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On a New Class of Finite Integrals Involving Generalized Hypergeometric Function

K. Shantha Kumari[1,*], Arjun K. Rathie[2] and Shwetha S. Shetty[1]

[1] *Department of Mathematics, A J Institute of Engineering and Technology (affiliated to Visveswaraya Technological University), Mangaluru, 575006*
e-mail : shanthakk99@gmail.com (K.S. Kumari); shwethashetty576@gmail.com (S.S. Shetty)

[2] *Department of Mathematics, Vedant College of Engineering & Technology (affiliated to Rajasthan Technical University), Bundi 323021, Rajasthan, INDIA*
e-mail : arjunkumarrathie@gmail.com (A.K. Rathie)

Abstract The main aim of this research paper is to obtain fifty new class of integrals involving generalized hypergeometric function in the form of two master formulas. One hundred interesting integrals have been obtained as special cases of our main findings. The results are established with the help of generalized Watson's summation theorem obtained earlier by Lavoie, et al. and an interesting integral due to MacRobert.

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1. Introduction

The natural generalization of the classical Gauss's hypergeometric function ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by [1–3]

$$ {}_pF_q\left[\begin{array}{c} a_1,\ \ldots,\ a_p \\ b_1,\ \ldots,\ b_q \end{array};\ z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \tag{1.1}$$

where $(a)_n$ is the well known Pochhammer's symbol (or the shifted or raised factorial) defined for every complex number a by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\ldots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} \tag{1.2}$$

in which

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx,$$

*Corresponding author.

denotes the well-known gamma function [4, p.500] for $\Re(z) > 0$.
The series on the right hand side of (1.1) is indeed a Taylor series expansion for a function, say f, as $\sum_{k=o}^{\infty} c_k z^k$ with $c_k = \frac{f^{(k)}(0)}{k!}$, for which the ratio of successive terms can be written as:

$$\frac{c_k + 1}{c_k} = \frac{(a_1 + k)(a_2 + k) \cdots (a_p + k)}{(b_1 + k)(b_2 + k) \cdots (b_q + k)}$$

The convergence of the series (1.1) depend on the relation between the parameters of both numerator and denominator. According to the ratio test we get the following cases [1–3] :

i) If $p \leq q$, then series(1.1) converge for all $|z| < \infty$.
ii) If $p = q + 1$, the series converges for $|z| < 1$, otherwise diverges.
iii) If $p = q + 1$, on the circle of convergence $|z| = 1$, the series is: absolutely convergent if

$$\operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0,$$

conditionally convergent if

$$-1 < \operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) \leq 0, \ z \neq 1,$$

divergent if

$$\operatorname{Re}\left(\sum_{n=1}^{q} b_n - \sum_{n=1}^{p} a_n\right) \leq -1.$$

It is very interesting to mention here that whenever hypergeometric function ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ expressed in terms of Gamma function, the results are very important from a theoretical and application point of view. Thus the classical summation theorem such as those of Gauss, Gauss's second, Kummer, and Bailey for the series ${}_2F_1$, Watson, Dixon, and Whipple for the series ${}_3F_2$ and others play an important role.

During 1992-1996, in a series of three interesting research papers, Lavoie, et al. [5–7] have generalized the above mentioned classical summation theorems.

However, in our present investigation, we are interested in the following classical Watson's summation theorem [2, 3]

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix} ;1\right] \\ &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+c\right)\ \Gamma\left(\frac{a+b+1}{2}\right)\ \Gamma\left(c+\frac{1-a-b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\ \Gamma\left(\frac{1+b}{2}\right)\ \Gamma\left(c+\frac{1-a}{2}\right)\ \Gamma\left(c+\frac{1-b}{2}\right)} \end{aligned} \tag{1.3}$$

