_F$ are interval-valued fuzzy set in X, which are called an interval truth membership function, an interval indeterminacy membership function and an interval falsity membership function, respectively.

Definition 2.9. [9] Let X be the space of points. We define a neutrosophic cubic set(NCS), $A(x) = \{\langle \widetilde{\mu}, \lambda\rangle |x \in X\}$ where $\widetilde{\mu}(x) = \{\langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F\rangle |x \in X\}$ and $\lambda(x) = \{\langle \lambda_T, \lambda_I, \lambda_F\rangle |x \in X\}$ with $\widetilde{\mu}_T : X \to D[0,1]$, $\widetilde{\mu}_I : X \to D[0,1]$, $\widetilde{\mu}_F : X \to D[0,1]$ and $\lambda_T : X \to [0,1]$, $\lambda_I : X \to [0,1]$, $\lambda_F : X \to [0,1]$. We will briefly denote by $A(x) = \{\langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle |x \in X\}$, where $[0,0] \preceq \widetilde{\mu}_T + \widetilde{\mu}_I + \widetilde{\mu}_F \preceq [3,3]$ and $0 \leq \lambda_T + \lambda_I + \lambda_F \leq 3$.

3. Operational Properties of Neutrosophic Cubic Sets

In this section, we define some basic operations on neutrosophic cubic sets. Shortly we use NC instead of neutrosophic cubic.

Definition 3.1. [9] Let X be non-empty set. A NC set A in X is the structure

$$A(x) = \{\langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle : x \in X\}.$$

Definition 3.2. For any non-empty subset G of set X, the characteristic NC function of G in X defined to be a structure

$$X_G = \{\langle x, \widetilde{\mu}_{TG}, \widetilde{\mu}_{IG}, \widetilde{\mu}_{FG}, \lambda_{TG}, \lambda_{IG}, \lambda_{FG}\rangle : x \in X\},$$

where

$$\widetilde{\mu}_{TG} = \begin{cases} [1\ 1], & \text{if } x \in G, \\ [0\ 0], & \text{otherwise,} \end{cases} \qquad \lambda_{TG} = \begin{cases} 0, & \text{if } x \in G, \\ 1, & \text{otherwise,} \end{cases}$$

$$\widetilde{\mu}_{IG} = \begin{cases} [1\ 1], & \text{if } x \in G, \\ [0\ 0], & \text{otherwise,} \end{cases} \qquad \lambda_{IG} = \begin{cases} 0, & \text{if } x \in G, \\ 1, & \text{otherwise,} \end{cases}$$

$$\widetilde{\mu}_{FG} = \begin{cases} [0\ 0], & \text{if } x \in G, \\ [1\ 1], & \text{otherwise,} \end{cases} \qquad \lambda_{FG} = \begin{cases} 1, & \text{if } x \in G, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.3. The whole NC set $\mathcal{S}$ in semigroup S is defined to be the structure

$$\mathcal{S} = \left\{\left\langle \widetilde{1}_{TS}, \widetilde{1}_{IS}, \widetilde{0}_{FS}, 0_{TS}, 0_{IS}, 1_{FS}\right\rangle : x \in S\right\},$$

where $\widetilde{1} = [1,1]$ and $\widetilde{0} = [0,0]$.

Definition 3.4. For two NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F\rangle$ in semigroup S, we define

$$A \subseteq B \Longleftrightarrow \widetilde{\mu}_T \preceq \widetilde{\nu}_T, \widetilde{\mu}_I \preceq \widetilde{\nu}_I, \widetilde{\mu}_F \succeq \widetilde{\nu}_F \text{ and } \lambda_T \geq \eta_T, \lambda_I \geq \eta_I, \lambda_F \leq \eta_F.$$

Definition 3.5. NC product of $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ is defined to be NC set

$$A \otimes B = \left\{ \left\langle \begin{array}{c} x, (\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x), (\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x), (\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x), \\ (\lambda_T \circ \eta_T)(x), (\lambda_I \circ \eta_I)(x), (\lambda_F \circ \eta_F)(x) : x \in S \end{array} \right\rangle \right\},$$

which briefly it is denoted by

$$A \otimes B = \left\langle \begin{array}{c} (\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x), (\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x), (\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x) \\ (\lambda_T \circ \eta_T)(x), (\lambda_I \circ \eta_I)(x), (\lambda_F \circ \eta_F)(x) \end{array} \right\rangle,$$

where $\widetilde{\mu}_T \circ \widetilde{\nu}_T, \widetilde{\mu}_I \circ \widetilde{\nu}_I, \widetilde{\mu}_F \circ \widetilde{\nu}_F$ and $\lambda_T \circ \eta_T, \lambda_I \circ \eta_I, \lambda_F \circ \eta_F$ are defined as follows, respectively

$$(\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x) = \begin{cases} \underset{x=yz}{rsup} \left[rmin \left\{ \widetilde{\mu}_T(y), \widetilde{\nu}_T(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ [0,0], & \text{otherwise,} \end{cases}$$

and

$$(\lambda_T \circ \eta_T)(x) = \begin{cases} \underset{x=yz}{\wedge} \left[\max \left\{ \lambda_T(y), \eta_T(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ 1, & \text{otherwise,} \end{cases}$$

$$(\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x) = \begin{cases} \underset{x=yz}{rsup} \left[rmin \left\{ \widetilde{\mu}_I(y), \widetilde{\nu}_I(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ [0,0], & \text{otherwise,} \end{cases}$$

and

$$(\lambda_I \circ \eta_I)(x) = \begin{cases} \underset{x=yz}{\wedge} \left[\max \left\{ \lambda_I(y), \eta_I(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ 1, & \text{otherwise,} \end{cases}$$

$$(\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x) = \begin{cases} \underset{x=yz}{rinf} \left[rmax \left\{ \widetilde{\mu}_F(y), \widetilde{\nu}_F(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ [1,1], & \text{otherwise,} \end{cases}$$

and

$$(\lambda_F \circ \eta_F)(x) = \begin{cases} \underset{x=yz}{\vee} \left[\min \left\{ \lambda_F(y), \eta_F(z) \right\} \right], & \text{if } x = yz \text{ for some } y, z \in S, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in S$.

Definition 3.6. Let A and B be a two NC sets in X. The intersection of A and B, denoted by $A \cap B$, is the NC set

$$A \cap B = \left\langle \widetilde{\mu}_T \widetilde{\cap} \widetilde{\nu}_T, \widetilde{\mu}_I \widetilde{\cap} \widetilde{\nu}_I, \widetilde{\mu}_F \widetilde{\cup} \widetilde{\nu}_F, \lambda_T \vee \eta_T, \lambda_I \vee \eta_I, \lambda_F \wedge \eta_F \right\rangle,$$

where

$$\begin{aligned} \left(\widetilde{\mu}_T \widetilde{\cap} \widetilde{\nu}_T\right)(x) &= rmin\{\widetilde{\mu}_T(x), \widetilde{\nu}_T(x)\}, \left(\widetilde{\mu}_I \widetilde{\cap} \widetilde{\nu}_I\right)(x) \\ &= rmin\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\}, \left(\widetilde{\mu}_F \widetilde{\cup} \widetilde{\nu}_F\right)(x) \\ &= rmax\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\} \end{aligned}$$

and

$$\begin{aligned} (\lambda_T \vee \eta_T)(x) &= \max\left\{\lambda_T(x), \eta_T(x)\right\}, (\lambda_I \vee \eta_I)(x) \\ &= \max\left\{\lambda_I(x), \eta_I(x)\right\}, (\lambda_F \wedge \eta_F)(x) \\ &= \min\left\{\lambda_F(x), \eta_F(x)\right\}. \end{aligned}$$

Definition 3.7. The union of A and B, denoted by $A \cup B$, is the NC set

$$A \cup B = \left\langle \widetilde{\mu}_T \widetilde{\cup} \widetilde{\nu}_T, \widetilde{\mu}_I \widetilde{\cup} \widetilde{\nu}_I, \widetilde{\mu}_F \widetilde{\cap} \widetilde{\nu}_F, \lambda_T \wedge \eta_T, \lambda_I \wedge \eta_I, \lambda_F \vee \eta_F \right\rangle,$$

where

$$\begin{aligned} \left(\widetilde{\mu}_T \widetilde{\cup} \widetilde{\nu}_T\right)(x) &= rmax\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\} \\ \left(\widetilde{\mu}_I \widetilde{\cap} \widetilde{\nu}_I\right)(x) &= rmax\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\} \\ \left(\widetilde{\mu}_F \widetilde{\cap} \widetilde{\nu}_F\right)(x) &= rmin\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\} \end{aligned}$$

and

$$\begin{aligned} (\lambda_T \wedge \eta_T)(x) &= \min\{\lambda_T(x), \eta_T(x)\}, \\ (\lambda_I \wedge \eta_I)(x) &= \min\{\lambda_I(x), \eta_I(x)\}, \\ (\lambda_F \vee \eta_F)(x) &= \max\{\lambda_F(x), \eta_F(x)\}. \end{aligned}$$

Proposition 3.8. *For any NC sets* $A = \langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$, $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ *and* $C = \left\langle \widetilde{\zeta}_T, \widetilde{\zeta}_I, \widetilde{\zeta}_F, \delta_T, \delta_I, \delta_F \right\rangle$ *in semigroup* S. *We have*

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(3) $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$.
(4) $A \otimes (B \cap C) \sqsubseteq (A \otimes B) \cap (A \otimes C)$.

