Thai Journal of Mathematics
Volume 20 Number 1 (2022)
Pages 235–243

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Hyperstability of Orthogonally Pexider Lie Functional Equation: An Orthogonally Fixed Point Approach

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Abstract In the present paper, by using orthogonally fixed point methods we prove the stability and hyperstability of orthogonally Pexider Lie homomorphisms and derivations on orthogonally Lie Banach algebras.

MSC: 39B82; 39B55; 47H10; 46L05

Keywords: orthogonally Pexider Lie homomorphism; orthogonally Pexider Lie derivation; orthogonally fixed point; hyperstability

Submission date: 03.03.2019 / Acceptance date: 18.01.2022

1. Introduction and Preliminaries

The study of stability problem of functional equations which had been raised by Ulam [1] have been done by several authors on different functional equations (see [2–6]). In 1941 [7], Hyers solved the approximately additive mappings on the setting of Banach spaces. Hyers theorem was generalized by Th. M. Rassias [8]. A generalization of the theorem of Th. M. Rassias was obtained by Găvruta [9] by replacing a general control function $\varphi : X \times X \longrightarrow [0, \infty)$.
One of the generalization version of Cauchy equation is the Pexider type $g(a + b) = g_1(a) + g_2(b)$. Jun et al. [10] proved the stability of Pexider equation. In [11], Eshaghi Gordji et al. introduced the concept of orthogonally set.

Definition 1.1. [11] Let $\mathcal{A} \neq \emptyset$ and $\perp \subseteq \mathcal{A} \times \mathcal{A}$ be a binary relation. $\perp$ is called an orthogonally set (briefly O-set) and denotes it by $(\mathcal{A}, \perp)$ if $\perp$ satisfies in

$$\exists a_0; (\forall b; b \perp a_0) \quad or \quad (\forall b; a_0 \perp b),$$

If $(\mathcal{A}, \perp)$ is an O-set and $(\mathcal{A}, d)$ is a generalized metric space, then $(\mathcal{A}, \perp, d)$ is called orthogonally generalized metric space.

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Published by The Mathematical Association of Thailand.

Definition 1.2. [11] Let $(\mathcal{A}, \perp, d)$ be an orthogonally metric space.
(i) A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if

$$(\forall n; a_n \perp a_{n+1}) \;\; or \;\; (\forall n; a_{n+1} \perp a_n).$$

(ii) A mapping g from $\mathcal{A}$ into $\mathcal{A}$ is $\perp-$continuous in $a \in \mathcal{A}$ if for all O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $a_n \to a$, $g(a_n) \to g(a)$. Clearly, every continuous map is $\perp-$continuous at any $a \in \mathcal{A}$.
(iii) $(\mathcal{A}, \perp, d)$ is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent to a point in $\mathcal{A}$.
(iv)A mapping g from $\mathcal{A}$ into $\mathcal{A}$ is $\perp$-preserving if for all $a, b \in \mathcal{A}$ with $a \perp b$, then $g(a) \perp g(b)$.
(v) A mapping g from $\mathcal{A}$ into $\mathcal{A}$ is called orthogonality contraction with Lipschitz constant $0 < \lambda < 1$ if

$$d(g(a), g(b)) \leq \lambda \, d(a, b) \quad where \ a \perp b.$$

Theorem 1.3 ([12])**.** *Suppose that $(\mathcal{A}, d, \perp)$ is an O-complete generalized metric space. Let $J : \mathcal{A} \to \mathcal{A}$ be a $\perp$-preserving, $\perp$-continuous and $\perp$-λ-contraction. Let $a_0 \in \mathcal{A}$ satisfies for all $b \in \mathcal{A}$, $a_0 \perp b$ or for all $b \in \mathcal{A}$, $b \perp a_0$, and consider the "O-sequence of successive approximations with initial element a_0 ": a_0, $J(a_0)$, $J^2(a_0)$, ..., $J^n(a_0)$, ... Then, either*