provided $\operatorname{Re}(2c - a - b) > -1$ and its following generalization due to Lavoie, et al. [3]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};1\right] \\
&= \frac{A_{i,j}\, 2^{a+b+i-2}\,\Gamma\left(\frac{a+b+i+1}{2}\right)\,\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\,\Gamma\left(c-\frac{(a+b+|i+j|-j-1)}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\Gamma(b)} \\
&\times\left\{\frac{B_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{(1-(-1)^i)}{4}\right)\,\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\,\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}\right. \\
&\left.+\frac{C_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{(1+(-1)^i)}{4}\right)\,\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\,\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\}
\end{aligned}
\tag{1.4}
$$

for $i, j = 0, \pm 1, \pm 2$

Here, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$ and $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are as in Tables (1, 2), (3, 4) and (5, 6).

i\j	-2	-1	0
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1
-1	$\frac{1}{(c-1)}$	1	2
-2	$\frac{1}{2(c-1)}$	1	1

TABLE 1. Table for $A_{i,j}$ $i = 0, \pm 1, \pm 2$ and $j = -2, -1, 0$

i\j	1	2
2	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	1	$\frac{1}{2(c+1)}$
-1	2	$\frac{2}{(c+1)}$
-2	2	$\frac{2}{(c+1)}$

TABLE 2. Table for $A_{i,j}$ $i = 0, \pm 1, \pm 2$ and $j = 1, 2$

i\j	-2	-1	0
2	$c(a+b-1)$ $-(a+1)(b+1)+2$	$a+b-1$	$a(2c-a)+b(2c-b)$ $-2c+1$
1	$c-b-1$	1	1
0	$(c-a-1)(c-b-1)$ $+(c-1)(c-2)$	1	1
-1	$2(c-1)(c-2)$ $-(a-b)(c-b-1)$	$2c-a+b-2$	1
-2	$B_{-2,-2}$	$2(c-1)(a+b-1)$ $-(a-b)^2+1$	$a(2c-a)+b(2c-b)$ $-2c+1$

$$\begin{aligned}B_{-2,-2} = {} & 2(c-1)(c-2)[(2c-1)(a+b-1)-a(a+1)-b(b+1)+2] \\ & -(a-b-1)(a-b+1)[(c-1)(2c-a-b-3)+ab]\end{aligned}$$

TABLE 3. Table for $B_{i,j}$ $i=0,\pm1,\pm2$ and $j=-2,-1,0$

i\j	1	2
2	$2c(a+b-1)-(a-b)^2+1$	$B_{2,2}$
1	$2c-a+b$	$2c(c+1)-(a-b)(c-b+1)$
0	1	$(c-a+1)(c-b+1)+c(c+1)$
-1	1	$c-b+1$
-2	$a+b-1$	$c(a+b-1)-(a-1)(b-1)$

TABLE 4. Table for $B_{i,j}$ $i=0,\pm1,\pm2$ and $j=1,2$

$$\begin{aligned}B_{2,2} = {} & 2c(c+1)[(2c+1)(a+b-1)-a(a-1)-b(b-1)] \\ & -(a-b-1)(a-b+1)[(c+1)(2c-a-b+1)+ab]\end{aligned}$$

$i\diagdown j$	-2	-1	0
2	-4	$-(4c-a-b-3)$	-8
1	$-(c-a-1)$	-1	-1
0	4	1	0
-1	$(c-1)(c-2)+(a-b)(c-a-1)$	$2c+a-b-2$	1
-2	$(2c-a+b-3)(2c+a-b-3)$	$C_{-2,-1}$	8

$$C_{-2,-1} = 8c^2-2(c-1)(a+b+7)-(a-b)^2-7$$

TABLE 5. Table for $C_{i,j}$ $i=0,\pm1,\pm2$ and $j=-2,-1,0$

$i \diagdown j$	1	2
2	$-[8c^2 - 2c(a+b-1) - (a-b)^2 + 1]$	$-4(2c+a-b+1)(2c-a+b+1)$
1	$-(2c+a-b)$	$-[2c(c+1)+(a-b)(c-a+1)]$
0	-1	-4
-1	1	$c-a+1$
-2	$4c-a-b+1$	4