Proof. (1) and (2) are straightforward.
(3) Let X be any element of S. If x is not expressed as $x = yz$, then

$$\begin{aligned} (\widetilde{\mu}_T \circ (\widetilde{\nu}_T \widetilde{\cup} \widetilde{\zeta}_T))(x) &= [0,\ 0] = ((\widetilde{\mu}_T \circ \widetilde{\nu}_T) \widetilde{\cup} (\widetilde{\mu}_T \circ \widetilde{\zeta}_T))(x), \\ (\widetilde{\mu}_I \circ (\widetilde{\nu}_I \widetilde{\cup} \widetilde{\zeta}_I))(x) &= [0, 0] = ((\widetilde{\mu}_I \circ \widetilde{\nu}_I) \widetilde{\cup} (\widetilde{\mu}_I \circ \widetilde{\zeta}_I))(x), \\ (\widetilde{\mu}_F \circ (\widetilde{\nu}_F \widetilde{\cap} \widetilde{\zeta}_F))(x) &= [1, 1] = ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \widetilde{\cap} (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x) \end{aligned}$$

and

$$\begin{aligned} (\lambda_T \circ (\eta_T \wedge \delta_T))(x) &= 1 = ((\lambda_T \circ \eta_T) \wedge (\lambda_T \circ \delta_T))(x), \\ (\lambda_I \circ (\eta_I \wedge \delta_I))(x) &= 1 = ((\lambda_I \circ \eta_I) \wedge (\lambda_I \circ \delta_I))(x), \\ (\lambda_F \circ (\eta_F \vee \delta_F))(x) &= 0 = ((\lambda_F \circ \eta_F) \vee (\lambda_F \circ \delta_F))(x). \end{aligned}$$

Therefore $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$. Assume that x is expressed as $x = yz$. Then

$$\begin{aligned}
(\widetilde{\mu}_T \circ (\widetilde{\nu}_T \cup \widetilde{\zeta}_T))(x) &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_T(y), (\widetilde{\nu}_T \cup \widetilde{\zeta}_T)(z)\right\}\right] \\
&= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_T(y), rmax\left\{\widetilde{\nu}_T(z), \widetilde{\zeta}_T(z)\right\}\right\}\right] \\
&= \underset{x=yz}{rsup}\left[rmax\left\{rmin\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}, rmin\left\{\widetilde{\mu}_T(y), \widetilde{\zeta}_T(z)\right\}\right\}\right] \\
&= rmax\left\{\underset{x=yz}{rsup}\left\{[rmin\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}, rmin\left\{\widetilde{\mu}_T(y), \widetilde{\zeta}_T(z)\right\}\right\}]\right\} \\
&= ((\widetilde{\mu}_T \circ \widetilde{\nu}_T) \cup (\widetilde{\mu}_T \circ \widetilde{\zeta}_T))(x) \\
(\widetilde{\mu}_I \circ (\widetilde{\nu}_I \cup \widetilde{\zeta}_I))(x) &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_I(y), (\widetilde{\nu}_I \cup \widetilde{\zeta}_I)(z)\right\}\right] \\
&= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_I(y), rmax\left\{\widetilde{\nu}_I(z), \widetilde{\zeta}_I(z)\right\}\right\}\right] \\
&= \underset{x=yz}{rsup}\left[rmax\left\{rmin\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}, rmin\left\{\widetilde{\mu}_I(y), \widetilde{\zeta}_I(z)\right\}\right\}\right] \\
&= rmax\left\{\underset{x=yz}{rsup}\left\{[rmin\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}, rmin\left\{\widetilde{\mu}_I(y), \widetilde{\zeta}_I(z)\right\}\right\}]\right\} \\
&= ((\widetilde{\mu}_I \circ \widetilde{\nu}_I) \cup (\widetilde{\mu}_I \circ \widetilde{\zeta}_I))(x) \\
(\widetilde{\mu}_F \circ (\widetilde{\nu}_F \cap \widetilde{\zeta}_F))(x) &= \underset{x=yz}{rinf}\left[rmax\left\{\widetilde{\mu}_F(y), (\widetilde{\nu}_F \cap \widetilde{\zeta}_F)(z)\right\}\right] \\
&= \underset{x=yz}{rinf}\left[rmax\left\{\widetilde{\mu}_F(y), rmin\left\{\widetilde{\nu}_F(z), \widetilde{\zeta}_F(z)\right\}\right\}\right] \\
&= \underset{x=yz}{rinf}\left[rmin\left\{rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}, rmax\left\{\widetilde{\mu}_F(y), \widetilde{\zeta}_F(z)\right\}\right\}\right] \\
&= rmin\left\{\underset{x=yz}{rinf}\left\{[rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}, rmax\left\{\widetilde{\mu}_F(y), \widetilde{\zeta}_F(z)\right\}\right\}]\right\} \\
&= ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \cap (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_T \circ (\eta_T \wedge \delta_T))(x) &= \underset{x=yz}{\wedge} \max\{\lambda_T(y), (\eta_T \wedge \delta_T)(z)\} \\
&= \underset{x=yz}{\wedge} \max\{\lambda_T(y), \min\{\eta_T(z), \delta_T(z)\}\} \\
&= \min\left\{\underset{x=yz}{\wedge} \max\{\lambda_T(y), \eta_T(z)\}, \underset{x=yz}{\wedge} \max\{\lambda_T(y), \delta_T(z)\}\right\} \\
&= \min\{(\lambda_T \circ \eta_T), (\lambda_T \circ s_T)\} \\
&= ((\lambda_T \circ \eta_T) \wedge (\lambda_T \circ \delta_T))(x).
\end{aligned}$$

$$\begin{aligned}(\lambda_I\circ(\eta_I\wedge\delta_I))(x) &= \underset{x=yz}{\wedge}\max\{\lambda_I(y),(\eta_I\wedge\delta_I)(z)\}\\ &= \underset{x=yz}{\wedge}\max\{\lambda_I(y),\min\{\eta_I(z),\delta_I(z)\}\}\\ &= \min\left\{\underset{x=yz}{\wedge}\max\{\lambda_I(y),\eta_I(z)\},\underset{x=yz}{\wedge}\max\{\lambda_I(y),\delta_I(z)\}\right\}\\ &= ((\lambda_I\circ\eta_I)\wedge(\lambda_I\circ\delta_I))(x).\end{aligned}$$

$$\begin{aligned}(\lambda_F\circ(\eta_F\vee\delta_F))(x) &= \underset{x=yz}{\vee}\min\{\lambda_F(y),(\eta_F\vee\delta_F)(z)\}\\ &= \underset{x=yz}{\vee}\min\{\lambda_F(y),\max\{\eta_F(z),\delta_F(z)\}\}\\ &= \max\left\{\underset{x=yz}{\vee}\min\{\lambda_F(y),\eta_F(z)\},\underset{x=yz}{\vee}\min\{\lambda_F(y),\delta_F(z)\}\right\}\\ &= ((\lambda_F\circ\eta_F)\vee(\lambda_F\circ\delta_F))(x).\end{aligned}$$

Hence (3) holds.

(4) Let $x\in S$. If x is not expressed as $x=yz$, then it is clear that $A\otimes(B\cap C)\sqsubseteq(A\otimes B)\cap(A\otimes C)$. Assume that there exist $y,z\in S$ such that $x=yz$. Then

$$\begin{aligned}(\widetilde{\mu}_T\circ(\widetilde{\nu}_T\cap\widetilde{\zeta}_T))(x) &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_T(y),(\widetilde{\nu}_T\cap\widetilde{\zeta}_T)(z)\right\}\right]\\ &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_T(y),rmin\left\{\widetilde{\nu}_T(z),\widetilde{\zeta}_T(z)\right\}\right\}\right]\\ &= \underset{x=yz}{rsup}\left[rmin\left\{rmin\{\widetilde{\mu}_T(y),\widetilde{\nu}_T(z)\},rmin\left\{\widetilde{\mu}_T(y),\widetilde{\zeta}_T(z)\right\}\right\}\right]\\ &\preceq rmin\left\{\underset{x=yz}{rsup}[rmin\{\widetilde{\mu}_T(y),\widetilde{\nu}_T(z)\}],\underset{x=yz}{rsup}[rmin\left\{\widetilde{\mu}_T(y),\widetilde{\zeta}_T(z)\right\}]\right\}\\ &= ((\widetilde{\mu}_T\circ\widetilde{\nu}_T)\cap(\widetilde{\mu}_T\circ\widetilde{\zeta}_T))(x)\end{aligned}$$

$$\begin{aligned}(\widetilde{\mu}_I\circ(\widetilde{\nu}_I\cap\widetilde{\zeta}_I))(x) &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_I(y),(\widetilde{\nu}_I\cap\widetilde{\zeta}_I)(z)\right\}\right]\\ &= \underset{x=yz}{rsup}\left[rmin\left\{\widetilde{\mu}_I(y),rmin\left\{\widetilde{\nu}_I(z),\widetilde{\zeta}_I(z)\right\}\right\}\right]\\ &= \underset{x=yz}{rsup}\left[rmin\left\{rmin\{\widetilde{\mu}_I(y),\widetilde{\nu}_I(z)\},rmin\left\{\widetilde{\mu}_I(y),\widetilde{\zeta}_I(z)\right\}\right\}\right]\\ &\preceq rmin\left\{\underset{x=yz}{rsup}[rmin\{\widetilde{\mu}_I(y),\widetilde{\nu}_I(z)\}],\underset{x=yz}{rsup}[rmin\left\{\widetilde{\mu}_I(y),\widetilde{\zeta}_I(z)\right\}]\right\}\\ &= ((\widetilde{\mu}_I\circ\widetilde{\nu}_I)\cap(\widetilde{\mu}_I\circ\widetilde{\zeta}_I))(x)\end{aligned}$$