$$d(J^n(a_0), J^{n+1}(a_0)) = \infty \qquad \forall \ n \geq 0$$

or there exists a positive integer n_0 such that

$$d(J^n(a_0), J^{n+1}(a_0)) < \infty$$

for all $n > n_0$. If the second alternative holds, then
(1) The O-sequence of $\{J^n(a_0)\}$ converges to a fixed point a^* of J.
(2) a^* is the unique fixed point of J in $\mathcal{A}^* = \{b \in \mathcal{A} : d(J^n(a_0), b) < \infty\}$.
(3) If $b \in \mathcal{A}$, then

$$d(b, a^*) \leq \frac{1}{1-\lambda} d(b, J(b)).$$

In this paper, by using the concept of orthogonally sets and orthogonally fixed point theorem we prove the stability of orthogonally Pexider Lie homomorphisms and orthogonally Pexider Lie derivations in orthogonally Lie Banach algebras in section 2 and 3 respectively.

2. Orthogonally Pexider Lie Homomorphism

Let $\mathcal{A}$ be a Lie Banach algebra and $\perp \subseteq \mathcal{A} \times \mathcal{A}$ be an orthogonally set. $(\mathcal{A}, \perp)$ is called orthogonally Lie Banach algebra.

Definition 2.1. Let $(\mathcal{A}, \perp)$ be an orthogonally Lie Banach algebra and let $g, g_1, g_2 : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings satisfying

$$\begin{cases} g(a+b) = g_1(a) + g_2(b), \\ \\ g([a,b]) = [g_1(a), g_2(b)], \end{cases} \tag{2.1}$$

for all $a, b \in \mathcal{A}$ with $a \perp b$. The system of equations (2.1) is called an orthogonally Pexider Lie homomorphism.

Throughout the paper, suppose that $\mathcal{A}$ is an orthogonally Lie Banach algebra, $\phi, \varphi : \mathcal{A}^2 \to [0, \infty)$ are mappings, $i \in \{-1, 1\}$ and $0 < L < 1$.

Theorem 2.2. *Let $g, g_1, g_2 : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings and g be an odd mapping for which*

$$\|g(a+b) - g_1(a) - g_2(b)\| \leq \varphi(a,b), \tag{2.2}$$

$$\|g([a,b]) - [g_1(a), g_2(b)]\| \leq \phi(a,b), \tag{2.3}$$

where

$$\lim_{n\to\infty} \frac{\varphi(2^{ni}a, 2^{ni}b)}{2^{ni}} = 0, \tag{2.4}$$

$$\lim_{n\to\infty} \frac{\phi(2^{ni}a, b)}{2^{ni}} = \lim_{n\to\infty} \frac{\phi(a, 2^{ni}b)}{2^{ni}} = 0 \tag{2.5}$$

where $a \perp b$. If there exists $0 < L < 1$ such that for any fixed $a \in \mathcal{A}$, and some $v_a \in \mathcal{A}$ with $a \perp v_a$ the mapping

$$\begin{aligned} a \mapsto \psi(a, v_a) =& \varphi(\frac{a+v_a}{2}, \frac{a-v_a}{2}) + \varphi(0, \frac{a-v_a}{2}) + \varphi(\frac{a+v_a}{2}, 0) + \varphi(\frac{a}{2}, \frac{v_a}{2}) \\ &+ \varphi(\frac{a}{2}, \frac{-v_a}{2}) + 2\varphi(\frac{a}{2}, 0) + \varphi(0, \frac{v_a}{2}) + \varphi(0, \frac{-v_a}{2}) \end{aligned} \tag{2.6}$$

has the property

$$\psi(a, v_a) \leq L 2^i \psi(\frac{a}{2^i}, \frac{v_a}{2^i}). \tag{2.7}$$

Then there exists is a unique $G : \mathcal{A} \longrightarrow \mathcal{A}$ orthogonally Lie homomorphism such that

$$\begin{aligned} \|g(a) - G(a)\| &\leq \frac{L^{\frac{1+i}{2}}}{1-L} \psi(a, v_a), \\ \|g_1(a) - g_1(0) - G(a)\| &\leq \frac{L^{\frac{1+i}{2}}}{1-L} \psi(a, v_a) + \varphi(a, 0), \\ \|g_2(a) - g_2(0) - G(a)\| &\leq \frac{L^{\frac{1+i}{2}}}{1-L} \psi(a, v_a) + \varphi(0, a) \end{aligned} \tag{2.8}$$

for all $a \in \mathcal{A}$.