TABLE 6. Table for $C_{i,j}$ $i = 0, \pm 1, \pm 2$ and $j = 1, 2$

For $i = j = 0$, the result (1.4) reduce to classical Watson's summation theorem (1.3). In addition to this, we also require the following interesting integral due to MacRobert [8]

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta} \, (\sin\theta)^{\alpha-1} \, (\cos\theta)^{\beta-1} \, d\theta = e^{\frac{\omega\pi\alpha}{2}} \, \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.5}$$

provided $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$ and $\omega = \sqrt{-1}$.

The aim of this research paper is to obtain fifty new class of integrals involving generalized hypergeometric function in the form of two master formulas.

More than one hundred interesting integrals have also been obtained as special cases of our main findings. The results are established with the help of generalized Watson's summation theorem (1.4) and an interesting integral (1.5). The results obtained in this paper are simple, significant, easily established and may be potentially useful.

2. Two Master Formulas

The two master formulas to be proved in this paper are given in the following theorems.

Theorem 2.1. *For $\ell = 0, \pm 1, \pm 2, \ldots$ and $i = j = 0, \pm 1, \pm 2$, the following results holds true*

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta} (\sin\theta)^{d+\ell} (\cos\theta)^{d-1} \\
&\times {}_4F_3\left[\begin{matrix} a, b, c, 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), 2c+j, d \end{matrix} ; ; e^{\omega\theta}\cos\theta\right] d\theta \\
&= \frac{A_{i,j}\, e^{\frac{\omega\pi(d+\ell+1)}{2}} \Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\
&\times \frac{2^{a+b+i-2}\Gamma\left(\frac{a+b+i+1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c-\frac{(a+b+|i+j|-j-1)}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\
&\times \left\{ \frac{B_{i,j}\Gamma\left(\frac{a}{2}+\frac{(1-(-1)^i)}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)} \right. \\
&\left. + \frac{C_{i,j}\Gamma\left(\frac{a}{2}+\frac{(1+(-1)^i)}{4}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\}.
\end{aligned} \tag{2.1}$$

provided $\mathrm{Re}(d) > 0,\ \mathrm{Re}(d+\ell+1) > 0$ *and* $\mathrm{Re}(2c-a-b+i+2j+1) > 0$.

Theorem 2.2. *For $\ell = 0, \pm 1, \pm 2, \ldots$ and $i = j = 0, \pm 1, \pm 2,$ the following results holds true*

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d-1}(\cos\theta)^{d+\ell} \\
&\times\ {}_4F_3\left[\begin{matrix} a, b, c, 2d+\ell+1 \\ \frac{1}{2}(a+b+i+1), 2c+j, d \end{matrix}; e^{\omega(\theta-\frac{\pi}{2})}\sin\theta\right] d\theta \\
&= \frac{A_{i,j}\ e^{\frac{\omega\pi d}{2}}\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\
&\times \frac{2^{a+b+i-2}\Gamma\left(\frac{a+b+i+1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c-\frac{(a+b+|i+j|-j-1)}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\
&\times \left\{\frac{B_{i,j}\Gamma\left(\frac{a}{2}+\frac{(1-(-1)^i)}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}\right. \\
&\left.+\frac{C_{i,j}\Gamma\left(\frac{a}{2}+\frac{(1+(-1)^i)}{4}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\}.
\end{aligned} \tag{2.2}$$

provided $\mathrm{Re}(d) > 0,\ \mathrm{Re}(d+\ell+1) > 0$ *and* $\mathrm{Re}(2c-a-b+i+2j+1) > 0$.

Here, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficient $A_{i,j}, B_{i,j}$ and $C_{i,j}$ are as in Tables (1, 2), (3, 4) and (5, 6).