$$\begin{aligned}
(\widetilde{\mu}_F \circ (\widetilde{\nu}_F \cup \widetilde{\zeta}_F))(x) &= \underset{x=yz}{rinf}\left[rmax\left\{\widetilde{\mu}_F(y), (\widetilde{\nu}_F \cup \widetilde{\zeta}_F)(z)\right\}\right] \\
&= \underset{x=yz}{rinf}\left[rmax\left\{\widetilde{\mu}_F(y), rmax\left\{\widetilde{\nu}_F(z), \widetilde{\zeta}_F(z)\right\}\right\}\right] \\
&= \underset{x=yz}{rinf}\left[rmax\left\{rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}rmax\left\{\widetilde{\mu}_F(y), \widetilde{\zeta}_F(z)\right\}\right\}\right] \\
&\succeq rmax\left\{\underset{x=yz}{r\inf}[rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}], \underset{x=yz}{r\inf}[rmax\left\{\widetilde{\mu}_F(y), \widetilde{\zeta}_F(z)\right\}]\right\} \\
&= ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \cup (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_T \circ (\eta_T \vee \delta_T))(x) &= \bigwedge_{x=yz} \max\{\lambda_T(y), (\eta_T \vee \delta_T)(z)\} \\
&= \bigwedge_{x=yz} \max\{\lambda_T(y), \max\{\eta_T(z), \delta_T(z)\}\} \\
&= \bigwedge_{x=yz} \max\{\max\{\lambda_T(y), \eta_T(z)\}, \max\{\lambda_T(y), \delta_T(z)\}\} \\
&\geq \max\left\{\bigwedge_{x=yz} \max\{\lambda_T(y), \eta_T(z)\}, \bigwedge_{x=yz} \max\{\lambda_T(y), \delta_T(z)\}\right\} \\
&= ((\lambda_T \circ \eta_T) \vee (\lambda_T \circ \delta_T))(x).
\end{aligned}$$

$$\begin{aligned}
(\lambda_I \circ (\eta_I \vee \delta_I))(x) &= \bigwedge_{x=yz} \max\{\lambda_I(y), (\eta_I \vee \delta_I)(z)\} \\
&= \bigwedge_{x=yz} \max\{\lambda_I(y), \max\{\eta_I(z), \delta_I(z)\}\} \\
&= \bigwedge_{x=yz} \max\{\max\{\lambda_I(y), \eta_I(z)\}, \max\{\lambda_I(y), \delta_I(z)\}\} \\
&\geq \max\left\{\bigwedge_{x=yz} \max\{\lambda_I(y), \eta_I(z)\}, \bigwedge_{x=yz} \max\{\lambda_I(y), \delta_I(z)\}\right\} \\
&= ((\lambda_I \circ \eta_I) \vee (\lambda_I \circ \delta_I))(x).
\end{aligned}$$

$$\begin{aligned}
(\lambda_F \circ (\eta_F \wedge \delta_F))(x) &= \bigvee_{x=yz} \min\{\lambda_F(y), (\eta_F \wedge \delta_F)(z)\} \\
&= \bigvee_{x=yz} \min\{\lambda_F(y), \min\{\eta_F(z), \delta_F(z)\}\} \\
&= \bigvee_{x=yz} \min\{\min\{\lambda_F(y), \eta_F(z)\}, \min\{\lambda_F(y), \delta_F(z)\}\} \\
&\leq \min\left\{\bigvee_{x=yz} \min\{\lambda_F(y), \eta_F(z)\}, \bigvee_{x=yz} \min\{\lambda_F(y), \delta_F(z)\}\right\} \\
&= ((\lambda_F \circ \eta_F) \wedge (\lambda_F \circ \delta_F))(x).
\end{aligned}$$

Hence (4) holds. ■

Proposition 3.9. *For any NC sets* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$, $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ *and* $C = \left\langle \widetilde{\zeta}_T, \widetilde{\zeta}_I, \widetilde{\zeta}_F, \delta_T, \delta_I, \delta_F \right\rangle$ *in semigroup* S*, if* $A \sqsubseteq B$*, then* $A \otimes C \sqsubseteq B \otimes C$ *and* $C \otimes A \sqsubseteq C \otimes B$*.*

Proof. Straightforward. ■

Proposition 3.10. *For any non-empty subsets* G *and* H *of semigroup* S*, we have*

(1) $\chi G \otimes \chi H = \chi GH$

$= \langle \widetilde{\mu}_{T\chi G} \widetilde{\circ} \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \widetilde{\circ} \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \widetilde{\circ} \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \circ \lambda_{T\chi H}, \lambda_{I\chi G} \circ \lambda_{I\chi H}, \lambda_{F\chi G} \circ \lambda_{F\chi H} \rangle$
$= \langle \widetilde{\mu}_{T\chi GH}, \widetilde{\mu}_{I\chi GH}, , \widetilde{\mu}_{F\chi GH}, \lambda_{T\chi GH}, \lambda_{I\chi GH}, \lambda_{F\chi GH} \rangle$
(2) $\chi G \cap \chi H = \chi G \cap H$
$= \langle \widetilde{\mu}_{T\chi G} \cap \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \cap \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \cup \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \vee \lambda_{T\chi H}, \lambda_{I\chi G} \vee \lambda_{I\chi H}, \lambda_{F\chi G} \wedge \lambda_{F\chi H} \rangle$
$= \langle \widetilde{\mu}_{T\chi G\cap H}, \widetilde{\mu}_{I\chi G\cap H}, , \widetilde{\mu}_{F\chi G\cup H}, \lambda_{T\chi G\cap H}, \lambda_{I\chi G\cap H}, \lambda_{F\chi G\cup H} \rangle$
(3) $\chi G \cup \chi H = \chi G \cup H$
$= \langle \widetilde{\mu}_{T\chi G} \cup \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \cup \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \cap \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \wedge \lambda_{T\chi H}, \lambda_{I\chi G} \wedge \lambda_{I\chi H}, \lambda_{F\chi G} \vee \lambda_{F\chi H} \rangle$
$= \langle \widetilde{\mu}_{T\chi G\cup H}, \widetilde{\mu}_{F\chi G\cup H}, , \widetilde{\mu}_{F\chi G\cap H}, \lambda_{T\chi G\cup H}, \lambda_{I\chi G\cup H}, \lambda_{F\chi G\cap H} \rangle$.

Proof. (1) Let $a \in S$. If $a \in GH$, then $\widetilde{\mu}_{T\chi GH}(a) = [1,1], \widetilde{\mu}_{I\chi GH}(a) = [1,1], \widetilde{\mu}_{F\chi GH}(a) = [0,0], \lambda_{T\chi GH}(a) = 0, \lambda_{I\chi GH}(a) = 0, \lambda_{F\chi GH}(a) = 1$ and $a = xy$ for some $a \in G$ and $b \in H$. Thus

$$\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= \underset{a=xy}{rsup}\left[rmin\left\{\widetilde{\mu}_{T\chi G}(x), \widetilde{\mu}_{T\chi H}(y)\right\}\right] \\
&\succeq rmin\left\{\widetilde{\mu}_{T\chi G}(b), \widetilde{\mu}_{T\chi H}(c)\right\} = [1,1] \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= \underset{a=xy}{rsup}\left[rmin\left\{\widetilde{\mu}_{I\chi G}(x), \widetilde{\mu}_{I\chi H}(y)\right\}\right] \\
&\succeq rmin\left\{\widetilde{\mu}_{I\chi G}(b), \widetilde{\mu}_{I\chi H}(c)\right\} = [1,1] \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= \underset{a=xy}{rinf}\left[rmax\left\{\widetilde{\mu}_{F\chi G}(x), \widetilde{\mu}_{F\chi H}(y)\right\}\right] \\
&\preceq rmax\left\{\widetilde{\mu}_{F\chi G}(b), \widetilde{\mu}_{F\chi H}(c)\right\} = [0,0]
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= \underset{a=xy}{\wedge}\left[\max\left\{\lambda_{T\chi G}(x), \lambda_{T\chi H}(y)\right\}\right] \\
&\leq \max\left\{\lambda_{T\chi G}(b), \lambda_{T\chi H}(c)\right\} = 0 \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= \underset{a=xy}{\wedge}\left[\max\left\{\lambda_{I\chi G}(x), \lambda_{I\chi H}(y)\right\}\right] \\
&\leq \max\left\{\lambda_{I\chi G}(b), \lambda_{I\chi H}(c)\right\} = 0 \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= \underset{a=xy}{\vee}\left[\min\left\{\lambda_{F\chi G}(x), \lambda_{F\chi H}(y)\right\}\right] \\
&\geq \min\left\{\lambda_{F\chi G}(b), \lambda_{F\chi H}(c)\right\} = 1.
\end{aligned}$$