Proof. First of all, since g is an odd, then $\phi(0,0) = \varphi(0,0) = 0$. Let E be the set of all mappings $e : \mathcal{A} \to \mathcal{A}$ such that for all $a \in \mathcal{A}$, $e(a) \perp \frac{1}{2} e(2a)$ or $\frac{1}{2} e(2a) \perp e(a)$ and $e(0) = 0$. For any fixed $a \in \mathcal{A}$ and some $v_a \in \mathcal{A}$ with $a \perp v_a$ define $d : E^2 \longrightarrow [0, \infty)$ by

$$d(e_1, e_2) = inf\Big\{K \in \mathbb{R}_+ : \|e_1(a) - e_2(a)\| \leq K\psi(a, v_a) \qquad \forall \quad e_1 \perp e_2 \quad and \quad a \perp v_a\Big\}.$$

Clearly $(E, \perp, d)$ is an O-complete generalized metric space. Define $J : E \longrightarrow E$, by $Je(a) = \frac{1}{2^i} e(2^i a) \quad \forall\ a \in \mathcal{A}$. J is a strictly $\perp$-contractive mapping. Indeed, for given e_1 and e_2 in E with $d(e_1, e_2) < K$ where K is a real number and for any fixed $a, v_a \in \mathcal{A}$ with $a \perp v_a$, we have

$$\begin{aligned} \|e_1(a) - e_2(a)\| &\leq K\psi(a, v_a) \\ \|\frac{1}{2^i} e_1(2^i a) - \frac{1}{2^i} e_2(2^i a)\| &\leq \frac{1}{2^i} K\psi(2^i a, 2^i v_a) \\ \|\frac{1}{2^i} e_1(2^i a) - \frac{1}{2^i} e_2(2^i a)\| &\leq LK\psi(a, v_a) \\ d(Je_1, Je_2) &\leq LK. \end{aligned}$$

Put $K = d(e_1, e_2) + \frac{1}{n}$ for $n \in \mathbb{Z}^+$. Then $d(Je_1, Je_2) \leq L(d(e_1, e_2) + \frac{1}{n})$. Letting $n \to \infty$ then

$$d(Je_1, Je_2) \leq Ld(e_1, e_2) \qquad \forall\, e_1, e_2 \in E.$$

Let $a, b = 0$ in (2.2) and (2.3), we get

$$g(0) = 0 \quad , \quad g_1(0) = g_2(0) = 0. \tag{2.9}$$

Put $b = 0$ in (2.2) and (2.9), respectively, so we obtain

$$\|g(a) - g_1(a) - g_2(0)\| \leq \varphi(a, 0),$$

$$\big\|g(a) - g_1(a) - g_1(0)\big\| \leq \varphi(a, 0). \tag{2.10}$$

Now by setting $a = 0$ in (2.2) and (2.9), respectively, we have

$$\|g(b) - g_1(0) - g_2(b)\| \leq \varphi(0, b)$$

$$\big\|g(b) - g_2(b) - g_2(0)\big\| \leq \varphi(0, b). \tag{2.11}$$

So , $d(g, Jg) \leq L = L^1 < \infty$. By using the prove of Theorem 2.1 in [13], we have

$$\|g(a) - 2g(\frac{a}{2})\| \leq \psi(a, v_a),$$

that is, $d(g, Jg) \leq 1 = L^0 < \infty$.
By Theorem 1.3, since $\lim_{n\to\infty} d(J^n g, d) = 0$, there exists $G : \mathcal{A} \longrightarrow \mathcal{A}$ which is the unique fixed point of J in the set