Proof. The proof of our theorems are quite straight forward. For this, in order to prove the result (2.1), denoting the left-hand side of (2.1) by I,expressing the ${}_3F_2$ function as a series, we have

$$\begin{aligned}
I = &\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d-1} \\
&\times \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(2d+\ell+1)_n\ (c)_n\ e^{n\omega\theta}\ (\cos\theta)^n}{(d)_n\ \left(\frac{a+b+1+i}{2}\right)_n(2c+j)_n\ n!}d\theta,
\end{aligned}$$

Changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series in the interval, so we have

$$\begin{aligned}
I = &\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(2d+\ell+1)_n}{\left(\frac{a+b+1+i}{2}\right)_n(2c+j)_n\ n!} \\
&\times \int_0^{\frac{\pi}{2}} e^{\omega(2d+n+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d+n-1}d\theta,
\end{aligned}$$

Evaluating the integral with the help of the MacRobert's result (1.5), we have, after some simplification

$$I = e^{\frac{\omega\pi}{2}(d+\ell+1)}\,\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{\left(\frac{a+b+1+i}{2}\right)_n(2c+j)_n\; n!},$$

Summing up the series; we get

$$\begin{aligned} I &= e^{\frac{\omega\pi}{2}(d+\ell+1)}\,\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\times\ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix};1\right]. \end{aligned}$$

The ${}_3F_2$ can now be evaluated with the help of generalized Watson's summation theorem (1.4) and we easily arrive at the right- hand side of (2.1). ∎

This completes the proof of theorem(2.1). In exactly the same manner, we can establish theorem(2.2).

3. Special Cases

In this section, we shall mention one hundred interesting results in the form of two general integrals.

1. We observe here that if in (2.1)we let $b=-2n$ and replace a by $a+2n$ or we let $b=-2n-1$ and replace a by $a+2n+1$, in each case, one of the two terms appearing on the right-hand side of (2.1) will vanish and under the same conditions of convergence, we get fifty interesting special cases, which are given below in the form of two corollaries.

Corollary 3.1. *For $\ell = 0, \pm1, \pm2, \ldots$ and $i = j = 0, \pm1, \pm2$, the following twenty five results holds true.*

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d-1} \\ &\quad\times\ {}_4F_3\left[\begin{matrix} -2n, a+2n, c, 2d+\ell+1 \\ \frac{1}{2}(a+i+1), 2c+j, d \end{matrix}; e^{\omega\theta}\cos\theta\right] d\theta \\ &\quad= e^{\frac{\omega\pi}{2}(d+\ell+1)}\ D_{i,j}\,\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\quad\frac{\left(\frac{1}{2}\right)_n\left(\frac{a}{2}-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{(1-(-1)^i)}{4}\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{a}{2}+\frac{(1+(-1)^i)}{4}\right)_n} \end{aligned} \tag{3.1}$$

where the coefficient $D_{i,j}$ are as in Tables (7) and (8)

i\j	-2	-1	0
2	$\frac{(a+1)[(c-1)(a-1)+2n(a+2n)]}{(c-1)(a+4n-1)(a+4n+1)}$	$\frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-4n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$
0	$1-\frac{2n(a+2n)}{(c-1)(2c-a-3)}$	1	1
-1	$1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)}$	$1-\frac{4n}{(2c-a-2)}$	1
-2	$D_{-2,-2}$	$1-\frac{8n(a+2n)}{(a-1)(2c-a-3)}$	$1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}$

TABLE 7. Table for $D_{i,j},\ i=0,\pm1,\pm2$ and $j=-2,-1,0$

$$D_{-2,-2}=1-\frac{2an(6c+a-7)(2c-a-3)-4n^2[5a^2-4a-21-4c(3c-a-8)]-64n^3(a+n)}{(c-1)(a-1)(2c-a-3)(2c-a-5)}$$

i\j	1	2
2	$\frac{(a+1)[(a-1)(2c-a-1)-8n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$D_{2,2}$
1	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\frac{a[(c+1)(2c-a)-2n(2c+a+4n+2)]}{(c+1)(2c-a)(a+4n)}$
0	1	$1-\frac{2n(a+2n)}{(c+1)(2c-a+1)}$
-1	1	$1+\frac{2n}{(c+1)}$
-2	1	$1+\frac{2n(a+2n)}{(c+1)(a-1)}$