It follows that

$$\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= [1,1], \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= [1,1], \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= [0,0]
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= 0, \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= 0, \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H}, \lambda_{T\chi G} \circ \lambda_{T\chi H} \rangle &= \langle \widetilde{\mu}_{T\chi GH}, \lambda_{T\chi GH} \rangle \\
\langle \widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H}, \lambda_{I\chi G} \circ \lambda_{I\chi H} \rangle &= \langle \widetilde{\mu}_{I\chi GH}, \lambda_{I\chi GH} \rangle \\
\langle \widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H}, \lambda_{F\chi G} \circ \lambda_{F\chi H} \rangle &= \langle \widetilde{\mu}_{F\chi GH}, \lambda_{F\chi GH} \rangle
\end{aligned}$$

that is, $\chi G \otimes \chi H = \chi GH$. Assume that $a \notin GH$. Then

$$\widetilde{\mu}_{T\chi GH}(a) = [0,0], \widetilde{\mu}_{I\chi GH}(a) = [0,0], \widetilde{\mu}_{F\chi GH}(a) = [1,1]$$

and

$$\lambda_{T\chi GH}(a) = 1, \lambda_{I\chi GH}(a) = 1, \lambda_{F\chi GH}(a) = 0.$$

Let $y, z \in S$ be such that $a = yz$. Then we know that $y \notin G$ or $z \notin H$.Assume that $y \notin G$. Then

$$\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= \underset{a=yz}{rsup}[rmin\{\widetilde{\mu}_{T\chi G}(y), \widetilde{\mu}_{T\chi H}(z)\}] \\
&= \underset{a=yz}{rsup}[rmin\{[0\ 0], \widetilde{\mu}_{T\chi H}(z)\}] \\
&= [0,0] = \widetilde{\mu}_{T\chi GH}(a) \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= \underset{a=yz}{rsup}[rmin\{\widetilde{\mu}_{I\chi G}(y), \widetilde{\mu}_{I\chi H}(z)\}] \\
&= \underset{a=yz}{rsup}[rmin\{[0\ 0], \widetilde{\mu}_{I\chi H}(z)\}] \\
&= [0,0] = \widetilde{\mu}_{I\chi GH}(a) \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= \underset{a=yz}{rinf}[rmax\{\widetilde{\mu}_{F\chi G}(y), \widetilde{\mu}_{F\chi H}(z)\}] \\
&= \underset{a=yz}{rinf}[rmax\{[1,1], \widetilde{\mu}_{F\chi H}(z)\}] \\
&= [1,1] = \widetilde{\mu}_{F\chi GH}(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= \bigwedge_{a=yz}[\max\{\lambda_{T\chi G}(y), \lambda_{T\chi H}(z)\}] \\
&= \bigwedge_{a=yz}[\max\{1, \lambda_{T\chi H}(z)\}] = 1 = \lambda_{T\chi GH}(a) \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= \bigwedge_{a=yz}[\max\{\lambda_{I\chi G}(y), \lambda_{I\chi H}(z)\}] \\
&= \bigwedge_{a=yz}[\max\{1, \lambda_{I\chi H}(z)\}] = 1 = \lambda_{I\chi GH}(a) \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= \bigvee_{a=yz}[\min\{\lambda_{F\chi G}(y), \lambda_{F\chi H}(z)\}] \\
&= \bigvee_{a=yz}[\min\{0, \lambda_{F\chi H}(z)\}] = 0 = \lambda_{F\chi GH}(a)
\end{aligned}$$

Similarly, if $z \notin H$, then

$$\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= [0,0] = \widetilde{\mu}_{T\chi GH}(a), \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= [0,0] = \widetilde{\mu}_{I\chi GH}(a), \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= [1,1] = \widetilde{\mu}_{F\chi GH}(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= 1 = \lambda_{T\chi GH}(a), \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= 1 = \lambda_{I\chi GH}(a), \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= 0 = \lambda_{F\chi GH}(a).
\end{aligned}$$

Therefore $\chi G \otimes \chi H = \chi GH$. The proof of (2) and (3) are straightforward. ■

4. Neutrosophic Cubic Subsemigroups and Ideals

In this section, we define different types of Neutrosophic cubic ideals in semigroups.

Definition 4.1. A NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in semigroup S is called a NC subsemigroup of S if it satisfies:

$$\begin{aligned} (\forall x, y \in S) \quad \widetilde{\mu}_T(xy) &\succeq rmin\{\widetilde{\mu}_T(x), \widetilde{\mu}_T(y)\}, \\ \widetilde{\mu}_I(xy) &\succeq rmin\{\widetilde{\mu}_I(x), \widetilde{\mu}_I(y)\}, \\ \widetilde{\mu}_F(xy) &\preceq rmax\{\widetilde{\mu}_F(x), \widetilde{\mu}_F(y)\}. \\ \lambda_T(xy) &\leq \max\{\lambda_T(x), \lambda_T(y)\}, \\ \lambda_I(xy) &\leq \max\{\lambda_I(x), \lambda_I(y)\}, \\ \lambda_F(xy) &\geq \min\{\lambda_F(x), \lambda_F(y)\}. \end{aligned}$$

Example 4.2. Consider a semigroup $S = \{a, b, c\}$ with the following Cayley's table

.	a	b	c
a	c	c	c
b	c	c	a
c	c	b	c

Define a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.3, 0.6]$	$[0.5, 0.7]$	$[0.6, 0.7]$	0.4	0.6	0.8
b	$[0.2, 0.4]$	$[0.3, 0.4]$	$[0.8, 0.9]$	0.6	0.7	0.6
c	$[0.7, 0.9]$	$[0.8, 0.9]$	$[0.5, 0.6]$	0.2	0.3	0.9

Theorem 4.3. *A NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in semigroup S is a NC subsemigroup of S if and only if $A \otimes A \sqsubseteq A$.*

Proof. Straightforward. ∎

Definition 4.4. A NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in a semigroup S is called a right($left$) NC ideal of S if $\forall\ x, y \in S$ it satisfies:

$$\begin{aligned} \widetilde{\mu}_T(xy) &\succeq \widetilde{\mu}_T(x)\,(\widetilde{\mu}_T(xy) \succeq \widetilde{\mu}_T(y)), \\ \widetilde{\mu}_I(xy) &\succeq \widetilde{\mu}_I(x)\,(\widetilde{\mu}_I(xy) \succeq \widetilde{\mu}_I(y)), \\ \widetilde{\mu}_F(xy) &\preceq \widetilde{\mu}_F(x)\,(\widetilde{\mu}_F(xy) \preceq \widetilde{\mu}_F(y)) \\ \lambda_T(xy) &\leq \lambda_T(x)\,(\lambda_T(xy) \leq \lambda_T(y)), \\ \lambda_I(xy) &\leq \lambda_I(x)\,(\lambda_I(xy) \leq \lambda_I(y)), \\ \lambda_F(xy) &\geq \lambda_F(x)(\lambda_F(xy) \geq \lambda_F(y)). \end{aligned}$$

By a (two sided) NC ideal we mean a left and right NC ideal.

Example 4.5. Consider a semigroup $S = \{a, b, c\}$ with the following Cayley's table

.	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

Define a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.7, 0.9]$	$[0.8, 0.9]$	$[0.1, 0.3]$	0.2	0.3	0.9
b	$[0.2, 0.4]$	$[0.3, 0.4]$	$[0.2, 0.5]$	0.6	0.7	0.7
c	$[0.1, 0.3]$	$[0.5, 0.6]$	$[0.2, 0.4]$	0.7	0.5	0.8

It is easy to verify that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC ideal of S. Obviously, every left (resp. right) NC ideal is a NC subsemigroup. But the converse may not be true as seen in the following example.

Example 4.6. Consider a semigroup $S = \{a, b, c, d\}$ with the following Cayley's table

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Define a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.5, 0.8]$	$[0.7, 0.9]$	$[0.1, 0.3]$	0.2	0.1	0.3
b	$[0.3, 0.6]$	$[0.4, 0.7]$	$[0.3, 0.5]$	0.6	0.7	0.4
c	$[0.5, 0.8]$	$[0.5, 0.6]$	$[0.2, 0.3]$	0.4	0.5	0.4
d	$[0.2, 0.4]$	$[0.2, 0.4]$	$[0.5, 0.6]$	0.6	0.8	0.5

It is easy to verify that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC subsemigroup of S, but it is not a left NC ideal of S since $\widetilde{\mu}_T(dc) = \widetilde{\mu}_T(b) = [0.3, 0.6] \not\succeq [0.5, 0.8] = \widetilde{\mu}_T(c)$ and $\lambda_T(dc) = \lambda_T(b) = 0.6 > 0.4 = \lambda_T(c)$.