$$M = \{e \in E : d(g, e) < \infty\}$$

such that

$$G(a) = \lim_{n\to\infty} \frac{g(2^{ni}a)}{2^{ni}}.$$

Thus we have

$$d(g, G) \leq \frac{1}{1 - L} d(g, Jg)$$

which yields

$$\|g(a) - G(a)\| \leq \frac{L^{\frac{1+i}{2}}}{1 - L} \psi(a, v_a).$$

Further, using inequalities (2.10) and (2.11), we have

$$\begin{aligned} \|g_1(a) - g_1(0) - G(a)\| &\leq \big\|g(a) - g_1(a) - g_1(0)\big\| + \|g(a) - G(a)\| \\ &\leq \frac{L^{\frac{1+i}{2}}}{1 - L} \psi(a, v_a) + \varphi(a, 0), \end{aligned}$$

$$\begin{aligned} \|g_2(a) - g_2(0) - G(a)\| &\leq \big\|g(a) - g_2(a) - g_2(0)\big\| + \|g(a) - G(a)\| \\ &\leq \frac{L^{\frac{1+i}{2}}}{1 - L} \psi(a, v_a) + \varphi(0, a) \end{aligned}$$

as desired.

The inequalities (2.10) and (2.11), imply that

$$\|2^{-ni}g(2^{ni}a) - 2^{-ni}g_1(2^{ni}a) - 2^{-nj}g_1(0)\| \leq 2^{-ni}\varphi(2^{ni}a, 0),$$

$$\|2^{-ni}g(2^{ni}a) - 2^{-ni}g_2(2^{ni}a) - 2^{-ni}g_2(0)\| \leq 2^{-ni}\varphi(0, 2^{ni}a)$$

for all $a \in \mathcal{A}$, whence

$$G(a) = \lim_{n\to\infty} \frac{g_1(2^{ni}a) - g_1(0)}{2^{ni}} = \lim_{n\to\infty} \frac{g_2(2^{ni}a) - g_2(0)}{2^{ni}}. \tag{2.12}$$

Let $a, b \in \mathcal{A}$ with $a \perp b$. By Definition 1.1 we have $2^{ni}a \perp 2^{ni}b$ for all $n \in \mathbb{N}$ and from (2.2), (2.4) and (2.12), we get

$$\begin{aligned}
&\|2^{-nj}g(2^{ni}(a+b)) - 2^{-ni}(g_1(2^{ni}a) + g_1(0)) - 2^{-nj}(g_2(2^{ni}b) + g_2(0))\| \\
&\qquad = \|2^{-ni}g\big(2^{ni}(a+b)\big) - 2^{-ni}g_1(2^{ni}a) - 2^{-ni}g_2(2^{ni}b)\| \\
&\qquad \leq 2^{-ni}\varphi(2^{ni}a, 2^{ni}b).
\end{aligned}$$

Letting $n \to \infty$, then $G(a+b) - G(a) - G(b) = 0$. Therefore G is an orthogonally additive.

Now, we claim that the mapping G is an orthogonally Lie homomorphism. Define $t : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$t([a, b]) = g([a, b]) - [g_1(a), g_2(b)] \qquad \forall\ a, b \in \mathcal{A},\ a \perp b.$$

By (2.5), we have

$$\lim_{n\to\infty} \frac{t(2^{ni}a, b)}{2^{ni}} = 0. \tag{2.13}$$

Utilizing the relations (2.12), (2.13), we have

$$\begin{aligned}
G([a, b]) &= \lim_{n\to\infty} \frac{g\big(2^{ni}([a, b])\big)}{2^{ni}} = \lim_{n\to\infty} \frac{g\big([2^{ni}a, b]\big)}{2^{ni}} \\
&= \lim_{n\to\infty} \frac{[2^{ni}g_1(2^{ni}a), g_2(b)] + t([2^{ni}a, b])}{2^{na}} \\
&= \lim_{n\to\infty} \left([g_1(2^{ni}a), g_2(b)] + \frac{t([2^{ni}a, b])}{2^{ni}}\right) = [G(a), g_2(b)],
\end{aligned} \tag{2.14}$$