TABLE 8. Table for $D_{i,j},\ i=0,\pm1,\pm2$ and $j=1,2$

$$D_{2,2}=\frac{(a+1)\left(\begin{array}{c}(a-1)(c+1)(2c-a+1)(2c-a-1)-2an(6c+a+5)(2c-a+1)\\ +4n^2(5a^2+4a-5-4c(3c-a+4))+64n^3(a+n)]\end{array}\right)}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)}$$

Corollary 3.2. *For $\ell=0,\pm1,\pm2,\dots$ and $i=j=0,\pm1,\pm2$, the following twenty five results holds true.*

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d-1}\\ &\quad\times\ {}_4F_3\left[\begin{array}{c}-2n-1,a+2n+1,c,2d+\ell+1\\ \frac{1}{2}(a+i+1),2c+j,d\end{array};e^{\omega\theta}\cos\theta\right]d\theta\\ &\quad= e^{\frac{\omega\pi}{2}(d+\ell+1)}\ E_{i,j}\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)}\\ &\quad\frac{\left(\frac{3}{2}\right)_n\left(\frac{a}{2}-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{(1+(-1)^i)}{4}\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{a}{2}+\frac{(3-(-1)^i)}{4}\right)_n}\end{aligned}\tag{3.2}$$

where the coefficient $E_{i,j}$ are as in Tables (9) and (10).

$i \diagdown j$	*1*	*2*
2	$E_{2,1}$	$\frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(c+1)(2c-a-1)(a+4n+1)(a+4n+3)}$
1	$\frac{(2c+a+4n+2)}{(2c+1)(a+4n+2)}$	$\frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)}$
0	$\frac{1}{(2c+1)}$	$\frac{1}{(c+1)}$
-1	$\frac{-(2c-a)}{a(2c+1)}$	$\frac{-(c-a-2n)}{a(c+1)}$
-2	$\frac{-(4c-a+1)}{(a-1)(2c+1)}$	$\frac{-(2c-a+1)}{(a-1)(c+1)}$

$$E_{2,1} = \frac{(a+1)[(4c+a+3)(2c-a-1)-8n(a+2n+2)]}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)}$$

TABLE 10. Table for $E_{i,j}$ $i = 0, \pm 1, \pm 2$ and $j = 1, 2$

$i \diagdown j$	*-2*	*-1*	*0*
2	$\frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)}$	$\frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}$	$\frac{2(a+1)}{(a+4n+1)(a+4n+3)}$
1	$\frac{(c-a-2n-2)}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(a+4n+2)(2c-1)}$	$\frac{1}{a+4n+2}$
0	$\frac{-1}{(c-1)}$	$\frac{-1}{(2c-1)}$	*0*
-1	$E_{-1,-2}$	$\frac{-(2c+a+4n)}{a(2c-1)}$	$\frac{-1}{a}$
-2	$\frac{-(2c+a+4n-1)(2c-a-4n-5)}{(a-1)(c-1)(2c-a-5)}$	$E_{-2,-1}$	$\frac{-2}{(a-1)}$

TABLE 9. Table for $E_{i,j}$ $i = 0, \pm 1, \pm 2$ and $j = -2, -1, 0$

$$E_{-2,-1} = -\frac{[(4c+a-1)(2c-a-3)-8n(a+2n+2)]}{(a-1)(2c-1)(2c-a-3)}$$

$$E_{-1,-2} = -\frac{[(c+a)(2c-a-4)-2n(3a-2c+4n+6)]}{a(c-1)(2c-a-4)}$$

In particular, in (3.1), if we take $i = j = 0$, we get the following result

$$\int_0^{\frac{\pi}{2}} e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d-1} \times {}_4F_3\left[\begin{matrix} -2n, a+2n, c, 2d+\ell+1 \\ \frac{1}{2}(a+1), 2c, d \end{matrix} ; e^{\omega\theta}\cos\theta\right] d\theta = e^{\frac{\omega\pi}{2}(d+\ell+1)} \frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{a}{2}-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n} \tag{3.3}$$