Theorem 4.7. *For NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in semigroup S, the following statements are equivalent:*

(1) *$A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S.*

(2) *$S \otimes A \sqsubseteq A$.*

Proof. Assume that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S. Let $a \in S$. If

$$(S \otimes A)(a) = \left\langle \widetilde{0}, \widetilde{0}, \widetilde{1}, 1, 1, 0 \right\rangle,$$

then it is clear that $S \otimes A \sqsubseteq A$. Otherwise, there exist $x, y \in S$ such that $a = xy$. Since $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S, we have

$$\begin{aligned}
\left(\widetilde{1}_S \circ \widetilde{\mu}_T\right)(a) &= \underset{a=xy}{rsup}\left[rmin\left\{\widetilde{1}_S(x), \widetilde{\mu}_T(y)\right\}\right] \\
&\preceq rmin\left\{[1,1], \widetilde{\mu}_T(xy)\right\} = \widetilde{\mu}_T(a) \\
\left(\widetilde{1}_S \circ \widetilde{\mu}_I\right)(a) &= \underset{a=xy}{rsup}\left[rmin\left\{\widetilde{1}_S(x), \widetilde{\mu}_I(y)\right\}\right] \\
&\preceq rmin\left\{[1,1], \widetilde{\mu}_I(xy)\right\} = \widetilde{\mu}_I(a) \\
\left(\widetilde{0}_S \circ \widetilde{\mu}_F\right)(a) &= \underset{a=xy}{rinf}\left[rmax\left\{\widetilde{0}_S(x), \widetilde{\mu}_F(y)\right\}\right] \\
&\succeq rmax\left\{[0,0], \widetilde{\mu}_F(xy)\right\} = \widetilde{\mu}_F(a)
\end{aligned}$$

and

$$\begin{aligned}
(0_S \circ \lambda_T)(a) &= \underset{a=xy}{\wedge} \max\{0_S(x), \lambda_T(y)\} \\
&\geq \max\{\lambda_T(xy)\} = \lambda_T(a) = \lambda_T(a) \\
(0_S \circ \lambda_I)(a) &= \underset{a=xy}{\wedge} \max\{0_S(x), \lambda_I(y)\} \\
&\geq \max\{\lambda_I(xy)\} = \lambda_I(a) = \lambda_I(a) \\
(1_S \circ \lambda_F)(a) &= \underset{a=xy}{\vee} \min\{1_S(x), \lambda_F(y)\} \\
&\leq \min\{\lambda_F(xy)\} = \lambda_F(a) = \lambda_F(a).
\end{aligned}$$

Therefore $S \otimes A \sqsubseteq A$. Conversely, suppose that $S \otimes A \sqsubseteq A$. For any elements x, y of S, let $a = xy$. Then

$$\begin{aligned}
\widetilde{\mu}_T(xy) &= \widetilde{\mu}_T(a) \succeq \left(\widetilde{1}_S \circ \widetilde{\mu}_T\right)(a) = \underset{a=bc}{rsup}\left[rmin\left\{\widetilde{1}_S(b), \widetilde{\mu}_T(c)\right\}\right] \\
&\succeq rmin\left\{\widetilde{1}_S(x), \widetilde{\mu}_T(y)\right\} = \widetilde{\mu}_T(y) \\
\widetilde{\mu}_I(xy) &= \widetilde{\mu}_I(a) \succeq \left(\widetilde{1}_S \circ \widetilde{\mu}_I\right)(a) = \underset{a=bc}{rsup}\left[rmin\left\{\widetilde{1}_S(b), \widetilde{\mu}_I(c)\right\}\right] \\
&\succeq rmin\left\{\widetilde{1}_S(x), \widetilde{\mu}_I(y)\right\} = \widetilde{\mu}_I(y) \\
\widetilde{\mu}_F(xy) &= \widetilde{\mu}_F(a) \preceq \left(\widetilde{0}_S \circ \widetilde{\mu}_F\right)(a) = \underset{a=bc}{rinf}\left[rmax\left\{\widetilde{0}_S(b), \widetilde{\mu}_F(c)\right\}\right] \\
&\preceq rmax\left\{\widetilde{0}_S(x), \widetilde{\mu}_F(y)\right\} = \widetilde{\mu}_F(y)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_T(xy) &= \lambda_T(a) \leq (0_S \circ \lambda_T)(a) = \underset{a=bc}{\wedge} \max\{0_S(b), \lambda_T(c)\} \\
&\leq \max\{0_S(x), \lambda_T(y)\} = \lambda_T(y) \\
\lambda_I(xy) &= \lambda_I(a) \leq (0_S \circ \lambda_I)(a) = \underset{a=bc}{\wedge} \max\{0_S(b), \lambda_I(c)\} \\
&\leq \max\{0_S(x), \lambda_I(y)\} = \lambda_I(y) \\
\lambda_F(xy) &= \lambda_F(a) \geq (1_S \circ \lambda_F)(a) = \underset{a=bc}{\vee} \max\{1_S(b), \lambda_F(c)\} \\
&\geq \min\{1_S(x), \lambda_F(y)\} = \lambda_F(y)
\end{aligned}$$

Hence $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S. ■

Similarly, we can induce the following theorem.

Theorem 4.8. *For a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in a semigroup S, the following are equivalent:*
(1) *$A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a right NC ideal of S.*
(2) *$A \otimes S \sqsubseteq A$.*

Theorem 4.9. *If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC set in a semigroup S, then $S \otimes A$ (resp. $A \otimes S$) is a left (resp. right) NC ideal of S.*

Proof. Since $S \otimes (S \otimes A) = (S \otimes S) \otimes A \sqsubseteq S \otimes A$, it follows from Theorem 4.7, that $S \otimes A$ is a left NC ideal of S. Similarly $A \otimes S$ is a right NC ideal of S. ■

Now we will consider the conditions for a left (resp. right) NC ideal to be constant.

Proposition 4.10. *Let U be a left zero subsemigroup of a semigroup S. If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S, then $A(x) = A(y)$ for all $x, y \in U$.*

Proof. Let $x, y \in U$. Then $xy = x$ and $yx = y$. Thus

$$\begin{aligned} \widetilde{\mu}_T(x) &= \widetilde{\mu}_T(xy) \succeq \widetilde{\mu}_T(y) = \widetilde{\mu}_T(yx) \succeq \widetilde{\mu}_T(x) \\ \widetilde{\mu}_I(x) &= \widetilde{\mu}_I(xy) \succeq \widetilde{\mu}_I(y) = \widetilde{\mu}_I(yx) \succeq \widetilde{\mu}_I(x) \\ \widetilde{\mu}_F(x) &= \widetilde{\mu}_F(xy) \preceq \widetilde{\mu}_F(y) = \widetilde{\mu}_F(yx) \preceq \widetilde{\mu}_F(x) \end{aligned}$$

and

$$\begin{aligned} \lambda_T(x) &= \lambda_T(xy) \leq \lambda_T(y) = \lambda_T(xy) \leq \lambda_T(x) \\ \lambda_I(x) &= \lambda_I(xy) \leq \lambda_I(y) = \lambda_I(xy) \leq \lambda_I(x) \\ \lambda_F(x) &= \lambda_F(xy) \geq \lambda_F(y) = \lambda_F(xy) \geq \lambda_F(x) \end{aligned}$$

Therefore $A(x) = A(y)$ for all $x, y \in U$. ∎

Similarly, we have the following proposition.

Proposition 4.11. *Let U be a right zero subsemigroup of a semigroup S. If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a right NC ideal of S, then $A(x) = A(y)$ for all $x, y \in U$.*

Theorem 4.12. *Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of a semigroup S. If the set of all idempotent elements of S forms a left zero subsemigroup of S, then $A(u) = A(v)$ for all idempotent elements u and v of S.*

Proof. Let $Idm(S)$ be the set of all idempotent elements of S and assume that $Idm(S)$ is a left zero subsemigroup of S. For any $u, v \in Idm(S)$, we have $uv = u$ and $uv = v$. Hence

$$\begin{aligned} \widetilde{\mu}_T(u) &= \widetilde{\mu}_T(uv) \succeq \widetilde{\mu}_T(v) = \widetilde{\mu}_T(uv) \succeq \widetilde{\mu}_T(u) \\ \widetilde{\mu}_I(u) &= \widetilde{\mu}_I(uv) \succeq \widetilde{\mu}_I(v) = \widetilde{\mu}_I(uv) \succeq \widetilde{\mu}_I(u) \\ \widetilde{\mu}_F(u) &= \widetilde{\mu}_F(uv) \preceq \widetilde{\mu}_F(v) = \widetilde{\mu}_F(uv) \preceq \widetilde{\mu}_F(u) \end{aligned}$$

and

$$\begin{aligned} \lambda_T(u) &= \lambda_T(uv) \leq \lambda_T(v) = \lambda_T(uv) \leq \lambda_T(u) \\ \lambda_I(u) &= \lambda_I(uv) \leq \lambda_I(v) = \lambda_I(uv) \leq \lambda_I(u) \\ \lambda_F(u) &= \lambda_F(uv) \geq \lambda_F(v) = \lambda_F(uv) \geq \lambda_F(u) \end{aligned}$$

Therefore $A(u) = A(v)$ for all $u, v \in Idm(S)$. ∎

Similarly, we have the following theorem.