for all $a, b \in \mathcal{A}$ witha $\perp b$. Now, let $n \in \mathbb{N}$ be fixed and $a, b \in \mathcal{A}$ with $a \perp b$. By using (2.14) and orthogonal additivity of G, one obtains

$$[G(a), g_2(2^{ni}b)] = G\big([a, (2^{ni}b])\big) = G\big([(2^{ni}a), b]\big) = [G(2^{ni}a), g_2(b)] = 2^{ni}[G(a), g_2(b)]$$

which yields

$$[G(a), \frac{g_2(2^{ni}b)}{2^{ni}}] = [G(a), g_2(b)]. \tag{2.15}$$

A comparison of the (2.15) relation with (2.14) shows that

$$G([a, b]) = [G(a), \frac{g_2(2^{ni}b)}{2^{ni}}].$$

Taking the limit as $n \to \infty$, then $G([a, b]) = [G(a), G(b)]$. ∎

Corollary 2.3. *Let $\theta, \varepsilon \geq 0$ and $g, g_1, g_2 : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings such that g is an odd mapping for which*

$$\|g(a+b) - g_1(a) - g_2(b)\| \leq \varepsilon(\|a\|^q + \|b\|^q)$$

$$\|g([a,b]) - [g_1(a), g_2(b)]\| \leq \theta \|a\|^s \|b\|^r$$

where $a \perp b$ and q, s, r are real numbers. Then there exists a unique orthogonally Lie homomorphism $G : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\|g(a) - G(a)\| \leq \frac{2^{\frac{i(1+i)(q-1)}{2}}}{1 - 2^{i(q-1)}} \varepsilon \Big(2\|a + v_a\|^q + 2\|a - v_a\|^q + 4\|a\|^q + 4\|v_a\|^p \Big),$$

$$\|g_1(a) - g_1(0) - G(a)\| \leq \varepsilon \Big\{ \frac{2^{\frac{i(1+i)(q-1)}{2}}}{1 - 2^{i(q-1)}} (2\|a + v_a\|^q + 2\|a - v_a\|^q + 4\|a\|^q + 4\|v_a\|^q)$$

$$+ (\|a\|^q) \Big\},$$

$$\|g_2(a) - g_2(0) - G(a)\| \leq \varepsilon \Big\{ \frac{2^{\frac{i(1+i)(q-1)}{2}}}{1 - 2^{i(q-1)}} (2\|a + v_a\|^q + 2\|a - v_a\|^q + 4\|a\|^q + 4\|v_a\|^q)$$

$$+ (\|a\|^q) \Big\}$$

for any fixed $a \in \mathcal{A}$ and some $v_a \in \mathcal{A}$ with $a \perp v_a$, where $q, s < 1$ for $i = 1$ and $q, s > 1$ for $i = -1$.

In the next theorem, we investigate the hyperstability of orthogonally Pexider Lie homomorphisms.

Theorem 2.4. *Assume that $g, g_1, g_2 : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the system*

$$\|g(a+b) - g_1(a) - g_2(b)\| \leq \varphi(a,b),$$

$$\|g([a,b]) - [g_1(a), g_2(b)]\| \leq \phi(a,b),$$

where ϕ, φ satisfying in (2.4) and (2.5) for all $a, b \in \mathcal{A}$ with $a \perp b$. Let $g_1(0) = g_2(0) = 0$ and $\mathcal{A}$ be an orthogonally Lie Banach algebra without order, i.e. $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$ implies that $a = 0$. If g is an odd mapping, $\phi(0,0) = \varphi(0,0) = 0$ such that for all $a \in \mathcal{A}$ that is fixed, and some $v_a \in \mathcal{A}$ with $a \perp v_a$ and ψ ((2.6) in Theorem 2.2) has the property

$$\psi(a, v_a) \leq L 2^i \psi(\frac{a}{2^i}, \frac{v_a}{2^i}). \tag{2.16}$$

Then two mappings g_1, g_2 are orthogonally Lie homomorphisms, and if either $\varphi(0,a) = 0$ or $\varphi(a,0) = 0$, then g is an orthogonally Lie homomorphism.