Further, if we take $\ell = -1$, it reduces to

$$\int_0^{\frac{\pi}{2}} e^{2d\omega\theta}(\sin\theta\cos\theta)^{d-1} {}_3F_2\left[\begin{matrix} -2n,\ a+2n,\ c,\ 2d \\ \frac{1}{2}(a+1),\ 2c,\ d \end{matrix} ;\ e^{\omega\theta}\cos\theta\right] d\theta = e^{\omega\pi d}\frac{\Gamma(d)\Gamma(d)}{\Gamma(2d)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{a}{2}-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n} \tag{3.4}$$

Similarly, in (3.2), if we take $i = j = 0$, we get the following interesting result

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d+\ell}(\cos\theta)^{d-1} \\ & \times\ {}_4F_3\left[\begin{array}{c}-2n-1,\ a+2n+1,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+1),\ 2c,\ d\end{array};e^{\omega\theta}\cos\theta\right]d\theta \\ & = 0 \end{aligned} \tag{3.5}$$

for all $\ell = 0, \pm1, \pm2, \ldots$.

2. In (2.2), we let $b = -2n$ and replace a by $a + 2n$ or we let $b = -2n - 1$ and replace a by $a+2n+1$, in each case, one of the two terms appearing on the right-hand side of (2.2)will vanish and under the same conditions of convergence, we get fifty interesting special cases, which are given below in the form of two corollaries.

Corollary 3.3. *For $\ell = 0, \pm1, \pm2, \ldots$ and $i = j = 0, \pm1, \pm2$, the following twenty five results holds true*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d-1}(\cos\theta)^{d+\ell} \\ & \times\ {}_4F_3\left[\begin{array}{c}-2n,\ a+2n,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+i+1),\ 2c+j,\ d\end{array};e^{\omega(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta \\ & = e^{\frac{\omega\pi d}{2}}\ D_{i,j}\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ & \frac{\left(\frac{1}{2}\right)_n\left(\frac{a}{2}-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{(1-(-1)^i)}{4}\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{a}{2}+\frac{(1+(-1)^i)}{4}\right)_n}\end{aligned} \tag{3.6}$$

where the coefficient $D_{i,j}$ are as in Tables (7) and (8)

Corollary 3.4. *For $\ell = 0, \pm1, \pm2, \ldots$ and $i = j = 0, \pm1, \pm2$, the following twenty five results holds true*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} & e^{\omega(2d+\ell+1)\theta}(\sin\theta)^{d-1}(\cos\theta)^{d+\ell} \\ & \times\ {}_4F_3\left[\begin{array}{c}-2n-1,\ a+2n+1,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+i+1),\ 2c+j,\ d\end{array};e^{\omega(\theta-\frac{\pi}{2})}\sin\theta\right]d\theta \\ & = e^{\frac{\omega\pi}{2}(d+\ell+1)}\ E_{i,j}\frac{\Gamma(d)\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ & \frac{\left(\frac{3}{2}\right)_n\left(\frac{a}{2}-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{(1+(-1)^i)}{4}\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{a}{2}+\frac{(3-(-1)^i)}{4}\right)_n}\end{aligned} \tag{3.7}$$

where the coefficient $E_{i,j}$ are as in Table (9) and (10)

3. In the results(2.1) and (2.2) of theorems 1 and 2, if we take $d = c$, we get two general results obtained very recently by Rakha et al. [9]

4. In the results (3.1) to (3.7), if we take $d = c$, we get results obtained recently by Rakha et al. [9]

Similarly, other results can be obtained.

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