Theorem 4.13. *Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a right NC ideal of a semigroup S. If the set of all idempotent elements of S forms a right zero subsemigroup of S, then $A(u) = A(v)$ for all idempotent elements u and v of S.*

Theorem 4.14. *Let S be a semigroup. Then the following properties hold:*

(1) *The intersection of two NC subsemigroups of S is a NC subsemigroup of S.*
(2) *The intersection of two left (resp. right) NC ideals of S is a left (resp. right) NC ideals of S.*

Proof. (1) Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ be two NC subsemigroups of S. Let x and y be any elements of S. Then

$$\begin{aligned}
(\widetilde{\mu}_T \cap \widetilde{\nu}_T)(xy) &= rmin\{\widetilde{\mu}_T(xy), \widetilde{\nu}_T(xy)\} \\
&\succeq rmin\{rmin\{\widetilde{\mu}_T(x), \widetilde{\mu}_T(y)\}, rmin\{\widetilde{\nu}_T(x), \widetilde{\nu}_T(y)\}\} \\
&= rmin\{rmin\{\widetilde{\mu}_T(x), \widetilde{\nu}_T(x)\}, rmin\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(y)\}\} \\
&= rmin\{(\widetilde{\mu}_T \cap \widetilde{\nu}_T)(x), (\widetilde{\mu}_T \cap \widetilde{\nu}_T)(y)\} \\
(\widetilde{\mu}_I \cap \widetilde{\nu}_I)(xy) &= rmin\{\widetilde{\mu}_I(xy), \widetilde{\nu}_I(xy)\} \\
&\succeq rmin\{rmin\{\widetilde{\mu}_I(x), \widetilde{\mu}_I(y)\}, rmin\{\widetilde{\nu}_I(x), \widetilde{\nu}_I(y)\}\} \\
&= rmin\{rmin\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\}, rmin\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(y)\}\} \\
&= rmin\{(\widetilde{\mu}_I \cap \widetilde{\nu}_I)(x), (\widetilde{\mu}_I \cap \widetilde{\nu}_I)(y)\} \\
(\widetilde{\mu}_F \cup \widetilde{\nu}_F)(xy) &= rmax\{\widetilde{\mu}_F(xy), \widetilde{\nu}_F(xy)\} \\
&\preceq rmax\{rmax\{\widetilde{\mu}_F(x), \widetilde{\mu}_F(y)\}, rmax\{\widetilde{\nu}_F(x), \widetilde{\nu}_F(y)\}\} \\
&= rmax\{rmax\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\}, rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(y)\}\} \\
&= rmax\{(\widetilde{\mu}_F \cup \widetilde{\nu}_F)(x), (\widetilde{\mu}_F \cup \widetilde{\nu}_F)(y)\}
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_T \vee \eta_T)(xy) &= \max\{\lambda_T(xy), \eta_T(xy)\} \\
&\leq \max\{\max\{\lambda_T(x), \lambda_T(y)\}, \max\{\eta_T(x), \eta_T(y)\}\} \\
&= \max\{\max\{\lambda_T(x), \eta_T(x)\}, \max\{\lambda_T(y), \eta_T(y)\}\} \\
&= \max\{(\lambda_T \vee \eta_T)(x), (\lambda_T \vee \eta_T)(y)\} \\
(\lambda_I \vee \eta_I)(xy) &= \max\{\lambda_I(xy), \eta_I(xy)\} \\
&\leq \max\{\max\{\lambda_I(x), \lambda_I(y)\}, \max\{\eta_I(x), \eta_I(y)\}\} \\
&= \max\{\max\{\lambda_I(x), \eta_I(x)\}, \max\{\lambda_I(y), \eta_I(y)\}\} \\
&= \max\{(\lambda_I \vee \eta_I)(x), (\lambda_I \vee \eta_I)(y)\} \\
(\lambda_F \wedge \eta_F)(xy) &= \min\{\lambda_F(xy), \eta_F(xy)\} \\
&\geq \min\{\min\{\lambda_F(x), \lambda_F(y)\}, \min\{\eta_F(x), \eta_F(y)\}\} \\
&= \min\{\min\{\lambda_F(x), \eta_F(x)\}, \min\{\lambda_F(y), \eta_F(y)\}\} \\
&= \min\{(\lambda_F \wedge \eta_F)(x), (\lambda_F \wedge \eta_F)(y)\}
\end{aligned}$$

Therefore $A \sqcap B = \langle \widetilde{\mu}_T \cap \widetilde{\nu}_T, \widetilde{\mu}_I \cap \widetilde{\nu}_I, \widetilde{\mu}_F \cup \widetilde{\nu}_F, \lambda_T \vee \eta_T, \lambda_I \vee \eta_I, \lambda_F \wedge \eta_F \rangle$ is a NC subsemigroup of S.

The second property can be proved in a similar manner. ■

Proposition 4.15. *If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is right NC ideal and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ is a left NC ideal of a semigroup S, then $A \otimes B \sqsubseteq A \sqcap B$.*

Proof. Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is right NC ideal and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ is any NC ideal of S. Then by Theorem 4.7 and Theorem 4.8 we have $A \otimes B \sqsubseteq A \otimes S \sqsubseteq A$ and $A \otimes B \sqsubseteq S \otimes B \sqsubseteq B$. Thus $A \otimes B \sqsubseteq A \sqcap B$. ■

Proposition 4.16. *If S is a regular semigroup, then $A \otimes B = A \sqcap B$ for every right NC ideal $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and every left NC ideal $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ of S.*

Proof. Let a be any element of S. Since S is regular, there exist an element $x \in S$ such that $a = axa$. Hence we have

$$\begin{aligned}
(\widetilde{\mu}_T \circ \widetilde{\nu}_T)(a) &= \underset{a=yz}{rsup}\{rmin\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}\} \succeq rmin\{\widetilde{\mu}_T(ax), \widetilde{\nu}_T(a)\} \\
&\succeq rmin\{\widetilde{\mu}_T(a), \widetilde{\nu}_T(a)\} = (\widetilde{\mu}_T \cap \widetilde{\nu}_T)(a) \\
(\widetilde{\mu}_I \circ \widetilde{\nu}_I)(a) &= \underset{a=yz}{rsup}\{rmin\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}\} \succeq rmin\{\widetilde{\mu}_I(ax), \widetilde{\nu}_I(a)\} \\
&\succeq rmin\{\widetilde{\mu}_I(a), \widetilde{\nu}_I(a)\} = (\widetilde{\mu}_I \cap \widetilde{\nu}_I)(a) \\
(\widetilde{\mu}_F \circ \widetilde{\nu}_F)(a) &= \underset{a=yz}{rinf}\{rmax\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}\} \preceq rmax\{\widetilde{\mu}_F(ax), \widetilde{\nu}_F(a)\} \\
&\preceq rmax\{\widetilde{\mu}_F(a), \widetilde{\nu}_F(a)\} = (\widetilde{\mu}_F \cup \widetilde{\nu}_F)(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_T \circ \eta_T)(a) &= \bigwedge_{a=yz} \max\{\lambda_T(y), \eta_T(z)\} \leq \max\{\lambda_T(ax), \eta_T(a)\} \\
&\leq \max\{\lambda_T(a), \eta_T(a)\} = (\lambda_T \cap \eta_T)(a) \\
(\lambda_I \circ \eta_I)(a) &= \bigwedge_{a=yz} \max\{\lambda_I(y), \eta_I(z)\} \leq \max\{\lambda_I(ax), \eta_I(a)\} \\
&\leq \max\{\lambda_I(a), \eta_I(a)\} = (\lambda_I \cap \eta_I)(a) \\
(\lambda_F \circ \eta_F)(a) &= \bigvee_{a=yz} \min\{\lambda_F(y), \eta_F(z)\} \geq \min\{\lambda_F(ax), \eta_F(a)\} \\
&\geq \min\{\lambda_F(a), \eta_F(a)\} = (\lambda_F \cup \eta_F)(a)
\end{aligned}$$

and so $A \otimes B \sqsupseteq A \sqcap B$. It follows from Proposition 4.15 that $A \otimes B = A \sqcap B$. ∎

We now discuss the converse of Proposition 4.16. We first consider the following lemmas.

Lemma 4.17. *[2] For a semigroup S, the following conditions are equivalent.*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R of S and every left ideal L of S.*

Lemma 4.18. *For a non-empty subset G of a semigroup S, we have*

(1) G is a subsemigroup of S if and only if the characteristic NC set $\chi = \langle \widetilde{\mu}_{T\chi}, \widetilde{\mu}_{I\chi}, \widetilde{\mu}_{F\chi}, \lambda_{T\chi}, \lambda_{I\chi}, \lambda_{F\chi} \rangle$ of G in S is a NC subsemigroup of S.
(2) G is a left (right) ideal of S if and only if the characteristic NC set $\chi = \langle \widetilde{\mu}_{T\chi}, \widetilde{\mu}_{I\chi}, \widetilde{\mu}_{F\chi}, \lambda_{T\chi}, \lambda_{I\chi}, \lambda_{F\chi} \rangle$ of G in S is a left (resp. right) NC ideal of S.