Proof. By Theorem 2.2, mapping $G : \mathcal{A} \longrightarrow \mathcal{A}$ is an orthogonally Lie homomorphism such that

$$G(a) = \lim_{n\to\infty} \frac{g(2^{ni}a)}{2^{ni}} = \lim_{n\to\infty} \frac{g_1(2^{ni}a)}{2^{ni}} = \lim_{n\to\infty} \frac{g_2(2^{ni}a)}{2^{ni}} \qquad \forall\ a \in \mathcal{A}. \tag{2.17}$$

Since $g_1(0) = g_2(0) = 0$, then by applying (2.17) in (2.15), we conclude that $G(a)(G(b) - g_2(b)) = 0$ for all $a, b \in \mathcal{A}$. Therefore, $G = g_2$.

Let $a, b \in \mathcal{A}$ with $a \perp b$ and t be the mapping which defined in Theorem 2.2. By (2.5)

$$\lim_{n\to\infty} \frac{r(a, 2^{ni}b)}{2^{ni}} = 0. \tag{2.18}$$

Using (2.17) and (2.18), we have

$$G([a, b]) = [g_1(a), g_2(a)]. \tag{2.19}$$

According to proof of Theorem 2.2, we get

$$[\frac{g_1(2^{ni}a)}{2^{ni}}, g_2(b)] = [g_1(a), g_2(b)].$$

By applying (2.17) in the above relation we conclude that $g_1 = G$. By applying (2.16) to the either relation (2.10) or relation (2.11) satisfy in last hypothesis of this theorem, then g is orthogonally Lie homomorphism. ■

3. Orthogonally Pexider Lie Derivation

In this section, we investigate the stability and hyperstability of the orthogonally Pexider Lie derivations on orthogonally Lie Banach algebras by using the orthogonally fixed point approach.

Definition 3.1. Let $(A, \perp)$ be an orthogonally Lie Banach algebra and let $d, d_1, d_2 : A \longrightarrow A$ be mappings satisfying

$$\begin{cases} d(a + b) = d_1(a) + d_2(b), \\ d([a, b]) = [d_1(a), b] + [a, d_2(b)], \end{cases} \tag{3.1}$$

for all $a, b \in \mathcal{A}$ with $a \perp b$. The system (3.1) is called an orthogonally Pexiderized Lie derivation.

Theorem 3.2. *Let $d, d_1, d_2 : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings, d is odd and satisfying inequalities*

$$\|d(a + b) - d_1(a) - d_2(b)\| \le \varphi(a, b), \tag{3.2}$$

$$\|d([a, b]) - [d_1(a), b] - [a, d_2(b)]\| \le \phi(a, b), \tag{3.3}$$

where ϕ, φ satisfy in (2.4) and (2.5) for all $a, b \in \mathcal{A}$ with $a \perp b$. If d is an odd mapping, $\phi(0, 0) = \varphi(0, 0) = 0$ and there exists $0 < L < 1$ such that for any fixed $a \in \mathcal{A}$ and some $v_a \in \mathcal{A}$ with $a \perp v_a$, the mapping ψ ((2.6) in Theorem 2.2) has the property

$$\psi(a, v_a) \le L2^i\psi(\frac{a}{2^i}, \frac{v_a}{2^i}).$$

Then there exists is a unique orthogonally Lie derivation $D : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$\|d(a) - D(a)\| \le \frac{L^{\frac{1+i}{2}}}{1 - L}\psi(a, v_a),$$

$$\|d_1(a) - d_1(0) - D(a)\| \le \frac{L^{\frac{1+i}{2}}}{1 - L}\psi(a, v_a) + \varphi(a, 0),$$

$$\|d_2(a) - d_2(0) - D(a)\| \le \frac{L^{\frac{1+i}{2}}}{1 - L}\psi(a, v_a) + \varphi(0, a).$$

Proof. According to the proof of Theorem 2.2, there exists a mapping D which is the fixed point of J and satisfying

$$\|f(a) - D(a)\| \leq \frac{L^{\frac{1+i}{2}}}{1-L}\psi(a, v_a).$$

for all fixed $a, v_a \in \mathcal{A}$ with $a \perp v_a$. Moreover,

$$D(a) = \lim_{n\to\infty} \frac{d(2^{ni}a)}{2^{ni}} = \lim_{n\to\infty} \frac{d_1(2^{ni}a)}{2^{ni}} = \lim_{n\to\infty} \frac{d_2(2^{ni}a)}{2^{ni}}. \tag{3.4}$$

for all $a \in \mathcal{A}$.

To show that the mapping D is a Lie derivation, define $t : \mathcal{A} \longrightarrow \mathcal{A}$ by $t([a, b]) = d([a, b]) - [d_1(a), b] - [a, d_2(b)]$ for all $a, b \in \mathcal{A}$ with $a \perp b$. It follows from (2.5) that

$$\lim_{n\to\infty} \frac{t([2^{ni}a, b])}{2^{ni}} = 0. \tag{3.5}$$

Therefore from (3.4) and (3.5), we get

$$\begin{aligned} D([a, b]) &= \lim_{n\to\infty} \frac{d\big([2^{ni}a, b]\big)}{2^{ni}} \\ &= \lim_{n\to\infty} \frac{[d_1(2^{ni}a), b] + [2^{ni}a, d_2(b)] + t([2^{ni}a, b])}{2^{ni}} \\ &= [D(a), b] + [a, d_2(b)] \end{aligned} \tag{3.6}$$

where $a \perp b$. Now let $n \in \mathbb{N}$ be fixed and $a, b \in \mathcal{A}$ with $a \perp b$. By (3.6) and additivity of D, it can be shown that

$$\begin{aligned} [D(a), 2^{ni}b] + [a, d_2(2^{ni}b)] = D([a, 2^{ni}b]) = D([2^{ni}a, b]) &= [D(2^{ni}a), b] + [2^{ni}a, d_2(b)] \\ &= [2^{ni}D(a), b] + [2^{ni}a, d_2(b)] \end{aligned}$$

and then $[D(a), b] + [a, \frac{d_2(2^{ni}a)}{2^{ni}}] = D([a, b])$. So

$$D([a, b] = [D(a), b] + [a, D(b)].$$

whenever $n \to \infty$. This completes the proof. ∎

In the Next corollary, the result of Th. M. Rassias can be generalized if, we define

$$\varphi(a, b) = \varepsilon(\|a\|^q + \|b\|^q),$$

$$\phi(a, b) = \theta\|a\|^s\|b\|^r$$

where q, s, r and $\theta, \varepsilon \geq 0$ are real numbers. As a consequence of Theorem 2.4, we have hyperstability of orthogonally Pexider Lie derivation.

Theorem 3.3. *Let mappings $d, d_1, d_2 : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying*

$$\|d(a + b) - d_1(a) - d_2(b)\| \leq \varphi(a, b),$$

$$\|d([a, b]) - [d_1(a), b] - [a, d_2(b)]\| \leq \phi(a, b),$$

where ϕ, φ satisfy in (2.4) and (2.5) for all $a, b \in \mathcal{A}$ with $a \perp b$. Let $d_1(0) = d_2(0) = 0$ and $\mathcal{A}$ be an orthogonally Lie Banach algebra without order, i.e. $\mathcal{A}a = 0$ or $a\mathcal{A} = 0$ implies that $a = 0$. If d is an odd mapping, $\phi(0, 0) = \varphi(0, 0) = 0$ and there exists $0 < L < 1$ such *that for any fixed* $a \in \mathcal{A}$ *and some* $v_a \in \mathcal{A}$ *with* $a \perp v_a$*, the mapping* ψ *((2.6) in Theorem 2.2) has the property*

$$\psi(a, v_a) \leq L2^i \psi(\frac{a}{2^i}, \frac{v_i}{2^i}),$$

then two mappings d_1, d_2 *are orthogonally Lie derivations, and if either* $\varphi(0, a) = 0$ *or* $\varphi(a, 0) = 0$ *for all* $a \in \mathcal{A}$*, then* d *is an orthogonally Lie derivation.*

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