Proof. Straightforward. ∎

Theorem 4.19. *For every right NC ideal $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and every left NC ideal $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ of a semigroup S, if $A \otimes B = A \cap B$, then S is regular.*

Proof. Assume that $A \otimes B = A \cap B$ for every right NC ideal $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and every left NC ideal $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ of a semigroup S. Let R and L be any right and left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let a be any element of $R \cap L$. Then the characteristic NC sets $\chi_R = \langle \widetilde{\mu}_{T\chi_R}, \widetilde{\mu}_{I\chi_R}, \widetilde{\mu}_{F\chi_R}, \lambda_{T\chi_R}, \lambda_{I\chi_R}, \lambda_{F\chi_R} \rangle$ and $\chi_L = \langle \widetilde{\mu}_{T\chi_L}, \widetilde{\mu}_{I\chi_L}, \widetilde{\mu}_{F\chi_L}, \lambda_{T\chi_L}, \lambda_{I\chi_L}, \lambda_{F\chi_L} \rangle$ are a right NC ideal and a left NC ideal of S, respectively, by Lemma 4.18. It follows from the hypothesis and Proposition 3.10, that

$$\begin{aligned}
\widetilde{\mu}_{T\chi_{RL}}(a) &= (\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L})(a) = (\widetilde{\mu}_{T\chi_R} \cap \widetilde{\mu}_{T\chi_L})(a) \\
&= \widetilde{\mu}_{T\chi_{R\cap L}}(a) = [1,1] \\
\widetilde{\mu}_{I\chi_{RL}}(a) &= (\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L})(a) = (\widetilde{\mu}_{T\chi_R} \cap \widetilde{\mu}_{T\chi_L})(a) \\
&= \widetilde{\mu}_{T\chi_{R\cap L}}(a) = [1,1] \\
\widetilde{\mu}_{F\chi_{RL}}(a) &= (\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L})(a) = (\widetilde{\mu}_{T\chi_R} \cup \widetilde{\mu}_{T\chi_L})(a) \\
&= \widetilde{\mu}_{T\chi_{R\cup L}}(a) = [0,0]
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{T\chi_{RL}}(a) &= \left(\lambda_{T\chi_R} \circ \lambda_{T\chi_L}\right)(a) = \left(\lambda_{T\chi_R} \cap \lambda_{T\chi_L}\right)(a) \\
&= \lambda_{T\chi_{R\cap L}}(a) = 0 \\
\lambda_{I\chi_{RL}}(a) &= \left(\lambda_{I\chi_R} \circ \lambda_{I\chi_L}\right)(a) = \left(\lambda_{I\chi_R} \cap \lambda_{I\chi_L}\right)(a) \\
&= \lambda_{I\chi_{R\cap L}}(a) = 0 \\
\lambda_{F\chi_{RL}}(a) &= \left(\lambda_{F\chi_R} \circ \lambda_{F\chi_L}\right)(a) = \left(\lambda_{F\chi_R} \cup \lambda_{F\chi_L}\right)(a) \\
&= \lambda_{F\chi_{R\cup L}}(a) = 1
\end{aligned}$$

and so that $a \in RL$. Thus $R \cap L \subseteq RL$. Since the inclusion in the other direction always holds, we obtain that $R \cap L = RL$. It follows from Lemma 4.17 that S is regular. ∎

Definition 4.20. Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ be a NC set in X. For any $s, o, g \in [0,1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0,1]$, we define $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ as follows:

$$\begin{aligned}
& U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right) \\
= \; & \left\{ x \in X \middle| \left\langle \widetilde{\mu}_T(x) \succeq \widetilde{t}, \widetilde{\mu}_I(x) \succeq \widetilde{i}, \widetilde{\mu}_F(x) \preceq \widetilde{f}, \lambda_T(x) \leq s, \lambda_I(x) \leq o, \lambda_F(x) \geq g \right\rangle \right\}
\end{aligned}$$

and we say it is a NC level set of $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$.

Theorem 4.21. *For a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in a semigroup S, the following statements are equivalent:*

(1) *$A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC subsemigroup of S.*

(2) *Every non-empty NC level set of $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a subsemigroup of S.*

Proof. Assume that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC subsemigroup of S. Let $x, y \in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ for all $s, o, g \in [0,1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0,1]$. Then

$$\begin{aligned}
\widetilde{\mu}_T(x) &\succeq \widetilde{t}, \widetilde{\mu}_I(x) \succeq \widetilde{i}, \widetilde{\mu}_F(x) \preceq \widetilde{f} \\
\lambda_T(x) &\leq s, \lambda_I(x) \leq o, \lambda_F(x) \geq g, \\
\widetilde{\mu}_T(y) &\succeq \widetilde{t}, \widetilde{\mu}_I(y) \succeq \widetilde{i}, \widetilde{\mu}_F(y) \preceq \widetilde{f} \\
\lambda_T(y) &\leq s, \lambda_I(y) \leq o, \lambda_F(y) \geq g.
\end{aligned}$$

It follows form Definition 4.1, that

$$\begin{aligned}\widetilde{\mu}_T(xy) &\succeq rmin\{\widetilde{\mu}_T(x), \widetilde{\mu}_T(y)\} \succeq \widetilde{t}\\ \widetilde{\mu}_I(xy) &\succeq rmin\{\widetilde{\mu}_I(x), \widetilde{\mu}_I(y)\} \succeq \widetilde{i}\\ \widetilde{\mu}_F(xy) &\preceq rmax\{\widetilde{\mu}_F(x), \widetilde{\mu}_F(y)\} \preceq \widetilde{f}\end{aligned}$$

and

$$\begin{aligned}\lambda_T(xy) &\leq \max\{\lambda_T(x), \lambda_T(y)\} \leq s,\\ \lambda_I(xy) &\leq \max\{\lambda_I(x), \lambda_I(y)\} \leq o,\\ \lambda_F(xy) &\geq \min\{\lambda_F(x), \lambda_F(y)\} \geq g.\end{aligned}$$

Hence $xy \in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$ and thus $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$ is a subsemigroup of S. Conversely, let $s, o, g \in [0, 1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0, 1]$ be a such that $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right) \neq \emptyset$, and $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$ is a subsemigroup of S. Suppose that Definition 4.1 is false. Then there exist $a, b \in S$ such that

$$\begin{aligned}\widetilde{\mu}_T(ab) &\not\succeq rmin\{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}\\ \widetilde{\mu}_I(ab) &\not\succeq rmin\{\widetilde{\mu}_I(a), \widetilde{\mu}_I(b)\}\\ \widetilde{\mu}_F(ab) &\not\preceq rmax\{\widetilde{\mu}_F(a), \widetilde{\mu}_F(b)\}\end{aligned}$$

and

$$\begin{aligned}\lambda_T(ab) &\nleq \max\{\lambda_T(a), \lambda_T(b)\}\\ \lambda_I(ab) &\nleq \max\{\lambda_I(a), \lambda_I(b)\}\\ \lambda_F(ab) &\ngeq \min\{\lambda_F(a), \lambda_F(b)\}.\end{aligned}$$

If $\widetilde{\mu}_T(ab) \not\succeq rmin\{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}$, then $\widetilde{\mu}_T(ab) \prec \widetilde{t} \preceq crmin\{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}$ for some $\widetilde{t} \in D[0, 1]$. Hence $a, b \in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$, but $ab \notin U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$. It is impossible. Similar results can be deduced for any component $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g\right\rangle\right)$. Hence Definition 4.1 is valid, and therefore $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle$ is a NC subsemigroup of S. ∎

Theorem 4.22. *For a NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle$ in a semigroup S, the following statements are equivalent:*

(1) *$A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle$ is a left (resp. right) NC ideal of S.*

(2) *Every non-empty NC level set of $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle$ is a left (resp. right) NC ideal of S.*

Proof. It can be easily verified by the similar way to the proof of Theorem 4.21. ∎

5. Neutrosophic Cubic Transformations of Semigroups

In this section, we present some results related with neutrosophic cubic transformations and inverse neutrosophic cubic transformations of semigroups.

Denote by $N(X)$ the family of NC sets in a set X. Let X and Y be two classical sets. A mapping $h : X \longrightarrow Y$ induces two mappings $N_h : N(X) \longrightarrow N(Y), A \longmapsto N_h(A)$, and

$N_h^{-1} : N(Y) \longrightarrow N(X), B \longmapsto N_h^{-1}(B)$, where $N_h(A)$ is given by

$$N_h(\widetilde{\mu}_T)(y) = \begin{cases} \underset{y=h(x)}{rsup}\, \widetilde{\mu}_T(x), & \text{if } h^{-1}(y) \neq 0, \\ [0,0], & \text{otherwise,} \end{cases}$$

$$N_h(\widetilde{\mu}_I)(y) = \begin{cases} \underset{y=h(x)}{rsup}\, \widetilde{\mu}_I(x), & \text{if } h^{-1}(y) \neq 0, \\ [0,0], & \text{otherwise,} \end{cases}$$

$$N_h(\widetilde{\mu}_F)(y) = \begin{cases} \underset{y=h(x)}{rinf}\, \widetilde{\mu}_F(x), & \text{if } h^{-1}(y) \neq 0, \\ [1,1], & \text{otherwise,} \end{cases}$$

$$N_h(\lambda_T)(y) = \begin{cases} \inf\limits_{y=h(x)} \lambda_T(x), & \text{if } h^{-1}(y) \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

$$N_h(\lambda_I)(y) = \begin{cases} \inf\limits_{y=h(x)} \lambda_I(x), & \text{if } h^{-1}(y) \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

$$N_h(\lambda_F)(y) = \begin{cases} \sup\limits_{y=h(x)} \lambda_F(x), & \text{if } h^{-1}(y) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$ and $N_h^{-1}(B)$ is defined by

$$N_h^{-1}(\widetilde{\nu}_T)(x) = \widetilde{\nu}_T(h(x)), N_h^{-1}(\widetilde{\nu}_I)(x) = \widetilde{\nu}_I(h(x)), N_h^{-1}(\widetilde{\nu}_F)(x) = \widetilde{\nu}_F(h(x))$$

and

$N_h^{-1}(\eta_T)(x) = \eta_T(h(x)), N_h^{-1}(\eta_I)(x) = \eta_I(h(x)), N_h^{-1}(\eta_F)(x) = \eta_F(h(x))$ for all $x \in X$.

Then the mapping N_h (*resp.* N_h^{-1}) is called a NC transformation (resp. inverse NC transformation) induced by h. A NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in X has the NC property if for any subset C of X there exists $x_0 \in C$ such that

$$\widetilde{\mu}_T(x_0) = \underset{x\in C}{rsup}\widetilde{\mu}_T(x), \widetilde{\mu}_I(x_0) = \underset{x\in C}{rsup}\widetilde{\mu}_I(x), \widetilde{\mu}_F(x_0) = \underset{x\in C}{rinf}\widetilde{\mu}_F(x)$$

and

$$\lambda_T(x_0) = \inf_{x\in C}\lambda_T(x), \lambda_I(x_0) = \inf_{x\in C}\lambda_I(x), \lambda_F(x_0) = \sup_{x\in C}\lambda_F(x).$$

Theorem 5.1. *For a homomorphism $h : X \longrightarrow Y$ of semigroups, let $N_h : N(X) \longrightarrow N(Y)$ and $N_h^{-1} : N(Y) \longrightarrow N(X)$ be the NC transformation and inverse NC transformation, respectively, induced by h.*

(1) *If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle \in N(X)$ is a NC subsemigroup of X which has the NC property, then $N_h(A)$ is a NC subsemigroup of Y.*

(2) *If $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T,, \eta_I, \eta_F \rangle \in N(Y)$ is a NC subsemigroup of Y, then $N_h^{-1}(B)$ is a NC subsemigroup of X.*

Proof. (1) Given $h(x), h(y) \in h(X)$, let $x_0 \in h^{-1}(h(x))$ and $y_0 \in h^{-1}(h(y))$ be such that

$$\begin{aligned}
\widetilde{\mu}_T(x_0) &= \underset{a\in h^{-1}(h(x))}{rsup} \widetilde{\mu}_T(a), \widetilde{\mu}_I(x_0) = \underset{a\in h^{-1}(h(x))}{rsup} \widetilde{\mu}_I(a), \widetilde{\mu}_F(x_0) = \underset{a\in h^{-1}(h(x))}{rinf} \widetilde{\mu}_F(a) \\
\lambda_T(x_0) &= \inf_{a\in h^{-1}(h(x))} \lambda_T(a), \lambda_I(x_0) = \inf_{a\in h^{-1}(h(x))} \lambda_I(a), \lambda_F(x_0) = \sup_{a\in h^{-1}(h(x))} \lambda_F(a)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mu}_T(y_0) &= \underset{b\in h^{-1}(h(y))}{rsup} \widetilde{\mu}_T(b), \widetilde{\mu}_I(y_0) = \underset{b\in h^{-1}(h(y))}{rsup} \widetilde{\mu}_I(b), \widetilde{\mu}_F(y_0) = \underset{b\in h^{-1}(h(y))}{rinf} \widetilde{\mu}_F(b) \\
\lambda_T(y_0) &= \inf_{b\in h^{-1}(h(y))} \lambda_T(b), \lambda_I(y_0) = \inf_{b\in h^{-1}(h(y))} \lambda_I(b), \lambda_F(y_0) = \sup_{b\in h^{-1}(h(y))} \lambda_F(b)
\end{aligned}$$

respectively. Then

$$\begin{aligned}
N_h(\widetilde{\mu}_T)(h(x)h(y)) &= \underset{z\in h^{-1}(h(x)h(y))}{rsup} \widetilde{\mu}_T(z) \\
&\succeq \widetilde{\mu}_T(x_0y_0) \succeq rmin\{\widetilde{\mu}_T(x_0), \widetilde{\mu}_T(y_0)\} \\
&= rmin\left\{\underset{a\in h^{-1}(h(x))}{rsup} \widetilde{\mu}_T(a), \underset{b\in h^{-1}(h(y))}{rsup} \widetilde{\mu}_T(b)\right\} \\
&= rmin\{N_h(\widetilde{\mu}_T)(h(x)), N_h(\widetilde{\mu}_T)(h(y))\} \\
N_h(\widetilde{\mu}_I)(h(x)h(y)) &= \underset{z\in h^{-1}(h(x)h(y))}{rsup} \widetilde{\mu}_I(z) \\
&\succeq \widetilde{\mu}_I(x_0y_0) \succeq rmin\{\widetilde{\mu}_I(x_0), \widetilde{\mu}_I(y_0)\} \\
&= rmin\left\{\underset{a\in h^{-1}(h(x))}{rsup} \widetilde{\mu}_I(a), \underset{b\in h^{-1}(h(y))}{rsup} \widetilde{\mu}_I(b)\right\} \\
&= rmin\{N_h(\widetilde{\mu}_I)(h(x)), N_h(\widetilde{\mu}_I)(h(y))\} \\
N_h(\widetilde{\mu}_F)(h(x)h(y)) &= \underset{z\in h^{-1}(h(x)h(y))}{rinf} \widetilde{\mu}_F(z) \\
&\preceq \widetilde{\mu}_F(x_0y_0) \preceq rmax\{\widetilde{\mu}_F(x_0), \widetilde{\mu}_F(y_0)\} \\
&= rmax\left\{\underset{a\in h^{-1}(h(x))}{rinf} \widetilde{\mu}_F(a), \underset{b\in h^{-1}(h(y))}{rinf} \widetilde{\mu}_F(b)\right\} \\
&= rmax\{N_h(\widetilde{\mu}_F)(h(x)), N_h(\widetilde{\mu}_F)(h(y))\}.
\end{aligned}$$

■

6. Application

In this section, we consider the problem of evaluation of its students by an institution. The committee form consisting of both internal and external evaluators is considered as a semigroup S. The internal evaluator is the subset A which may be dealt as subsemigroup if both evaluators are internal. The set A is ideal (left or right), if one is internal and the other is external evaluator. Before providing the example we give the following definitions.

Definition 6.1. The sum of two neutrosophic cubic sets

$$\begin{aligned}
A_1 &= \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle, \text{where } \widetilde{\mu}_T = [a^L, a^U], \widetilde{\mu}_I = [b^L, b^U], \widetilde{\mu}_F = [c^L, c^U], \\
A_2 &= \left\langle \widetilde{\Psi}_T, \widetilde{\Psi}_I, \widetilde{\Psi}_F, \phi_T, \phi_I, \phi_F \right\rangle, \ where\ \widetilde{\Psi}_T = [v^L, v^U], \widetilde{\Psi}_I, [w^L, w^U], \widetilde{\Psi}_F = [x^L, x^U]
\end{aligned}$$

is defined as

$$A_1 \oplus A_2 = \left\langle \begin{array}{c} [a^L + v^L - a^L v^L, a^U + v^U - a^U v^U], \\ [b^L + w^L - b^L w^L, b^U + w^U - b^U w^U], \\ [c^L x^L, c^U x^U], \\ \lambda_T \phi_T, \lambda_I \phi_I, \lambda_F + \phi_F - \lambda_F \phi_F \end{array} \right\rangle.$$

Definition 6.2. The scalar multiplication of a neutrosophic cubic set

$$A_1 = \left\langle \widetilde{\lambda}_T, \widetilde{\lambda}_I, \widetilde{\lambda}_F, , \lambda_T, \lambda_I, \lambda_F \right\rangle, \text{where } \widetilde{\lambda}_T = [a^L, a^U], \widetilde{\lambda}_I = [b^L, b^U], \widetilde{\lambda}_F = [c^L, c^U],$$

with a scalar k defined by

$$kA_1 = \left\langle \begin{array}{c} \left[1-(1-a^L)^k, 1-(1-a^U)^k\right], \\ [1-(1-b^L)^k, 1-(1-b^U)^k], \\ \left[\left(c^U\right)^k, \left(c^U\right)^k\right], \\ \lambda_T^k, \lambda_I^k, 1-(1-\lambda_F)^k \end{array} \right\rangle.$$

Definition 6.3. Neutrosophic cubic weighted average operator (NCWA) is defined as

$$NCWA : R^m \longrightarrow R \text{ by } NCW \quad A_w(A_1, A_2, ..., A_m) = \sum_{i=1}^{m} w_i A_i,$$

where $W = (w_1, w_2, ..., w_m)^T$ is weight of $A_i (i = 1, 2, 3, ..., m)$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{m} w_i = 1$, first all the neutrosophic cubic values are weighted then aggregated.

Example 6.4. $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in $S = \{a, b, c\}$ defined by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_T	λ_T
a	$[0.3, 0.6]$	$[0.5, 0.7]$	$[0.6, 0.7]$	0.4	0.6	0.8
b	$[0.2, 0.4]$	$[0.3, 0.4]$	$[0.8, 0.9]$	0.6	0.7	0.6
c	$[0.7, 0.9]$	$[0.8, 0.9]$	$[0.5, 0.6]$	0.2	0.3	0.9

is a subsemigroup (see Example 4.2). Let $W = (0.25, 0.35, 0.40)^T$ be given weight. Then the aggregated value of A is

$$Agg(A) = \langle [0.359, 0.606], [0.155, 0.755], [0.590, 0.678], 0.468, 0.484, 0.855 \rangle.$$

This idea can be extended to ideals (left, right) as well.

Conclusion: In this paper we proposed a new notion of neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are introduced and several properties are investigated. Relations between neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are discussed. Characterizations of neutrosophic cubic left (resp. right) ideals are considered and how the images or inverse images of neutrosophic cubic subsemigroups and cubic left (resp. right) ideals become neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals, respectively, are studied and aggregation operator is applied. In the future, our aim is to study neutrosophic cubic (α, β)-ideals in semigroups and neutrosophic cubic aggregations operators.

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